Chapter 1

Preliminaries

1. Well-orderings

A *well-ordering* of a set is a linear order $\prec$ of it such that every nonempty subset has a $\prec$-minimum. The natural ordering of $\mathbb{N}$ is a well-ordering; it is the shortest of all well-orderings of $\mathbb{N}$. To get an example of a longer well-ordering define $n \prec m$ iff 1) n is even and m is odd or 2) n and m are both even or both odd and $n < m$: we put all odd numbers after the even numbers but keep all even and all odd numbers in their natural order. This example illustrates one important difference between the natural order on $\mathbb{N}$ and most other well-orderings on $\mathbb{N}$: there will be elements, other than the minimum, without immediate predecessor. Indeed, 1 is not the $\prec$-minimum of $\mathbb{N}$, because $1024 \prec 1$, but if $n \prec 1$ then also $n + 2 \prec 1$ and $n \prec n + 2$. An element like 1 in the above example will be called a *limit*; elements with a direct predecessor will be called *successors*. Observe, however, that every element (other than the maximum) of a well-ordered set does have a direct successor. We shall denote the direct successor of an element x by $x + 1$.

▶ **1**. Every compact subset of the Sorgenfrey line is well-ordered by the natural order of $\mathbb{R}$. *Hint*: Let X be such a compact set and $A \subseteq X$. Note that X is also compact as a subset of $\mathbb{R}$ and hence bounded. Let $a = \inf A$ and consider a finite subcover of the open cover $\bigl\{(-\infty, a]\bigr\} \cup \bigl\{(b, \infty) : b > a\bigr\}$ of X; deduce that $a \in A$.

An *initial segment* of a well-ordered set $(X, \prec)$ is a subset I with the property that $x \in I$ and $y \prec x$ imply $y \in I$. Note that either $I = X$ or $I = \{x : x \prec p\}$, where $p = \min X \setminus I$. It follows that initial segments are comparable with respect to $\subset$. We let $\mathcal{I}_X$ denote the family of all proper initial segments of X and $\mathcal{I}_X^+$ denotes $\mathcal{I}_X \cup \{X\}$, the family of all initial segments of X. We shall often use $\hat{p}$ as a convenient shorthand for $\{x : x \prec p\}$.

Induction and recursion

Well-orderings enable us to do proofs by induction and perform constructions by recursion.

1.1. Theorem (Principle of Induction). *Let $(X, \prec)$ be a well-ordered set and let A be a subset of X such that $\{y : y \prec x\} \subseteq A$ implies $x \in A$ for all $x \in X$. Then $A = X$.*

This is essentially a reformulation of the definition of well-ordering. The straightforward proof is by contraposition: if $A \neq X$ then $x = \min X \setminus A$ satisfies $\hat{x} \subseteq A$ yet $x \notin A$.

There is an alternative formulation that bears closer resemblance to the familiar principle of mathematical induction.

▶ **2**. Let $(X, \prec)$ be a well-ordered set and let A be a subset of X that satisfies 1) $\min X \in A$, 2) if $x \in A$ then $x + 1 \in A$, and 3) if x is a limit element and $\hat{x} \subseteq A$ then $x \in A$. Then $A = X$.

The main use of this principle is in showing that all elements of a well-ordered set have a certain property. By way of example we consider isomorphisms of well-ordered sets.

▶ **3**. Isomorphisms between well-ordered sets are unique: if f and g are order-preserving bijections between $(X, \prec)$ and $(Y, \sqsubset)$ then $f = g$. *Hint*: Let $I = \bigl\{x : f(x) = g(x)\bigr\}$ and apply the principle of induction.

▶ **4**. The well-ordered sets $(X, \prec)$ and $(\mathcal{I}_X, \subset)$ are isomorphic.

1.2. THEOREM (Principle of Recursion). *Let $(X, \prec)$ be a well-ordered set, Y any set and $\mathcal{F}$ the family of all maps f whose domain is an initial segment of X and whose range is in Y. For every map $F : \mathcal{F} \to Y$ there is a unique map $f : X \to Y$ such that $f(x) = F\bigl(f \restriction \hat{x}\bigr)$ for every x.*

The proof of this principle offers a good exercise in working with well-orders.

▶ **5**. Let $\mathcal{G}$ be the subfamily of $\mathcal{F}$ consisting of all approximations of f: these are functions g that satisfy $g(x) = F\bigl(g \restriction \{y : y \prec x\}\bigr)$ for all x in their domains.

a. If $g, h \in \mathcal{G}$ and $\operatorname{dom} g \subseteq \operatorname{dom} h$ then $g = h \restriction \operatorname{dom} g$. *Hint*: Use the principle of induction.

b. The union $f = \bigcup \mathcal{G}$ is a function that belongs to $\mathcal{G}$.

c. The domain of f is equal to X.

d. If f and g are two functions that satisfy the conclusion of the principle of recursion then $f = g$.

The Principle of Recursion formalises the idea that a function can be constructed by specifying its initial segments. By way of example we show how it can be used to show that any two well-ordered sets are comparable.

We compare well-ordered sets by 'being an initial segment of', more precisely we say that $(X, \prec)$ is shorter than $(Y, \sqsubset)$ if there is $y \in Y$ such that $(X, \prec)$ is isomorphic to the initial segment $\{z : z \sqsubset y\}$ of Y.

