4. Real Numbers

The set of all real numbers $\boldsymbol{R}$ (the *real line* or the *continuum*) is the unique ordered field in which every nonempty bounded set has a least upper bound. The proof of the following theorem marks the beginning of Cantor's theory of sets.

Theorem 4.1 (Cantor). *The set of all real numbers is uncountable.*

Proof. Let us assume that the set $\boldsymbol{R}$ of all reals is countable, and let c_0, $c_1, \ldots, c_n, \ldots, n \in \boldsymbol{N}$, be an enumeration of $\boldsymbol{R}$. We shall find a real number different from each c_n.

Let $a_0 = c_0$ and $b_0 = c_{k_0}$ where k_0 is the least k such that $a_0 < c_k$. For each n, let $a_{n+1} = c_{i_n}$ where i_n is the least i such that $a_n < c_i < b_n$, and $b_{n+1} = c_{k_n}$ where k_n is the least k such that $a_{n+1} < c_k < b_n$. If we let $a = \sup\{a_n : n \in \boldsymbol{N}\}$, then $a \neq c_k$ for all k. □

The Cardinality of the Continuum

Let $\mathfrak{c}$ denote the cardinality of $\boldsymbol{R}$. As the set $\boldsymbol{Q}$ of all rational numbers is dense in $\boldsymbol{R}$, every real number r is equal to $\sup\{q \in \boldsymbol{Q} : q < r\}$ and because $\boldsymbol{Q}$ is countable, it follows that $\mathfrak{c} \leq |P(\boldsymbol{Q})| = 2^{\aleph_0}$.

Let $\boldsymbol{C}$ (the *Cantor set*) be the set of all reals of the form $\sum_{n=1}^{\infty} a_n/3^n$, where each $a_n = 0$ or 2. $\boldsymbol{C}$ is obtained by removing from the closed interval $[0, 1]$, the open intervals $(\frac{1}{3}, \frac{2}{3})$, $(\frac{1}{9}, \frac{2}{9})$, $(\frac{7}{9}, \frac{8}{9})$, etc. (the middle-third intervals). $\boldsymbol{C}$ is in a one-to-one correspondence with the set of all ω-sequences of 0's and 2's and so $|\boldsymbol{C}| = 2^{\aleph_0}$.

Therefore $\mathfrak{c} \geq 2^{\aleph_0}$, and so by the Cantor-Bernstein Theorem we have

$$\mathfrak{c} = 2^{\aleph_0}. \tag{4.1}$$

By Cantor's Theorem 4.1 (or by Theorem 3.1), $\mathfrak{c} > \aleph_0$. Cantor conjectured that every set of reals is either at most countable or has cardinality of the continuum. In ZFC, every infinite cardinal is an aleph, and so $2^{\aleph_0} \geq \aleph_1$. Cantor's conjecture then becomes the statement

$$2^{\aleph_0} = \aleph_1$$

known as the *Continuum Hypothesis* (CH).

Among sets of cardinality $\mathfrak{c}$ are the set of all sequences of natural numbers, the set of all sequences of real numbers, the set of all complex numbers. This is because $\aleph_0^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$, $2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$.

Cantor's proof of Theorem 4.1 yielded more than uncountability of $\boldsymbol{R}$; it showed that the set of all transcendental numbers has cardinality $\mathfrak{c}$ (cf. Exercise 4.5).

The Ordering of $\boldsymbol{R}$

A linear ordering $(P, <)$ is *complete* if every nonempty bounded subset of P has a least upper bound. We stated above that $\boldsymbol{R}$ is the unique complete ordered field. We shall generally disregard the field properties of $\boldsymbol{R}$ and will concern ourselves more with the order properties.

One consequence of being a complete ordered field is that $\boldsymbol{R}$ contains the set $\boldsymbol{Q}$ of all rational numbers as a dense subset. The set $\boldsymbol{Q}$ is countable and its ordering is dense.

Definition 4.2. A linear ordering $(P, <)$ is *dense* if for all $a < b$ there exists a c such that $a < c < b$.

A set $D \subset P$ is a *dense subset* if for all $a < b$ in P there exists a $d \in D$ such that $a < d < b$.

The following theorem proves the uniqueness of the ordered set $(\boldsymbol{R}, <)$. We say that an ordered set is *unbounded* if it has neither a least nor a greatest element.

Theorem 4.3 (Cantor).

(i) *Any two countable unbounded dense linearly ordered sets are isomorphic.*

(ii) $(\boldsymbol{R}, <)$ *is the unique complete linear ordering that has a countable dense subset isomorphic to* $(\boldsymbol{Q}, <)$.

