3. Cardinal Numbers

Cardinality

Two sets X, Y have the same *cardinality* (cardinal number, cardinal),

$$|X| = |Y|, \tag{3.1}$$

if there exists a one-to-one mapping of X onto Y.

The relation (3.1) is an equivalence relation. We assume that we can assign to each set X its *cardinal number* $|X|$ so that two sets are assigned the same cardinal just in case they satisfy condition (3.1). Cardinal numbers can be defined either using the Axiom of Regularity (via equivalence classes of (3.1)), or using the Axiom of Choice. In this chapter we define cardinal numbers of well-orderable sets; as it follows from the Axiom of Choice that every set can be well-ordered, this defines cardinals in ZFC.

We recall that a set X is *finite* if $|X| = |n|$ for some $n \in \boldsymbol{N}$; then X is said to *have n elements*. Clearly, $|n| = |m|$ if and only if $n = m$, and so we define *finite cardinals* as natural numbers, i.e., $|n| = n$ for all $n \in \boldsymbol{N}$.

The ordering of cardinal numbers is defined as follows:

$$|X| \le |Y| \tag{3.2}$$

if there exists a one-to-one mapping of X into Y. We also define the strict ordering $|X| < |Y|$ to mean that $|X| \le |Y|$ while $|X| \ne |Y|$. The relation $\le$ in (3.2) is clearly transitive. Theorem 3.2 below shows that it is indeed a partial ordering, and it follows from the Axiom of Choice that the ordering is linear—any two sets are comparable in this ordering.

The concept of cardinality is central to the study of infinite sets. The following theorem tells us that this concept is not trivial:

Theorem 3.1 (Cantor). *For every set X, $|X| < |P(X)|$.*

Proof. Let f be a function from X into $P(X)$. The set

$$Y = \{x \in X : x \notin f(x)\}$$

is not in the range of f: If $z \in X$ were such that $f(z) = Y$, then $z \in Y$ if and only if $z \notin Y$, a contradiction. Thus f is not a function of X onto $P(X)$. Hence $|P(X)| \ne |X|$.

The function $f(x) = \{x\}$ is a one-to-one function of X into $P(X)$ and so $|X| \le |P(X)|$. It follows that $|X| < |P(X)|$. □

In view of the following theorem, $<$ is a partial ordering of cardinal numbers.

Theorem 3.2 (Cantor-Bernstein). *If $|A| \le |B|$ and $|B| \le |A|$, then $|A| = |B|$.*

Proof. If $f_1 : A \to B$ and $f_2 : B \to A$ are one-to-one, then if we let $B' = f_2(B)$ and $A_1 = f_2(f_1(A))$, we have $A_1 \subset B' \subset A$ and $|A_1| = |A|$. Thus we may assume that $A_1 \subset B \subset A$ and that f is a one-to-one function of A onto A_1; we will show that $|A| = |B|$.

We define (by induction) for all $n \in \boldsymbol{N}$:

$$\begin{aligned} A_0 &= A, \qquad & A_{n+1} &= f(A_n), \\ B_0 &= B, \qquad & B_{n+1} &= f(B_n). \end{aligned}$$

Let g be the function on A defined as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in A_n - B_n \text{ for some } n, \\ x & \text{otherwise.} \end{cases}$$

Then g is a one-to-one mapping of A onto B, as the reader will easily verify. Thus $|A| = |B|$. □

The arithmetic operations on cardinals are defined as follows:

$$\begin{aligned} \text{(3.3)} \quad & \kappa + \lambda = |A \cup B| && \text{where } |A| = \kappa,\ |B| = \lambda, \text{ and } A,\ B \text{ are disjoint}, \\ & \kappa \cdot \lambda = |A \times B| && \text{where } |A| = \kappa,\ |B| = \lambda, \\ & \kappa^\lambda = |A^B| && \text{where } |A| = \kappa,\ |B| = \lambda. \end{aligned}$$

Naturally, the definitions in (3.3) are meaningful only if they are independent of the choice of A and B. Thus one has to check that, e.g., if $|A| = |A'|$ and $|B| = |B'|$, then $|A \times B| = |A' \times B'|$.

Lemma 3.3. *If $|A| = \kappa$, then $|P(A)| = 2^\kappa$.*

Proof. For every $X \subset A$, let χ_X be the function

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{if } x \in A - X. \end{cases}$$

The mapping $f : X \to \chi_X$ is a one-to-one correspondence between $P(A)$ and $\{0, 1\}^A$. □

Thus Cantor's Theorem 3.1 can be formulated as follows:

$$\kappa < 2^\kappa \quad \text{for every cardinal } \kappa.$$

A few simple facts about cardinal arithmetic:

(3.4) $+$ and $\cdot$ are associative, commutative and distributive.