▶ **6**. The well-ordered set $(X, \prec)$ is shorter than $(\mathcal{I}_X^+, \subset)$.

▶ **7**. Let $(X, \prec)$ and $(Y, \sqsubset)$ be well-ordered sets. Let $\mathcal{F}$ be the set of maps whose domain is an initial segment of X and whose range is contained in Y. Define

$F : \mathcal{F} \to Y \cup \{Y\}$ by $F(f) = \min(Y \setminus \operatorname{ran} f)$ if the right-hand side is nonempty and $F(f) = Y$ otherwise. Apply the principle of recursion to F to obtain a map $f : X \to Y \cup \{Y\}$.

a. If $x \prec y$ in X and $f(y) \neq Y$ then $f(x) \sqsubset f(y)$.
b. If there is an x in X such that $f(x) = Y$ then Y is shorter than X.
c. If $Y \notin \operatorname{ran} f$ then X and Y are isomorphic if $\operatorname{ran} f = Y$, and X is shorter than Y if $\operatorname{ran} f \subset Y$.

► **8**. Let $(X, \prec)$ and $(Y, \sqsubset)$ be well-ordered sets. Then either Y is shorter than or isomorphic to X or $\mathcal{I}_X^+$ is shorter than or isomorphic to Y.

In practice a construction by recursion proceeds less formally. One could, for example, describe the construction of the map f from the Exercise 1.7 as follows: "define $f(\min X) = \min Y$ and, assuming $f(y)$ has been found for all $y \prec x$, put $f(x) = \min Y \setminus \{f(y) : y \prec x\}$ if this set is nonempty, and $f(x) = Y$ otherwise". One would then go on to check that f had the required properties.

► **9**. The natural well-order of $\mathbb{N}$ is shortest among all well-orders of $\mathbb{N}$.
Hint: Apply the procedure from Exercise 1.7 to $(\mathbb{N}, \in)$ and $(\mathbb{N}, \prec)$, where $\prec$ is any other well-ordering of $\mathbb{N}$.

► **10**. Construct well-orders of $\mathbb{N}$ with the following properties (*Hint*: try to find compact subsets of $\mathbb{S}$):

a. with two limit elements;
b. with infinitely many limit elements;
c. where the set of limit elements is isomorphic to the set from a.

2. Ordinals

Let $(X, \prec)$ be a well-ordered set; we have already encountered a copy of X that is more set-like than X itself: $(X, \prec)$ is isomorphic to its family $\mathcal{I}_X$ of proper initial segments, which is well-ordered by $\subset$. We say that $(X, \prec)$ is an *ordinal* if $(X, \prec) = (\mathcal{I}_X, \subset)$; by this we mean that $X = \mathcal{I}_X$ and that $x \prec y$ iff $\hat{x} \subset \hat{y}$.

► **1**. The set $\mathbb{N}$, as described in Appendix B, is an ordinal.

► **2**. Let $(X, \prec)$ be an ordinal (with at least ten elements).

a. The minimum element of X is $\varnothing$.
b. The next element of X is $\{\varnothing\}$.
c. The one after that is $\{\varnothing, \{\varnothing\}\}$.
d. Write down the next few elements of X.

► **3**. Let $(X, \prec)$ be an ordinal.

a. For every $x \in X$ we have $x = \hat{x}$.
b. If $x \in X$ then its direct successor (if it exists) is $x \cup \{x\}$.

c. Every element of X is an ordinal.
d. For $x, y \in X$ the following are equivalent: $x \prec y$, $x \subset y$ and $x \in y$.
e. The set X is transitive, i.e., if $y \in X$ and $x \in y$ then $x \in X$.
f. The set X is well-ordered by $\in$.

The conjunction of the last parts of Exercise 2.3 actually characterises ordinals.

► **4**. Let X be a transitive set that is well-ordered by $\in$; then $(X, \in)$ is an ordinal.

This characterisation of ordinals can be simplified with the aid of the Axiom of Foundation (see Appendix A).

► **5**. A set is an ordinal iff it is transitive and linearly ordered by $\in$.

This last characterisation is now taken to be the definition of ordinals. This elegant way of singling out prototypical well-ordered sets is due to Von Neumann and has the advantage of using nothing but sets and the $\in$-relation. The Cantorian definition of an ordinal was 'order type of well-ordered set', which essentially meant that every ordinal was a proper class of sets.

We shall use (lower case) Greek letters to denote ordinals and we reserve the letter ω to denote the ordinal $\mathbb{N}$. The class of ordinals is well-ordered by $\in$. We extend the meaning of the word 'sequence' to include maps whose domein is some ordinal and we use similar notation. Thus $\langle x_\alpha : \alpha < \beta \rangle$ abbreviates the map f whose domain is β and whose value at α is x_α. We will call this a sequence of length β, or a β-sequence.

► **6**. Let α and β be ordinals.
a. $\alpha \cap \beta$ is an ordinal, call it γ;
b. $\gamma = \alpha$ or $\gamma = \beta$;
c. $\alpha \in \beta$ or $\alpha = \beta$ or $\beta \in \alpha$.

► **7**. a. If α is an ordinal then so is $\alpha \cup \{\alpha\}$.
b. If x is a set of ordinals then $\bigcup x$ is an ordinal.

► **8**. Let x be a set of ordinals.
a. $\bigcup x = \sup x$;
b. $\bigcap x = \min x$.

