Chapter 6

Rudin's Dowker space

Rudin's Dowker space is of a totally different nature than that of Balogh; it was based on an example of Misčenko's of a linearly Lindelöf space that is not Lindelöf.

1. Description of the space

We work in the product $P = \prod_{n=1}^{\infty}(\omega_n + 1)$ of the successors of the first $\aleph_0$ many uncountable ordinals. We give P the *box topology*, where each ordinal has its usual *order topology*. The box topology has the family of all open boxes as a base; an *open box* is simply a product $\prod_{n=1}^{\infty} O_n$, where O_n is open in $\omega_n + 1$.

We consider two subspaces of P:

$$X' = \{x \in P : (\forall n)(\operatorname{cf} x_n > \omega_0)\}$$

and its subset

$$X = \{x \in P : (\exists i)(\forall n)(\omega_i > \operatorname{cf} x_n > \omega_0)\}.$$

The space X is Rudin's Dowker space. The rest of this chapter will be devoted to verifying this.

A nice base

We need an easy-to-handle base for the topology of X' and X. To this end we introduce the following notation. For $x, y \in P$ we say $x < y$ if $x_n < y_n$ for all n and $x \leqslant y$ means $x_n \leqslant y_n$ for all n. For $x, y \in P$ with $x < y$ we use $(x, y]$ to denote the set $\{z \in X' : (\forall n)(x_n < z_n \leqslant y_n)\}$, i.e, $(x, y] = X' \cap \prod_{n=1}^{\infty}(x_n, y_n]$.

▶ **1**. If $x \in X'$ then $\{(y, x] : y < x\}$ is a local base at x.

We shall be using the family $\mathcal{B} = \{(x, y] : x, y \in P,\, x < y\}$ as a base for the open sets of X'. The following consequence of the choice of points in X' will be very useful.

▶ **2**. X' is a P-space, i.e., if $\mathcal{U}$ is a countable family of open sets then $\bigcap \mathcal{U}$ is open.

2. X is normal

To prove X is normal we prove two things:

1. every open cover of X' has a disjoint open refinement, and
2. if A and B are closed and disjoint *in* X then their closures *in* X' are disjoint too.

► **1**. The two statements above imply that X is indeed normal.

The property that every open cover has a disjoint open refinement is called *ultraparacompactness*; it is (much) stronger than ordinary paracompactness.

X' is ultraparacompact

Let $\mathcal{O}$ be an open cover of X'. We build a sequence $\langle \mathcal{U}_\alpha : \alpha < \omega_1 \rangle$ of open covers of X' such that

1. each $\mathcal{U}_\alpha$ is a disjoint open cover and a subfamily of $\mathcal{B}$,
2. if $\alpha < \beta$ then $\mathcal{U}_\beta$ is a refinement of $\mathcal{U}_\alpha$,
3. if $U \in \mathcal{U}_\alpha$ and $U \subseteq O$ for some $O \in \mathcal{O}$ then $U \in \mathcal{U}_{\alpha+1}$, and
4. if $U \in \mathcal{U}_\alpha$, say $U = (x, y]$, and $U \subseteq O$ for no $O \in \mathcal{O}$ then for every $V \in \mathcal{U}_{\alpha+1}$ with $V \subseteq U$ and $V = (u, v]$ there is some n such that $v_n < y_n$ or $V \subseteq O$ for some $O \in \mathcal{O}$.

► **2**. Let $y \in X'$ and denote for $\alpha < \omega_1$ the unique element of $\mathcal{U}_\alpha$ that contains y by $(u_\alpha, v_\alpha]$.

a. For every n there is an α_n such that $v_\alpha(n) = v_{\alpha_n}(n)$ whenever $\alpha \geqslant \alpha_n$.

Let $\alpha_y = \sup_n \alpha_n$ and $\beta = \alpha_y + 1$.

b. There is an $O \in \mathcal{O}$ with $(u_\beta, v_\beta] \subseteq O$.

c. If $\gamma \geqslant \beta$ then $(u_\gamma, v_\gamma] = (u_\beta, v_\beta]$.

► **3**. The family $\{(u_{\alpha_y}, v_{\alpha_y}] : y \in X'\}$ is a disjoint open refinement of $\mathcal{O}$.

To construct the sequence we start with $\mathcal{U}_0 = \{X'\}$. Note that $X' = (0, t]$, where $t_i = \omega_i$ for all i.

To make $\mathcal{U}_{\alpha+1}$ from $\mathcal{U}_\alpha$ let $U \in \mathcal{U}_\alpha$, say $U = (x, y]$. If there is an $O \in \mathcal{O}$ with $U \subseteq O$ put $\mathcal{I}_U = \{U\}$. If not then consider two cases.

$y \in X'$ Take $z < y$ so that $x < z$ and $(z, y] \subseteq O$ for some $O \in \mathcal{O}$. For every subset A of $\mathbb{N}$ put

$$V_A = \{u \in (x, y] : (\forall i \in A)(u_i \leqslant z_i) \wedge (\forall i \notin A)(u_i > z_i)\}.$$

Set $\mathcal{I}_U = \{V_A : A \subseteq \mathbb{N}\}$.