(3.5) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.

(3.6) $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$.

(3.7) $(\kappa^\lambda)^\mu = \kappa^{\lambda\cdot\mu}$.

(3.8) If $\kappa \le \lambda$, then $\kappa^\mu \le \lambda^\mu$.

(3.9) If $0 < \lambda \le \mu$, then $\kappa^\lambda \le \kappa^\mu$.

(3.10) $\kappa^0 = 1$; $1^\kappa = 1$; $0^\kappa = 0$ if $\kappa > 0$.

To prove (3.4)–(3.10), one has only to find the appropriate one-to-one functions.

Alephs

An ordinal α is called a *cardinal number* (a cardinal) if $|\alpha| \ne |\beta|$ for all $\beta < \alpha$. We shall use κ, λ, μ, ... to denote cardinal numbers.

If W is a well-ordered set, then there exists an ordinal α such that $|W| = |\alpha|$. Thus we let

$$|W| = \text{the least ordinal such that } |W| = |\alpha|.$$

Clearly, $|W|$ is a cardinal number.

Every natural number is a cardinal (a *finite cardinal*); and if S is a finite set, then $|S| = n$ for some n.

The ordinal ω is the least infinite cardinal. Note that all infinite cardinals are limit ordinals. The infinite ordinal numbers that are cardinals are called *alephs*.

Lemma 3.4.

(i) *For every α there is a cardinal number greater than α.*
(ii) *If X is a set of cardinals, then* $\sup X$ *is a cardinal.*

For every α, let α^+ be the least cardinal number greater than α, the *cardinal successor* of α.

Proof. (i) For any set X, let

(3.11) $$h(X) = \text{the least } \alpha \text{ such that there is no one-to-one function of } \alpha \text{ into } X.$$

There is only a set of possible well-orderings of subsets of X. Hence there is only a set of ordinals for which a one-to-one function of α into X exists. Thus $h(X)$ exists.

If α is an ordinal, then $|\alpha| < |h(\alpha)|$ by (3.11). That proves (i).

(ii) Let $\alpha = \sup X$. If f is a one-to-one mapping of α onto some $\beta < \alpha$, let $\kappa \in X$ be such that $\beta < \kappa \le \alpha$. Then $|\kappa| = |\{f(\xi) : \xi < \kappa\}| \le \beta$, a contradiction. Thus α is a cardinal. □

Using Lemma 3.4, we define the increasing enumeration of all alephs. We usually use $\aleph_\alpha$ when referring to the cardinal number, and ω_α to denote the order-type:

$$\aleph_0 = \omega_0 = \omega, \qquad \aleph_{\alpha+1} = \omega_{\alpha+1} = \aleph_\alpha^+,$$
$$\aleph_\alpha = \omega_\alpha = \sup\{\omega_\beta : \beta < \alpha\} \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Sets whose cardinality is $\aleph_0$ are called *countable*; a set is *at most countable* if it is either finite or countable. Infinite sets that are not countable are *uncountable.*

A cardinal $\aleph_{\alpha+1}$ is a *successor cardinal.* A cardinal $\aleph_\alpha$ whose index is a limit ordinal is a *limit cardinal.*

Addition and multiplication of alephs is a trivial matter, due to the following fact:

Theorem 3.5. $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.

To prove Theorem 3.5 we use a pairing function for ordinal numbers:

The Canonical Well-Ordering of $\alpha \times \alpha$

We define a well-ordering of the class $Ord \times Ord$ of ordinal pairs. Under this well-ordering, each $\alpha \times \alpha$ is an initial segment of Ord^2; the induced well-ordering of α^2 is called the *canonical well-ordering* of α^2. Moreover, the well-ordered class Ord^2 is isomorphic to the class Ord, and we have a one-to-one function Γ of Ord^2 onto Ord. For many α's the order-type of $\alpha \times \alpha$ is α; in particular for those α that are alephs.