Every well-ordered set is isomorphic to exactly one ordinal, which we refer to as its *type*. We write $\operatorname{tp} X = \alpha$ to express that α is the type of X.

► **9**. Isomorphic ordinals are identical. *Hint*: Let $f : X \to Y$ be an isomorphism between ordinals and $A = \{x \in X : x = f(x)\}$, show that $A = X$ (see Exercise 2.2 for inspiration).

Thus every well-ordered set is isomorphic to at most one ordinal, to show that every well-ordered set is isomorphic to some ordinal we shall need the Axiom of Replacement.

▶ **10**. Every well-ordered set is isomorphic to an ordinal. Let $(X, \prec)$ be a well-ordered set and $A = \{x \in X : \hat{x}$ is isomorphic to an ordinal$\}$.

a. $\min X \in A$ because $\widehat{\min X} = \varnothing$.

b. If $x \in A$ then $x + 1 \in A$: if $\operatorname{tp} \hat{x} = \alpha$ then $\operatorname{tp} \widehat{x+1} = \alpha \cup \{\alpha\}$.

c. If x is a limit and $\hat{x} \subseteq A$ then $x \in A$ because $\operatorname{tp} \hat{x} = \{\operatorname{tp} \hat{y} : y \prec x\}$.

d. $\operatorname{tp} X = \{\operatorname{tp} \hat{x} : x \in X\}$.

The Axiom of Replacement was used twice: in the limit case and when assigning a type to X itself. In both cases we had the assignment $y \mapsto \operatorname{tp} \hat{y}$ and a set S ($\hat{x}$ and X respectively); the Axiom of Replacement guarantees that the $\operatorname{tp} \hat{y}$ with $y \in S$ can be collected in a set. It still remains to prove of course that the set $\{\operatorname{tp} \hat{y} : y \in S\}$ is an ordinal that is isomorphic to S.

The countable ordinals

The well-orderings of the subsets of $\mathbb{N}$ all belong to $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ and hence form a set, which we denote by WO. By the Axiom of Replacement their types form a set as well. Thus the countable ordinals can be seen to form a set, we denote it by ω_1.

▶ **11**. The set ω_1 is an ordinal and hence uncountable. *Hint*: Apply Exercise 2.5 and the Axiom of Foundation.

▶ **12**. Let $\{\prec_k : k \in \mathbb{N}\}$ be a family of well-orderings of $\mathbb{N}$. Define $\langle k, l\rangle \sqsubset \langle m, n\rangle$ iff $k < m$ or $k = m$ and $l \prec_k n$.

a. The relation $\sqsubset$ is a well-ordering of $\mathbb{N}^2$.

b. For every k the well-ordering $\prec_k$ is shorter than $\sqsubset$. *Hint*: The procedure in Exercise 1.7 applied to $(\mathbb{N}, \prec_k)$ and $(\mathbb{N}^2, \sqsubset)$ yields a map of $\mathbb{N}$ into $k \times \mathbb{N}$.

c. If A is a countable subset of ω_1 then there is $\beta \in \omega_1$ such that $\alpha < \beta$ for all $\alpha \in A$. *Hint*: Use the Axiom of Choice.

▶ **13**. If $\alpha \in \omega_1$ is a limit then there is a strictly increasing sequence $\langle \alpha_n \rangle_n$ such that $\alpha = \sup_n \alpha_n$. *Hint*: $C = \{\beta : \beta < \alpha\}$ is countable; let $f : \mathbb{N} \to C$ be a bijection and recursively find $k_n \in \mathbb{N}$ such that $f(k_{n+1}) > f(n), f(k_n)$; put $\alpha_n = f(k_n)$.

We have two uncountable objects associated with $\mathbb{N}$: its power set $\mathcal{P}(\mathbb{N})$ and the set of countable ordinals ω_1. There is a natural map from $\mathcal{P}(\mathbb{N})$ (or rather $\mathcal{P}(\mathbb{N} \times \mathbb{N})$) onto ω_1: map $A \subseteq \mathbb{N}^2$ to the type of $(\mathbb{N}, A)$ if A is a well-ordering and to 0 otherwise. On the other hand, a choice of one well-order (of the right type) for each ordinal in ω_1 produces an injection of ω_1 into $\mathcal{P}(\mathbb{N})$. This was a blatant application of the Axiom of Choice and there is no easy description of such an injection from ω_1 into $\mathcal{P}(\mathbb{N})$ as it can be used to construct a nonmeasurable subset of $\mathbb{R}$. This should give some indication of the essential difference between the entities $\mathcal{P}(\mathbb{N})$ and ω_1.

On an elementary level we have the following nonexistence results.

▶ **14**.a. There is no map f from ω_1 into $\mathcal{P}(\mathbb{N})$ such that $f(\alpha)$ is a proper subset of $f(\beta)$ whenever $\alpha < \beta$. *Hint*: If f were such a map consider the set of α for which $f(\alpha+1) \setminus f(\alpha)$ is nonempty.

b. There is no map $f : \omega_1 \to \mathbb{R}$ such that $\alpha < \beta$ implies $f(\alpha) < f(\beta)$.