Proof. (i) Let $P_1 = \{a_n : n \in \boldsymbol{N}\}$ and let $P_2 = \{b_n : n \in \boldsymbol{N}\}$ be two such linearly ordered sets. We construct an isomorphism $f : P_1 \to P_2$ in the following way: We first define $f(a_0)$, then $f^{-1}(b_0)$, then $f(a_1)$, then $f^{-1}(b_1)$, etc., so as to keep f order-preserving. For example, to define $f(a_n)$, if it is not yet defined, we let $f(a_n) = b_k$ where k is the least index such that f remains order-preserving (such a k always exists because f has been defined for only finitely many $a \in P_1$, and because P_2 is dense and unbounded).

(ii) To prove the uniqueness of $\boldsymbol{R}$, let C and C' be two complete dense unbounded linearly ordered sets, let P and P' be dense in C and C', respectively, and let f be an isomorphism of P onto P'. Then f can be extended (uniquely) to an isomorphism f^* of C and C': For $x \in C$, let $f^*(x) = \sup\{f(p) : p \in P \text{ and } p \leq x\}$. □

The existence of $(\boldsymbol{R}, <)$ is proved by means of *Dedekind cuts* in $(\boldsymbol{Q}, <)$. The following theorem is a general version of this construction:

Theorem 4.4. *Let $(P, <)$ be a dense unbounded linearly ordered set. Then there is a complete unbounded linearly ordered set $(C, \prec)$ such that*:

(i) *$P \subset C$, and $<$ and $\prec$ agree on P;*
(ii) *P is dense in C.*

Proof. A *Dedekind cut* in P is a pair (A, B) of disjoint nonempty subsets of P such that

(i) $A \cup B = P$;
(ii) $a < b$ for any $a \in A$ and $b \in B$;
(iii) A does not have a greatest element.

Let C be the set of all Dedekind cuts in P and let $(A_1, B_1) \preccurlyeq (A_2, B_2)$ if $A_1 \subset A_2$ (and $B_1 \supset B_2$). The set C is complete: If $\{(A_i, B_i) : i \in I\}$ is a nonempty bounded subset of C, then $(\bigcup_{i \in I} A_i, \bigcap_{i \in I} B_i)$ is its supremum. For $p \in P$, let

$$A_p = \{x \in P : x < p\}, \qquad B_p = \{x \in P : x \geq p\}.$$

Then $P' = \{(A_p, B_p) : p \in P\}$ is isomorphic to P and is dense in C. $\square$

Suslin's Problem

The real line is, up to isomorphism, the unique linearly ordered set that is dense, unbounded, complete and contains a countable dense subset.

Since $\boldsymbol{Q}$ is dense in $\boldsymbol{R}$, every nonempty open interval of $\boldsymbol{R}$ contains a rational number. Hence if S is a disjoint collection of open intervals, S is at most countable. (Let $\langle r_n : n \in \boldsymbol{N} \rangle$ be an enumeration of the rationals. To each $J \in S$ assign $r_n \in J$ with the least possible index n.)

Let P be a dense linearly ordered set. If every disjoint collection of open intervals in P is at most countable, then we say that P satisfies the *countable chain condition.*

Suslin's Problem. *Let P be a complete dense unbounded linearly ordered set that satisfies the countable chain condition. Is P isomorphic to the real line?*

This question cannot be decided in ZFC; we shall return to the problem in Chapter 9.

The Topology of the Real Line

The real line is a metric space with the metric $d(a, b) = |a - b|$. Its metric topology coincides with the order topology of $(\boldsymbol{R}, <)$. Since $\boldsymbol{Q}$ is a dense set in $\boldsymbol{R}$ and since every Cauchy sequence of real numbers converges, $\boldsymbol{R}$ is a separable complete metric space. (A metric space is *separable* if it has a countable dense set; it is *complete* if every Cauchy sequence converges.)

Open sets are unions of open intervals, and in fact, every open set is the union of open intervals with rational endpoints. This implies that the number of all open sets in $\boldsymbol{R}$ is the continuum and so is the number of all closed sets in $\boldsymbol{R}$ (Exercise 4.6).

Every open interval has cardinality $\mathfrak{c}$, therefore every nonempty open set has cardinality $\mathfrak{c}$. We show below that every uncountable closed set has cardinality $\mathfrak{c}$. Proving this was Cantor's first step in the search for the proof of the Continuum Hypothesis. In Chapter 11 we show that CH holds for Borel and analytic sets as well.

A nonempty closed set is *perfect* if it has no isolated points. Theorems 4.5 and 4.6 below show that every uncountable closed set contains a perfect set.