We define:

$$(3.12) \qquad (\alpha,\beta) < (\gamma,\delta) \leftrightarrow \begin{aligned}[t] &\text{either } \max\{\alpha,\beta\} < \max\{\gamma,\delta\},\\ &\text{or } \max\{\alpha,\beta\} = \max\{\gamma,\delta\} \text{ and } \alpha < \gamma,\\ &\text{or } \max\{\alpha,\beta\} = \max\{\gamma,\delta\},\ \alpha = \gamma \text{ and } \beta < \delta.\end{aligned}$$

The relation $<$ defined in (3.12) is a linear ordering of the class $Ord \times Ord$. Moreover, if $X \subset Ord \times Ord$ is nonempty, then X has a least element. Also, for each α, $\alpha \times \alpha$ is the initial segment given by $(0, \alpha)$. If we let

$$\Gamma(\alpha,\beta) = \text{the order-type of the set } \{(\xi,\eta) : (\xi,\eta) < (\alpha,\beta)\},$$

then Γ is a one-to-one mapping of Ord^2 onto Ord, and

$$(3.13) \qquad (\alpha, \beta) < (\gamma, \delta) \quad \text{if and only if} \quad \Gamma(\alpha, \beta) < \Gamma(\gamma, \delta).$$

Note that $\Gamma(\omega \times \omega) = \omega$ and since $\gamma(\alpha) = \Gamma(\alpha \times \alpha)$ is an increasing function of α, we have $\gamma(\alpha) \geq \alpha$ for every α. However, $\gamma(\alpha)$ is also continuous, and so $\Gamma(\alpha \times \alpha) = \alpha$ for arbitrarily large α.

Proof of Theorem 3.5. Consider the canonical one-to-one mapping Γ of $Ord \times Ord$ onto Ord. We shall show that $\Gamma(\omega_\alpha \times \omega_\alpha) = \omega_\alpha$. This is true for $\alpha = 0$. Thus let α be the least ordinal such that $\Gamma(\omega_\alpha \times \omega_\alpha) \neq \omega_\alpha$. Let $\beta, \gamma < \omega_\alpha$ be such that $\Gamma(\beta, \gamma) = \omega_\alpha$. Pick $\delta < \omega_\alpha$ such that $\delta > \beta$ and $\delta > \gamma$. Since $\delta \times \delta$ is an initial segment of $Ord \times Ord$ in the canonical well-ordering and contains (β, γ), we have $\Gamma(\delta \times \delta) \supset \omega_\alpha$, and so $|\delta \times \delta| \geq \aleph_\alpha$. However, $|\delta \times \delta| = |\delta| \cdot |\delta|$, and by the minimality of α, $|\delta| \cdot |\delta| = |\delta| < \aleph_\alpha$. A contradiction. □

As a corollary we have

$$(3.14) \qquad \aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max\{\aleph_\alpha, \aleph_\beta\}.$$

Exponentiation of cardinals will be dealt with in Chapter 5. Without the Axiom of Choice, one cannot prove that $2^{\aleph_\alpha}$ is an aleph (or that $P(\omega_\alpha)$ can be well-ordered), and there is very little one can prove about $2^{\aleph_\alpha}$ or $\aleph_\alpha^{\aleph_\beta}$.

Cofinality

Let $\alpha > 0$ be a limit ordinal. We say that an increasing β-sequence $\langle \alpha_\xi : \xi < \beta \rangle$, β a limit ordinal, is *cofinal* in α if $\lim_{\xi \to \beta} \alpha_\xi = \alpha$. Similarly, $A \subset \alpha$ is *cofinal* in α if $\sup A = \alpha$. If α is an infinite limit ordinal, the *cofinality* of α is

$$\begin{aligned} \operatorname{cf} \alpha = {} & \text{the least limit ordinal } \beta \text{ such that there is an increasing} \\ & \beta\text{-sequence } \langle \alpha_\xi : \xi < \beta \rangle \text{ with } \lim_{\xi \to \beta} \alpha_\xi = \alpha. \end{aligned}$$

Obviously, $\operatorname{cf} \alpha$ is a limit ordinal, and $\operatorname{cf} \alpha \leq \alpha$. Examples: $\operatorname{cf}(\omega + \omega) = \operatorname{cf} \aleph_\omega = \omega$.

Lemma 3.6. $\operatorname{cf}(\operatorname{cf} \alpha) = \operatorname{cf} \alpha$.