However, we can get a sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{P}(\mathbb{N})$ that is strictly increasing with respect to almost containment. When working with subsets of $\mathbb{N}$ 'almost' means 'with finitely many exceptions'. We attach an asterisk to a relation to indicate that it holds almost. Thus, $a \subseteq^* b$ means that $a \subseteq b$ with possibly finitely many points of a not belonging to b, in other words that $a \setminus b$ is finite. Similarly, $a \subset^* b$ means $a \subseteq^* b$ but $b \setminus a$ is infinite, and $a \cap b =^* \varnothing$ is expressed by saying that that a and b are almost disjoint.

▶ **15**. Let $\langle a_n \rangle_n$ be a sequence in $\mathcal{P}(\mathbb{N})$ such that $a_n \subset^* a_{n+1}$ for all n. There is a set $a \in \mathcal{P}(\mathbb{N})$ such that $a_n \subset^* a$ for all n and $a \subset^* \mathbb{N}$. *Hint*: Note that $a_n \setminus \bigcup_{m<n} a_m$ is always infinite; pick k_n in this difference and let $a = \mathbb{N} \setminus \{k_n : n \in \mathbb{N}\}$.

▶ **16**. There is a sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{P}(\mathbb{N})$ such that $x_\alpha \subset^* x_\beta$ whenever $\alpha < \beta$. *Hint*: Construct the x_α by recursion, applying Exercises 2.13 and 2.15 in the limit case.

Every ordinal (indeed every linearly ordered set) carries a natural topology, the *order topology*, which has the sets of the form $(\leftarrow, x)$ and $(x, \rightarrow)$ as a subbase. Unless explicitly specified otherwise we always assume, in topological situations, that ordinals carry their order topologies. The space ω_1 features in many counterexamples.

▶ **17**. The space ω_1 has the following properties.

a. It is first-countable: $\{(\beta, \alpha] : \beta < \alpha\}$ is a countable local base at α.

b. It is sequentially compact, i.e., every sequence has a converging subsequence. *Hint*: Given a sequence $\langle \alpha_n \rangle_n$ consider the minimal α for which $\{n : \alpha_n \leqslant \alpha\}$ is infinite.

c. The space ω_1 is not compact.

▶ **18**. If $f : \omega_1 \to \mathbb{R}$ is continuous then there is an α in ω_1 such that f is constant on the final segment $[\alpha, \omega_1)$.

a. For every α the set $F_\alpha = f\bigl[[\alpha, \omega_1)\bigr]$ is compact.

b. The set of α for which $F_{\alpha+1} \subset F_\alpha$ is countable. *Hint*: Choose an interval I_α with rational endpoints that meets $F_\alpha \setminus F_{\alpha+1}$; observe that $\alpha \mapsto I_\alpha$ is one-to-one.

c. Fix α such that $F_\beta = F_{\beta+1}$ for all $\beta \geqslant \alpha$. Then $F_\beta = F_\alpha$ for all $\beta \geqslant \alpha$.

d. There is only one point in F_α. *Hint*: If $x, y \in F_\alpha$ then there is a sequence $\langle \alpha_n \rangle_n$ such that $\alpha \leqslant \alpha_0 < \alpha_1 < \alpha_2 < \cdots$ and $f(\alpha_i) = x$ if i is even and $f(\alpha_i) = y$ if i is odd. Let $\beta = \sup_i \alpha_i$ and note that $x = f(\beta) = y$.

Cardinal numbers

We single out one important class of ordinals: the cardinal numbers or cardinals for short. Cardinal numbers will be used to measure the sizes of sets, rather than their order types. Accordingly, a *cardinal number* is an ordinal that is an *initial ordinal*, a 'smallest among equals': α is an initial ordinal if whenever β is an ordinal and $f : \alpha \to \beta$ a bijection one has $\beta \geqslant \alpha$. The contrapositive of this formulation reads: if $\beta < \alpha$ then there is no bijection between α and β. We generally reserve the letters κ, λ and μ for cardinal numbers.

► **19**.a. Every natural number is a cardinal number. *Hint*: Consider $I = \{n : \text{there is no injection from } n+1 \text{ into } n\}$.
b. The ordinal ω is a cardinal number. *Hint*: A bijection between ω and n induces an injection from $n+1$ into n.
c. The ordinal ω_1 is a cardinal number.
d. Every infinite cardinal is a limit ordinal.

The construction of ω_1 can be generalised to show that there is no largest cardinal number.

► **20**. Let X be a set and let $\mathrm{WO}(X)$ be the set of all well-orders of subsets of X.
a. The set of types of the elements of $\mathrm{WO}(X)$ is an ordinal, which we denote by X^+.
b. The ordinal X^+ is a cardinal. *Hint*: Every $\alpha < X^+$ admits an injection into X, but X^+ itself does not.

An important property of infinite cardinal numbers is that they are equal to their own squares. To see define a relation $\prec$ between pairs of ordinals as follows: $\langle\alpha, \beta\rangle \prec \langle\gamma, \delta\rangle$ iff 1) $\max\{\alpha, \beta\} < \max\{\gamma, \delta\}$, or 2) $\beta < \delta$, or 3) $\beta = \delta$ and $\alpha < \gamma$.

► **21**.a. If ξ is an ordinal then $\prec$ is well-ordering of $\xi \times \xi$.
b. If κ is an infinite cardinal and $\alpha, \beta \in \kappa$ then the order type of $\bigl\{\langle\gamma, \delta\rangle : \langle\gamma, \delta\rangle \prec \langle\alpha, \beta\rangle\bigr\}$ is smaller than κ. *Hint*: Determine the type of $\bigl\{\langle\gamma, \delta\rangle : \max\{\gamma, \delta\} = \alpha\bigr\}$ and apply induction with respect to κ.
c. If κ is an infinite cardinal then the order type of $\kappa \times \kappa$, with respect to $\prec$, is κ.