Theorem 4.5. *Every perfect set has cardinality* $\mathfrak{c}$.

Proof. Given a perfect set P, we want to find a one-to-one function F from $\{0, 1\}^\omega$ into P. Let S be the set of all finite sequences of 0's and 1's. By induction on the length of $s \in S$ one can find closed intervals I_s such that for each n and all $s \in S$ of length n,

(i) $I_s \cap P$ is perfect,
(ii) the diameter of I_s is $\leq 1/n$,
(iii) $I_{s^\frown 0} \subset I_s$, $I_{s^\frown 1} \subset I_s$ and $I_{s^\frown 0} \cap I_{s^\frown 1} = \emptyset$.

For each $f \in \{0, 1\}^\omega$, the set $P \cap \bigcap_{n=0}^\infty I_{f \restriction n}$ has exactly one element, and we let $F(f)$ to be this element of P. □

The same proof gives a more general result: Every perfect set in a separable complete metric space contains a closed copy of the Cantor set (Exercise 4.19).

Theorem 4.6 (Cantor-Bendixson). *If F is an uncountable closed set, then $F = P \cup S$, where P is perfect and S is at most countable.*

Corollary 4.7. *If F is a closed set, then either $|F| \leq \aleph_0$ or $|F| = 2^{\aleph_0}$.* □

Proof. For every $A \subset \boldsymbol{R}$, let

$$A' = \text{the set of all limit points of } A$$

It is easy to see that A' is closed, and if A is closed then $A' \subset A$. Thus we let

$$F_0 = F, \qquad F_{\alpha+1} = F'_\alpha,$$
$$F_\alpha = \bigcap_{\gamma<\alpha} F_\gamma \quad \text{if } \alpha > 0 \text{ is a limit ordinal.}$$

Since $F_0 \supset F_1 \supset \ldots \supset F_\alpha \supset \ldots$, there exists an ordinal θ such that $F_\alpha = F_\theta$ for all $\alpha \geq \theta$. (In fact, the least θ with this property must be countable, by the argument below.) We let $P = F_\theta$.

If P is nonempty, then $P' = P$ and so it is perfect. Thus the proof is completed by showing that $F - P$ is at most countable.

Let $\langle J_k : k \in \boldsymbol{N} \rangle$ be an enumeration of rational intervals. We have $F - P = \bigcup_{\alpha<\theta}(F_\alpha - F'_\alpha)$; hence if $a \in F - P$, then there is a unique α such that a is an isolated point of F_α. We let $k(a)$ be the least k such that a is the only point of F_α in the interval J_k. Note that if $\alpha \leq \beta$, $b \neq a$ and $b \in F_\beta - F'_\beta$, then $b \notin J_{k(a)}$, and hence $k(b) \neq k(a)$. Thus the correspondence $a \mapsto k(a)$ is one-to-one, and it follows that $F - P$ is at most countable. □

A set of reals is called *nowhere dense* if its closure has empty interior. The following theorem shows that $\boldsymbol{R}$ is not the union of countably many nowhere dense sets ($\boldsymbol{R}$ is not of *the first category*).

Theorem 4.8 (The Baire Category Theorem). *If* $D_0, D_1, \ldots, D_n, \ldots$, $n \in \boldsymbol{N}$, *are dense open sets of reals, then the intersection* $D = \bigcap_{n=0}^\infty D_n$ *is dense in* $\boldsymbol{R}$.

Proof. We show that D intersects every nonempty open interval I. First note that for each n, $D_0 \cap \ldots \cap D_n$ is dense and open. Let $\langle J_k : k \in \boldsymbol{N} \rangle$ be an enumeration of rational intervals. Let $I_0 = I$, and let, for each n, $I_{n+1} = J_k = (q_k, r_k)$, where k is the least k such that the closed interval $[q_k, r_k]$ is included in $I_n \cap D_n$. Then $a \in D \cap I$, where $a = \lim_{k\to\infty} q_k$. □

Borel Sets

Definition 4.9. An *algebra of sets* is a collection $\mathcal{S}$ of subsets of a given set S such that

(4.2) (i) $S \in \mathcal{S}$,
(ii) if $X \in \mathcal{S}$ and $Y \in \mathcal{S}$ then $X \cup Y \in \mathcal{S}$,
(iii) if $X \in \mathcal{S}$ then $S - X \in \mathcal{S}$.

(Note that $\mathcal{S}$ is also closed under intersections.)

A *σ-algebra* is additionally closed under countable unions (and intersections):

(iv) If $X_n \in \mathcal{S}$ for all n, then $\bigcup_{n=0}^\infty X_n \in \mathcal{S}$.