Proof. If $\langle \alpha_\xi : \xi < \beta \rangle$ is cofinal in α and $\langle \xi(\nu) : \nu < \gamma \rangle$ is cofinal in β, then $\langle \alpha_{\xi(\nu)} : \nu < \gamma \rangle$ is cofinal in α. □

Two useful facts about cofinality:

Lemma 3.7. *Let $\alpha > 0$ be a limit ordinal.*

(i) *If $A \subset \alpha$ and $\sup A = \alpha$, then the order-type of A is at least $\operatorname{cf} \alpha$.*

(ii) *If* $\beta_0 \le \beta_1 \le \ldots \le \beta_\xi \le \ldots$, $\xi < \gamma$, *is a nondecreasing* γ*-sequence of ordinals in* α *and* $\lim_{\xi\to\gamma} \beta_\xi = \alpha$, *then* $\operatorname{cf}\gamma = \operatorname{cf}\alpha$.

Proof. (i) The order-type of A is the length of the increasing enumeration of A which is an increasing sequence with limit α.

(ii) If $\gamma = \lim_{\nu\to\operatorname{cf}\gamma} \xi(\nu)$, then $\alpha = \lim_{\nu\to\operatorname{cf}\gamma} \beta_{\xi(\nu)}$, and the nondecreasing sequence $\langle\beta_{\xi(\nu)} : \nu < \operatorname{cf}\gamma\rangle$ has an increasing subsequence of length $\le \operatorname{cf}\gamma$, with the same limit. Thus $\operatorname{cf}\alpha \le \operatorname{cf}\gamma$.

To show that $\operatorname{cf}\gamma \le \operatorname{cf}\alpha$, let $\alpha = \lim_{\nu\to\operatorname{cf}\alpha} \alpha_\nu$. For each $\nu < \operatorname{cf}\alpha$, let $\xi(\nu)$ be the least ξ greater than all $\xi(\iota)$, $\iota < \nu$, such that $\beta_\xi > \alpha_\nu$. Since $\lim_{\nu\to\operatorname{cf}\alpha} \beta_{\xi(\nu)} = \alpha$, it follows that $\lim_{\nu\to\operatorname{cf}\alpha} \xi(\nu) = \gamma$, and so $\operatorname{cf}\gamma \le \operatorname{cf}\alpha$. ☐

An infinite cardinal $\aleph_\alpha$ is *regular* if $\operatorname{cf}\omega_\alpha = \omega_\alpha$. It is *singular* if $\operatorname{cf}\omega_\alpha < \omega_\alpha$.

Lemma 3.8. *For every limit ordinal* α, $\operatorname{cf}\alpha$ *is a regular cardinal.*

Proof. It is easy to see that if α is not a cardinal, then using a mapping of $|\alpha|$ onto α, one can construct a cofinal sequence in α of length $\le |\alpha|$, and therefore $\operatorname{cf}\alpha < \alpha$.

Since $\operatorname{cf}(\operatorname{cf}\alpha) = \operatorname{cf}\alpha$, it follows that $\operatorname{cf}\alpha$ is a cardinal and is regular. ☐

Let κ be a limit ordinal. A subset $X \subset \kappa$ is *bounded* if $\sup X < \kappa$, and *unbounded* if $\sup X = \kappa$.

Lemma 3.9. *Let* κ *be an aleph.*

(i) *If* $X \subset \kappa$ *and* $|X| < \operatorname{cf}\kappa$ *then* X *is bounded.*
(ii) *If* $\lambda < \operatorname{cf}\kappa$ *and* $f : \lambda \to \kappa$ *then the range of* f *is bounded.*

It follows from (i) that every unbounded subset of a regular cardinal has cardinality κ.

Proof. (i) Lemma 3.7(i).

(ii) If $X = \operatorname{ran}(f)$ then $|X| \le \lambda$, and use (i). ☐

There are arbitrarily large singular cardinals. For each α, $\aleph_{\alpha+\omega}$ is a singular cardinal of cofinality ω.

Using the Axiom of Choice, we shall show in Chapter 5 that every $\aleph_{\alpha+1}$ is regular. (The Axiom of Choice is necessary.)

Lemma 3.10. *An infinite cardinal* κ *is singular if and only if there exists a cardinal* $\lambda < \kappa$ *and a family* $\{S_\xi : \xi < \lambda\}$ *of subsets of* κ *such that* $|S_\xi| < \kappa$ *for each* $\xi < \lambda$, *and* $\kappa = \bigcup_{\xi<\lambda} S_\xi$. *The least cardinal* λ *that satisfies the condition is* $\operatorname{cf}\kappa$.