► **22**. If κ is a cardinal then κ^+ is the smallest cardinal that is larger than κ. In addition, if κ is infinite and $f : \kappa \to \kappa^+$ is any map then there is an $\alpha < \kappa^+$ such that $f(\beta) < \alpha$ for all $\beta < \kappa$.

The cardinals are, as a subclass of the ordinals, well-ordered. The finite cardinals correspond to the natural numbers; we index the infinite cardinals by the ordinals. Thus $\omega_0 = \omega$, $\omega_1 = \omega_0^+$, $\omega_2 = \omega_1^+$ and generally $\omega_{\alpha+1} = \omega_\alpha^+$ for every α. If α is a limit then $\sup\{\omega_\beta : \beta < \alpha\}$ is a cardinal, denoted ω_α.

It is customary to distinguish between the two identities of the cardinal numbers ω_α: when we think of it as an ordinal we keep writing ω_α, we write $\aleph_\alpha$ when treating it as a cardinal. Every cardinal of the form $\aleph_{\alpha+1}$ is called a *successor cardinal*; if α is a limit ordinal then we call $\aleph_\alpha$ a *limit cardinal*. Finally then, the *cardinality* of a set X is the unique cardinal number κ such that there is a bijection between X and κ.

Cofinality

We have seen that for every countable limit ordinal α there is a strictly increasing sequence $\langle\alpha_n\rangle_n$ such that $\alpha = \sup_n \alpha_n$. The set $A = \{\alpha_n : n \in \omega\}$ is *cofinal* in α in that for every $\beta \in \alpha$ there is a $\gamma \in A$ such that $\beta < \gamma$. Since every ordinal has a cofinal subset, to wit itself, we can define the *cofinality* of α, denoted $\operatorname{cf}\alpha$, to be the minimal type of a cofinal subset of α.

► **23**.a. $\operatorname{cf}\operatorname{cf}\alpha = \operatorname{cf}\alpha$;
b. $\operatorname{cf}\alpha$ is a cardinal number.

There are two types of cardinal numbers, those that equal their cofinalities, like $\aleph_0$ and $\aleph_1$, and those that do not, e.g., $\aleph_\omega$ has cofinality ω_0. We call κ *regular* if the former applies, i.e., if $\kappa = \operatorname{cf}\kappa$, and *singular* if $\operatorname{cf}\kappa < \kappa$.

► **24**. Every infinite successor cardinal is regular.

As we shall see regular cardinals are very often easier to handle than singular ones. As will become apparent, many recursive constructions use up small portions of a set per step; if the cardinality, κ, of the set is regular and each step uses up fewer than κ elements then the set will not be exhausted until the end of the construction.

3. Some combinatorics

The Pressing-Down Lemma

Let κ be a cardinal; a function $f : \kappa \to \kappa$ is said to be *regressive* if $f(\alpha) < \alpha$ for all $\alpha > 0$. For example the function $n \mapsto \max\{0, n-1\}$, is regressive on ω; note that this function is one-to-one on $\omega \setminus \{0\}$. On regular uncountable cardinals this is not possible.

3.1. THEOREM. *Let κ be regular and uncountable and $f : \kappa \to \kappa$ a regressive function. Then f is constant on an unbounded subset of κ.*

► **1**. Assume that for all α the preimage $f^{\leftarrow}(\alpha)$ is bounded, say by β_α.
a. If $\gamma < \kappa$ then $\sup\{\beta_\alpha : \alpha \leqslant \gamma\} < \kappa$.
Define $\gamma_0 = 0$ and, recursively, $\gamma_{n+1} = \sup\{\beta_\alpha : \alpha \leqslant \gamma_n\}$.
b. $\gamma_0 < \gamma_1 < \cdots$. *Hint*: $\gamma_{n+1} \geqslant \beta_{\gamma_n}$.
c. $\gamma = \sup_n \gamma_n < \kappa$.
d. $f(\gamma) \geqslant \gamma$, so f is not regressive.

This can be used, for example, to redo Exercise 2.18.

► **2**. If κ is regular and uncountable and $f : \kappa \to \mathbb{R}$ is continuous then f is constant on $[\alpha, \kappa)$ for some α.
a. For every n there is an α_n such that $\bigl|f(\beta) - f(\alpha_n)\bigr| < 2^{-n}$ whenever $\beta \geqslant \alpha_n$. *Hint*: Define a regressive function f_n such that $\bigl|f(\beta) - f(\alpha)\bigr| < 2^{-n}$ whenever $f_n(\alpha) < \beta \leqslant \alpha$ and apply the pressing-down lemma.

b. The ordinal $\alpha = \sup_n \alpha_n$ is as required.

▶ **3**. Let α be an ordinal of countable cofinality; define a regressive $f : \alpha \to \alpha$ with all preimages $f^{\leftarrow}(\beta)$ bounded.

The Δ-system lemma

A family $\mathcal{D}$ of sets is a Δ-*system* if there is a single set R, the *root*, such that $D_1 \cap D_2 = R$ whenever $D_1, D_2 \in \mathcal{D}$ are distinct. So certainly a pairwise disjoint family is a Δ-system. The Δ-system lemma says that a large enough family of small enough sets can be thinned out to a large Δ-system.