$y \notin X'$ Fix n with $\operatorname{cf} y_n = \omega_0$ and fix an increasing cofinal sequence $\langle \alpha_i \rangle_i$ of ordinals in y_n with $\alpha_0 = x_n$. For $i \in \omega$ put $V_i = \{u \in (x, y] : \alpha_i < u_n \leqslant \alpha_{i+1}\}$ and let $\mathcal{I}_U = \{V_i : i \in \omega\}$.

Disjoint closed sets in X have disjoint closures in X'

Let A and B be closed and disjoint in X. Define $A_n = \{x \in A : (\forall i)(\operatorname{cf} x_i \leqslant \aleph_n)\}$ and define B_n similarly.

▶ **4**. It suffices to show that for every n the sets A_n and B_n have disjoint closures in X'. *Hint*: $A = \bigcup_n A_n$ and X' is a P-space.

Fix n and take $x \in X' \setminus X$. Let θ be large enough and take a countable elementary substructure M_0 of $H(\theta)$ with $A, B, x, X', X, P \in M_0$. Use M_0 as the starting point of a sequence $\langle M_\alpha < \alpha < \omega_n \rangle$ of elementary substructures of $H(\theta)$ such that $M_\alpha \cup \{M_\alpha\} \subseteq M_{\alpha+1}$ (and $|M_{\alpha+1}| \leqslant \max\{|M_\alpha|, \aleph_0\}$) for all α and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ whenever α is a limit. In the end let $M = \bigcup_{\alpha<\omega_n} M_\alpha$.

▶ **5**. a. For every limit ordinal β the set M_β is an elementary substructure of $H(\theta)$.
b. For every α we have $\alpha \subseteq M_\alpha$.
c. For every α we have $|M_\alpha| = \max\{|\alpha|, \aleph_0\}$.

Define $\hat{x}$ by $\hat{x}_i = \sup M \cap x_i$ and for every $\alpha < \omega_n$ define u_α by $u_\alpha(i) = \sup M_\alpha \cap x_i$.

▶ **6**. a. If $\operatorname{cf} x_i \leqslant \aleph_n$ then $\hat{x}_i = x_i$. *Hint*: There is $C \in M_0$ with $|C| \leqslant \aleph_n$ that is cofinal in $x(i)$. Show that $C \subseteq M$.
b. If $\operatorname{cf} x(i) > \aleph_n$ then $\hat{x}_n(i) < x(i)$ and $\operatorname{cf} \hat{x}_n(i) = \aleph_n$.
c. $\hat{x}_n \in X$ for all n.

▶ **7**. a. There is an α such that $(u_\alpha, \hat{x}] \cap A = \varnothing$ or $(u_\alpha, \hat{x}] \cap B = \varnothing$.
b. For this α we have $(u_\alpha, x] \cap A_n = \varnothing$ or $(u_\alpha, x] \cap B_n = \varnothing$.

▶ **8**. X is collectionwise normal, i.e., if $\mathcal{F}$ is a discrete collection of closed sets then there is a disjoint family $\{U_F : F \in \mathcal{F}\}$ of open sets with $F \subseteq U_F$ for all F. *Hint*: $\{\operatorname{cl}_{X'} F : F \in \mathcal{F}\}$ is discrete.

3. X is not countably paracompact

We apply Exercise 3.6. For $n \geqslant 1$ let $F_n = \{x \in X : (\forall i \leqslant n)(x_i = \omega_i)\}$; we show that $\bigcap_{n=1}^\infty U_n \neq \varnothing$ whenever $\langle U_n \rangle_n$ is a sequence of open sets with $U_n \supseteq F_n$ for all n.

▶ **1**. The sets F_n are indeed closed and $\bigcap_{n=1}^\infty F_n = \varnothing$.

The key to the proof is the following lemma. We let t denote the top of P, i.e., $t_i = \omega_i$ for all i.

3.1. Lemma. *If U is an open set around F_n then there is an $x < t$ such that $X \cap (x, t] \subseteq U$.*

Given this lemma, the rest of the proof is easy.

▶ **2**. $\bigcap_{n=1}^\infty U_n \neq \varnothing$. *Hint*: Apply Lemma 3.1 infinitely often to get an $x < t$ with $(x, t] \cap X \subseteq \bigcap_n U_n$.

Proof of Lemma 3.1

Let n and U be given. Choose θ large enough, so that $P, X, U \in H(\theta)$.

▶ **3**. There is $M \prec H(\theta)$ with $\omega_n \cup \{P, X, U\} \subseteq M$, $|M| = \aleph_n$ and such that $M \cap \prod_{i>n} \omega_i$ is cofinal in $\prod_{i>n}(M \cap \omega_n)$. ***Hint***: Obtain M as the union of a sequence $\langle M_\alpha : \alpha < \omega_n \rangle$, where $M_\alpha \in M_{\alpha+1}$ for all α.

Define $x \in P$ by $x_i = \omega_i$ for $i \leqslant n$ and $x_i = \sup M \cap \omega_i$ for $i > n$.

▶ **4**. $x \in F_n$.

Choose $x' < x$ so that $(x', x] \cap X \subseteq U$.

▶ **5**. a. There is $z \in M \cap P$ with $x' \leqslant z < x$.
b. For every $u \in M \cap (z, t] \cap X$ we have $t \in U$.
c. $(z, t] \cap X \subseteq U$.

This completes the proof. In the next chapter we shall see that X has a (much smaller) subspace that is also a Dowker space.