Proof. If κ is singular, then there is an increasing sequence $\langle\alpha_\xi : \xi < \operatorname{cf}\kappa\rangle$ with $\lim_\xi \alpha_\xi = \kappa$. Let $\lambda = \operatorname{cf}\kappa$, and $S_\xi = \alpha_\xi$ for all $\xi < \lambda$.

If the condition holds, let $\lambda < \kappa$ be the least cardinal for which there is a family $\{S_\xi : \xi < \lambda\}$ such that $\kappa = \bigcup_{\xi<\lambda} S_\xi$ and $|S_\xi| < \kappa$ for each $\xi < \lambda$. For every $\xi < \lambda$, let β_ξ be the order-type of $\bigcup_{\nu<\xi} S_\nu$. The sequence $\langle \beta_\xi : \xi < \lambda \rangle$ is nondecreasing, and by the minimality of λ, $\beta_\xi < \kappa$ for all $\xi < \lambda$. We shall show that $\lim_\xi \beta_\xi = \kappa$, thus proving that $\operatorname{cf} \kappa \le \lambda$.

Let $\beta = \lim_{\xi\to\lambda} \beta_\xi$. There is a one-to-one mapping f of $\kappa = \bigcup_{\xi<\lambda} S_\xi$ into $\lambda \times \beta$: If $\alpha \in \kappa$, let $f(\alpha) = (\xi, \gamma)$, where ξ is the least ξ such that $\alpha \in S_\xi$ and γ is the order-type of $S_\xi \cap \alpha$. Since $\lambda < \kappa$ and $|\lambda \times \beta| = \lambda \cdot |\beta|$, it follows that $\beta = \kappa$. □

One cannot prove without the Axiom of Choice that ω_1 is not a countable union of countable sets. Compare this with Exercise 3.13

The only cardinal inequality we have proved so far is Cantor's Theorem $\kappa < 2^\kappa$. It follows that $\kappa < \lambda^\kappa$ for every $\lambda > 1$, and in particular $\kappa < \kappa^\kappa$ (for $\kappa \ne 1$). The following theorem gives a better inequality. This and other cardinal inequalities will also follow from König's Theorem 5.10, to be proved in Chapter 5.

Theorem 3.11. *If κ is an infinite cardinal, then $\kappa < \kappa^{\operatorname{cf} \kappa}$.*

Proof. Let F be a collection of κ functions from $\operatorname{cf} \kappa$ to κ: $F = \{f_\alpha : \alpha < \kappa\}$. It is enough to find $f : \operatorname{cf} \kappa \to \kappa$ that is different from all the f_α. Let $\kappa = \lim_{\xi\to\operatorname{cf}\kappa} \alpha_\xi$. For $\xi < \operatorname{cf} \kappa$, let

$$f(\xi) = \text{least } \gamma \text{ such that } \gamma \ne f_\alpha(\xi) \text{ for all } \alpha < \alpha_\xi.$$

Such γ exists since $|\{f_\alpha(\xi) : \alpha < \alpha_\xi\}| \le |\alpha_\xi| < \kappa$. Obviously, $f \ne f_\alpha$ for all $\alpha < \kappa$. □

Consequently, $\kappa^\lambda > \kappa$ whenever $\lambda \ge \operatorname{cf} \kappa$.

An uncountable cardinal κ is *weakly inaccessible* if it is a limit cardinal and is regular. There will be more about inaccessible cardinals later, but let me mention at this point that existence of (weakly) inaccessible cardinals is not provable in ZFC.

To get an idea of the size of an inaccessible cardinal, note that if $\aleph_\alpha > \aleph_0$ is limit and regular, then $\aleph_\alpha = \operatorname{cf} \aleph_\alpha = \operatorname{cf} \alpha \le \alpha$, and so $\aleph_\alpha = \alpha$.

Since the sequence of alephs is a normal sequence, it has arbitrarily large fixed points; the problem is whether some of them are regular cardinals. For instance, the least fixed point $\aleph_\alpha = \alpha$ has cofinality ω:

$$\kappa = \lim \langle \omega, \omega_\omega, \omega_{\omega_\omega}, \ldots \rangle = \lim_{n\to\omega} \kappa_n$$
$$\text{where } \kappa_0 = \omega,\ \kappa_{n+1} = \omega_{\kappa_n}.$$

Exercises

3.1. (i) A subset of a finite set is finite.
(ii) The union of a finite set of finite sets is finite.
(iii) The power set of a finite set is finite.
(iv) The image of a finite set (under a mapping) is finite.