▶ **4**. Let κ be a regular uncountable cardinal and $\{F_\alpha : \alpha < \kappa\}$ a family of finite sets. There is an unbounded subset A of κ such that $\{F_\alpha : \alpha \in A\}$ is a Δ-system.

a. Without loss of generality $F_\alpha \subseteq \kappa$ for all α.

b. The function $f : \alpha \mapsto \max\bigl(\{0\} \cup (F_\alpha \cap \alpha)\bigr)$ is regressive.

c. There are $\beta < \kappa$ and an unbounded set C in κ such that $F_\alpha \cap \alpha \subseteq \beta$ whenever $\alpha \in C$.

d. There are a finite subset R of β and an unbounded subset B of C such that $F_\alpha \cap \alpha = R$ whenever $\alpha \in B$.

e. There is an unbounded subset A of B such that $\max F_\alpha < \gamma$ whenever $\alpha < \gamma$ in A.

f. The set A is as required.

▶ **5**. a. The family $\{n : n \in \omega\}$ is a countable family of finite sets without a three-element Δ-sytem in it.

b. Let κ be singular of cofinality λ and let $\langle \alpha_\eta : \eta < \lambda \rangle$ be increasing and cofinal in κ. Let $\mathcal{F} = \bigl\{\{\alpha_\eta, \beta\} : \alpha_\eta < \beta < \alpha_{\eta+1}\bigr\}$. Every Δ-system in $\mathcal{F}$ is of cardinality less than κ.

Closed unbounded sets

We consider the order-topology on ordinals and in particular on regular cardinal numbers. Let κ be regular and uncountable. A *closed and unbounded set* in κ is one that is closed in the order topology and cofinal in κ. We write $\mathcal{C} = \{C : C \text{ is closed and unbounded in } \kappa\}$.

For the moment we concentrate on $\kappa = \omega_1$.

▶ **6**. $\mathcal{C}$ is closed under countable intersections.

a. If $C_0, C_1 \in \mathcal{C}$, then $C_0 \cap C_1 \in \mathcal{C}$. *Hint*: To show unboundedness let α be arbitrary and choose $\langle \alpha_n : n \in \omega \rangle$ strictly increasing with $\alpha_0 > \alpha$ and $\alpha_{2n+i} \in C_i$; consider $\sup_n \alpha_n$.

b. Let $\{C_n : n \in \omega\} \subseteq \mathcal{C}$, then $\bigcap_n C_n \in \mathcal{C}$. *Hint*: As above but now choose α_n in $\bigcap_{i \leqslant n} C_i$.

We can improve this exercise by using another kind of intersection. Let $\{A_\alpha : \alpha \in \omega_1\}$ be a family of subsets of ω_1. The *diagonal intersection* of this family is defined as

$$\mathop{\triangle}_{\alpha} A_\alpha = \bigl\{\delta : (\forall\gamma < \delta)(\delta \in A_\gamma)\bigr\}.$$

For families of closed unbounded sets this intersection is never empty.

▶ **7**. Let $\{C_\alpha : \alpha \in \omega_1\} \subseteq \mathcal{C}$, then $C = \triangle_\alpha C_\alpha \in \mathcal{C}$.

a. C is unbounded. *Hint*: Given α choose α_n recursively above α such that $\alpha_{n+1} \in \bigcap_{\beta\leqslant\alpha_n} C_\beta$ and consider $\sup_n \alpha_n$.

b. C is closed. *Hint*: If $\langle\alpha_n : n \in \omega\rangle$ is strictly increasing in C then $\alpha = \sup_n \alpha_n$ belongs to $\bigcap_{\beta<\alpha_n} C_\beta$ for all n.

▶ **8**. Generalize the exercises above to larger κ.

Stationary sets

A subset of a regular cardinal κ is said to be *stationary* if it meets every closed and unbounded subset of κ. Stationary sets play a large role in many set-theoretic and topological arguments, a we shall see later. As an example we show how the Pressing-Down Lemma can be strengthened.

▶ **9**. Let S be a stationary subset of a regular cardinal κ and $f : S \to \kappa$ a regressive function. Then f is constant on a stationary set. *Hint*: Assume that for every α there is a $C_\alpha \in \mathcal{C}$ that is disjoint from $f^{\leftarrow}(\alpha)$; consider a point δ in $\triangle_\alpha C_\alpha$.

Stationary subsets share a topological property with regular uncountable cardinals.

▶ **10**. Let S be an unbounded subset of some regular uncountable cardinal κ. Then S is stationary iff every continuous function $f : S \to \mathbb{R}$ is constant on a tail.

a. For $C \in \mathcal{C}$ (with unbounded complement) define a continuous function $f : \kappa \setminus C \to \mathbb{R}$ that is not constant on any tail.

b. If S is stationary and $f : S \to \mathbb{R}$ is continuous then there is an α such that f is constant on $S \setminus \alpha$. *Hint*: Apply the strong form of the Pressing-Down Lemma.

We can find large disjoint families of stationary sets on ω_1.

▶ **11**. For each $\alpha < \omega_1$ let $f_\alpha : \omega \to \alpha$ be a surjection. Define $A_{\beta,n} = \{\alpha : f_\alpha(n) = \beta\}$.

a. For all n: if $\beta \neq \gamma$ then $A_{\beta,n} \cap A_{\gamma,n} = \varnothing$.

b. For all β we have $(\beta, \omega_1) \subseteq \bigcup_n A_{\beta,n}$.

c. For every β there is an n_β such that A_{β,n_β} is stationary. *Hint*: If not we get $\{C_n : n \in \omega\} \subseteq \mathcal{C}$ with $\bigcap_n C_n \subseteq \beta + 1$.

d. There is $n \in \omega$ for which $S = \{\beta : n = n_\beta\}$ is stationary.

e. $\{A_{\beta,n} : \beta \in S\}$ is a disjoint family of stationary subsets.

4. Trees

A *tree* is a partially ordered set in which every set of predecessors is well-ordered. More formally, consider a partially ordered set $(P, \preccurlyeq)$ and, for $x \in P$ put $\hat{x} = \{y \in P : y \prec x\}$. We say that $(P, \preccurlyeq)$ is a tree if every set $\hat{x}$ is well-ordered.

▶ **1**. Let $\mathcal{Z}(\mathbb{S})$ denote the set of compact subsets of the Sorgenfrey line. Order $\mathcal{Z}(\mathbb{S})$ by 'being an initial segment of', i.e., $C \preccurlyeq D$ iff $C = D \cap (-\infty, \max C]$.
  a. The relation $\preccurlyeq$ is a partial order.
  b. $\hat{D} = \{C : C \prec D\}$ is well-ordered and isomorphic to $D \setminus \{\max D\}$.

▶ **2**. Let α be an ordinal and X a set. Let ${}^{<\alpha}X$ denote the set of functions with domain some β less than α and range contained in X; in short ${}^{<\alpha}X = \bigcup_{\beta<\alpha} {}^{\beta}X$. Then ${}^{<\alpha}X$ is a tree when ordered by inclusion. For every $s \in {}^{<\alpha}X$ the order type of $\hat{s}$ is its domain.

A special example is ${}^{<\omega}2$, the tree of finite sequences of zeros and ones, ordered by extension.

A tree is divided into levels: if $(T, \preccurlyeq)$ is a tree and α is an ordinal then T_α denotes the set of $t \in T$ for which $\hat{t}$ has type α. We write $\operatorname{ht} t = \operatorname{tp} \hat{t}$; thus $T_\alpha = \{t : \operatorname{ht} t = \alpha\}$. The minimal ordinal α for which $T_\alpha = \varnothing$ is called the *height* of T. The height of T exists because of the Axiom of Replacement.

▶ **3**. The height of a tree T is equal to $\sup\{\operatorname{ht} t + 1 : t \in T\}$.

▶ **4**. The height of the tree $\mathcal{Z}(\mathbb{S})$ is ω_1.

A *branch* of a tree (or a *path*) is a maximal linearly ordered subset.

▶ **5**. A subset B of a tree is a branch iff it is linearly ordered, contains $\hat{t}$ whenever $t \in B$, and there is no t such that $B \subseteq \hat{t}$.

▶ **6**. The tree $\mathcal{Z}(\mathbb{S})$ has no branches of type ω_1.

König's Lemma

A very useful result about infinite trees is the following.

4.1. Theorem (König's Lemma). *Let T be an infinite tree in which for every $n \in \omega$ the level T_n is finite. Then there is a sequence $\langle t_n \rangle_n$ in T such that $t_n \in T_n$ and $t_n < t_{n+1}$ for all n.*

▶ **7**. Prove König's Lemma. *Hint*: Construct $\langle t_n \rangle_n$ by recursion: choose $t_0 \in T_0$ with $\{s : t_0 < s\}$ infinite, then $t_1 \in T_1$ with $t_1 > t_0$ and $\{s : t_1 < s\}$ infinite, ...

König's Lemma has many applications.

▶ **8**. The topological product ${}^{\omega}2$ is compact. Let $\mathcal{U}$ be a family of open sets, no finite subfamily of which covers ${}^{\omega}2$ and let T be the set of s for which $[s]$ is not covered by a finite subfamily of $\mathcal{U}$, where for $s \in {}^{<\omega}2$ we put $[s] = \{x \in {}^{\omega}2 : s \subset x\}$.

a. The family $\{[s] : s \in {}^{<\omega}2\}$ is a base for the topology of ${}^{\omega}2$.
b. T is an infinite subtree of ${}^{<\omega}2$. *Hint*: If $s \in T$ then $s^\frown 0 \in T$ or $s^\frown 1 \in T$.
c. König's Lemma implies that $\mathcal{U}$ does not cover ${}^{\omega}2$.

This exercise has a converse.

▶ **9**. König's Lemma can be deduced from the compactness of ${}^{\omega}2$. Let T be an infinite tree with finite levels and consider the topological product $X = \prod_{n\in\omega} T_n$, where each T_n is given the discrete topology. For $n \in \omega$ put $F_n = \{x \in X : (\forall i < n)\big(x(i) < x(i+1)\big)\}$.
a. X is compact. *Hint*: X can be embedded into ${}^{\omega}2$ as a closed subset.
b. If $x \in \bigcap_{n\in\omega} F_n$ then $\langle x(n) : n \in \omega\rangle$ satisfies the conclusion of König's Lemma.
c. For each n the set F_n is clopen and nonempty; in addition $F_{n+1} \subseteq F_n$.
d. $\bigcap_{n\in\omega} F_n \neq \varnothing$.