3.2. (i) A subset of a countable set is at most countable.
(ii) The union of a finite set of countable sets is countable.
(iii) The image of a countable set (under a mapping) is at most countable.

3.3. $\boldsymbol{N} \times \boldsymbol{N}$ is countable.
[$f(m, n) = 2^m(2n+1) - 1$.]

3.4. (i) The set of all finite sequences in $\boldsymbol{N}$ is countable.
(ii) The set of all finite subsets of a countable set is countable.

3.5. Show that $\Gamma(\alpha \times \alpha) \leq \omega^\alpha$.

3.6. There is a well-ordering of the class of all finite sequences of ordinals such that for each α, the set of all finite sequences in ω_α is an initial segment and its order-type is ω_α.

We say that a set B is a *projection* of a set A if there is a mapping of A onto B. Note that B is a projection of A if and only if there is a partition P of A such that $|P| = |B|$. If $|A| \geq |B| > 0$, then B is a projection of A. Conversely, using the Axiom of Choice, one shows that if B is a projection of A, then $|A| \geq |B|$. This, however, cannot be proved without the Axiom of Choice.

3.7. If B is a projection of ω_α, then $|B| \leq \aleph_\alpha$.

3.8. The set of all finite subsets of ω_α has cardinality $\aleph_\alpha$.
[The set is a projection of the set of finite sequences.]

3.9. If B is a projection of A, then $|P(B)| \leq |P(A)|$.
[Consider $g(X) = f_{-1}(X)$, where f maps A onto B.]

3.10. $\omega_{\alpha+1}$ is a projection of $P(\omega_\alpha)$.
[Use $|\omega_\alpha \times \omega_\alpha| = \omega_\alpha$ and project $P(\omega_\alpha \times \omega_\alpha)$: If $R \subset \omega_\alpha \times \omega_\alpha$ is a well-ordering, let $f(R)$ be its order-type.]

3.11. $\aleph_{\alpha+1} < 2^{2^{\aleph_\alpha}}$.
[Use Exercises 3.10 and 3.9.]

3.12. If $\aleph_\alpha$ is an uncountable limit cardinal, then $\operatorname{cf} \omega_\alpha = \operatorname{cf} \alpha$; ω_α is the limit of a cofinal sequence $\langle \omega_\xi : \xi < \operatorname{cf} \alpha \rangle$ of cardinals.

3.13 (ZF). Show that ω_2 is not a countable union of countable sets.
[Assume that $\omega_2 = \bigcup_{n<\omega} S_n$ with S_n countable and let α_n be the order-type of S_n. Then $\alpha = \sup_n \alpha_n \leq \omega_1$ and there is a mapping of $\omega \times \alpha$ onto ω_2.]

A set S is *Dedekind-finite* (D-finite) if there is no one-to-one mapping of S onto a proper subset of S. Every finite set is D-finite. Using the Axiom of Choice, one proves that every infinite set is D-infinite, and so D-finiteness is the same as finiteness. Without the Axiom of Choice, however, one cannot prove that every D-finite set is finite.

The set $\boldsymbol{N}$ of all natural numbers is D-infinite and hence every S such that $|S| \geq \aleph_0$, is D-infinite.

3.14. S is D-infinite if and only if S has a countable subset.

[If S is D-infinite, let $f : S \to X \subset S$ be one-to-one. Let $x_0 \in S - X$ and $x_{n+1} = f(x_n)$. Then $S \supset \{x_n : n < \omega\}$.]

3.15. (i) If A and B are D-finite, then $A \cup B$ and $A \times B$ are D-finite.
(ii) The set of all finite one-to-one sequences in a D-finite set is D-finite.
(iii) The union of a disjoint D-finite family of D-finite sets is D-finite.

On the other hand, one cannot prove without the Axiom of Choice that a projection, power set, or the set of all finite subsets of a D-finite set is D-finite, or that the union of a D-finite family of D-finite sets is D-finite.

3.16. If A is an infinite set, then $PP(A)$ is D-infinite.

[Consider the set $\{\{X \subset A : |X| = n\} : n < \omega\}$.]

Historical Notes

Cardinal numbers and alephs were introduced by Cantor. The proof of the Cantor-Bernstein Theorem is Bernstein's; see Borel [1898], p. 103. (There is an earlier proof by Dedekind.) The first proof of $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$ appeared in Hessenberg [1906], p. 593. Regularity of cardinals was investigated by Hausdorff, who also raised the question of existence of regular limit cardinals. D-finiteness was formulated by Dedekind.