As a further example we prove the simplest version of Ramsey's theorem. For this we establish some notation: $[\omega]^2$ denotes the family of 2-element subsets of ω. A map $c : [\omega]^2 \to 2$ is said to be a colouring of $[\omega]^2$ and a subset A of ω is said to be *c-homogeneous* or just *homogeneous* if c is constant on $[A]^2$. For ease of notation we identify $[\omega]^2$ with $\{\langle i, j\rangle : i < j < \omega\}$.

4.2. THEOREM (Ramsey's theorem). *For every colouring of $[\omega]^2$ there is an infinite homogeneous set.*

▶ **10**. Prove Ramsey's theorem. Given a colouring $c : [\omega]^2 \to 2$ define a subtree $T = \{t_n : n \in \omega\}$ of ${}^{<\omega}2$ as follows: $t_0 = \varnothing$; if $n > 0$ and the t_i for $i \in n$ have been found define $t_n \restriction m$ by recursion: if $t_n \restriction m = t_i$ for some $i \in n$ then put $t_n(m) = c(i, n)$, if $t_n \restriction m \neq t_i$ for all i then stop: $t_n = t_n \restriction m$.
a. The map $n \mapsto t_n$ is one-to-one.
b. If $s < t_n$ then $s = t_i$ for some $i \in n$.
c. If $t_i < t_m < t_n$ then $c(i, m) = c(i, n)$.
d. If $\{t_n : n \in A\}$ is a branch through T then A is *prehomogeneous*, i.e., if $i \in j \in k$ in A then $c(i, j) = c(i, k)$.
e. An infinite prehomogeneous set contains an infinite homogeneous set.

▶ **11**. Every sequence in $\mathbb{R}$ (or any linearly ordered set) has a monotone subsequence. *Hint*: Given such a sequence $\langle x_n\rangle_n$ define $f : [\omega]^2 \to 2$ by $f(i, j) = 1$ if $x_i < x_j$ and $f(i, j) = 0$ if $x_i \geqslant x_j$ (where $i \in j$ is tacitly assumed).

Aronszajn trees

The proof of König's Lemma was a fairly easy recursion and it may seem that a straightforward adaptation will show that every tree of height ω_1 with countable levels has a branch of type ω_1.

▶ **12**. Investigate where such an adaptation of the proof of König's Lemma is liable to break down.

An *Aronszajn tree* is a tree of height ω_1 with all levels countable, but without a branch of type ω_1.

▶ **13**. There is an Aronszajn tree T contained in $\{t \in \mathcal{Z}(\mathbb{S}) : t \subseteq \mathbb{Q}\}$. Construct T by recursion, one level at a time and maintaining the following property $\dagger_\alpha$: if $\gamma < \beta \leqslant \alpha$, $s \in T_\gamma$ and $q > \max s$, where $q \in \mathbb{Q}$, then there is $t \in T_\beta$ such that $s \preccurlyeq t$ and $\max t = q$. Set $T_0 = \{\varnothing\}$.

a. Given T_α put $T_{\alpha+1} = \{t \cup \{q\} : t \in T_\alpha, q \in \mathbb{Q}, q > \max t\}$. If T_α is countable then so is $T_{\alpha+1}$. If $\dagger_\alpha$ holds then so does $\dagger_{\alpha+1}$.

b. If α is a limit and T_β has been found for $\beta < \alpha$ such that $\dagger_\beta$ holds for all $\beta < \alpha$ choose an increasing sequence $\langle \alpha_n \rangle_n$ with $\alpha = \sup_n \alpha_n$. For every pair $\langle t, q \rangle$, where $t \in \bigcup_{\beta<\alpha} T_\beta$ and $q \in \mathbb{Q}$ with $q > \max t$, let n_0 be minimal so that $t \in \bigcup_{\beta<\alpha_{n_0}} T_\beta$ and choose a sequence $\langle t_n \rangle_{n \geqslant n_0}$ with $t_n \in T_{\alpha_n}$, $t \preccurlyeq t_n \preccurlyeq t_{n+1}$ and $q - 2^{-n} \leqslant \max t_n < q$ for all n. Put $s_{t,q} = \bigcup_n t_n \cup \{q\}$ and let T_α be the set of all the $s_{t,q}$ thus obtained. Then T_α is countable and $\dagger_\alpha$ holds.

c. If $n \in \omega$ and $t \in T_n$ then $\operatorname{tp} t = n$.

d. If $\alpha \geqslant \omega$ and $t \in T_\alpha$ then $\operatorname{tp} t = \alpha + 1$.

e. The tree $T = \bigcup_{\alpha\in\omega_1} T_\alpha$ is an Aronszajn tree.

▶ **14**. An alternative construction. There is a sequence $\langle r_\alpha : \alpha < \omega_1 \rangle$ such that r_α is an injective map from α into $\mathbb{N}$ and such that $r_\alpha =^* r_\beta \restriction \alpha$ whenever $\alpha < \beta$. The tree $T = \{r_\beta \restriction \alpha : \alpha \leqslant \beta < \omega_1\}$ is an Aronszajn tree. *Hint*: Refer to Exercise 2.16. Choose the r_α by recursion, making sure that $r_\alpha[\alpha] \subseteq x_\alpha$.