For any collection $\mathcal{X}$ of subsets of S there is a smallest algebra (σ-algebra) $\mathcal{S}$ such that $\mathcal{S} \supset \mathcal{X}$; namely the intersection of all algebras (σ-algebras) $\mathcal{S}$ of subsets of S for which $\mathcal{X} \subset \mathcal{S}$.

Definition 4.10. A set of reals B is *Borel* if it belongs to the smallest σ-algebra $\mathcal{B}$ of sets of reals that contains all open sets.

In Chapter 11 we investigate Borel sets in more detail. In particular, we shall classify Borel sets by defining a hierarchy of ω_1 levels. For that we need however a weak version of the Axiom of Choice that is not provable in ZF alone. At this point we mention the lowest level of the hierarchy (beyond open sets and closed sets): The intersections of countably many open sets are called *G_δ sets*, and the unions of countably many closed sets are called *F_σ sets.*

Lebesgue Measure

We assume that the reader is familiar with the basic theory of Lebesgue measure. As we shall return to the subject in Chapter 11 we do not define the concept of measure at this point. We also caution the reader that some of the basic theorems on Lebesgue measure require the Countable Axiom of Choice (to be discussed in Chapter 5).

Lebesgue measurable sets form a σ-algebra and contain all open intervals (the measure of an interval is its length). Thus all Borel sets are Lebesgue measurable.

The Baire Space

The *Baire space* is the space $\mathcal{N} = \omega^\omega$ of all infinite sequences of natural numbers, $\langle a_n : n \in \boldsymbol{N} \rangle$, with the following topology: For every finite sequence $s = \langle a_k : k < n \rangle$, let

$$O(s) = \{f \in \mathcal{N} : s \subset f\} = \{\langle c_k : k \in \boldsymbol{N} \rangle : (\forall k < n)\, c_k = a_k\}. \tag{4.3}$$

The sets (4.3) form a basis for the topology of $\mathcal{N}$. Note that each $O(s)$ is also closed.

The Baire space is separable and is metrizable: consider the metric $d(f, g) = 1/2^{n+1}$ where n is the least number such that $f(n) \neq g(n)$. The countable set of all eventually constant sequences is dense in $\mathcal{N}$. This separable metric space is complete, as every Cauchy sequence converges.

Every infinite sequence $\langle a_n : n \in \boldsymbol{N} \rangle$ of positive integers defines a *continued fraction* $1/(a_0 + 1/(a_1 + 1/(a_2 + \ldots)))$, an irrational number between 0 and 1. Conversely, every irrational number in the interval $(0, 1)$ can be so represented, and the one-to-one correspondence is a homeomorphism. It follows that the Baire space is homeomorphic to the space of all irrational numbers.

For various reasons, modern descriptive set theory uses the Baire space rather than the real line. Often the functions in ω^ω are called reals.

Clearly, the space $\mathcal{N}$ satisfies the Baire Category Theorem; the proof is similar to the proof of Theorem 4.8 above. The Cantor-Bendixson Theorem holds as well. For completeness we give a description of perfect sets in $\mathcal{N}$.

Let *Seq* denote the set of all finite sequences of natural numbers. A (*sequential*) *tree* is a set $T \subset \mathit{Seq}$ that satisfies

$$\text{if } t \in T \text{ and } s = t{\restriction}n \text{ for some } n, \text{ then } s \in T. \tag{4.4}$$

If $T \subset \mathit{Seq}$ is a tree, let $[T]$ be the set of all *infinite paths* through T:

$$[T] = \{f \in \mathcal{N} : f{\restriction}n \in T \text{ for all } n \in \boldsymbol{N}\}. \tag{4.5}$$

The set $[T]$ is a closed set in the Baire space: Let $f \in \mathcal{N}$ be such that $f \notin [T]$. Then there is $n \in \boldsymbol{N}$ such that $f{\restriction}n = s$ is not in T. In other words, the open set $O(s) = \{g \in \mathcal{N} : g \supset s\}$, a neighborhood of f, is disjoint from $[T]$. Hence $[T]$ is closed.

Conversely, if F is a closed set in $\mathcal{N}$, then the set

$$T_F = \{s \in \mathit{Seq} : s \subset f \text{ for some } f \in F\} \tag{4.6}$$

is a tree, and it is easy to verify that $[T_F] = F$: If $f \in \mathcal{N}$ is such that $f{\restriction}n \in T$ for all $n \in \boldsymbol{N}$, then for each n there is some $g \in F$ such that $g{\restriction}n = f{\restriction}n$; and since F is closed, it follows that $f \in F$.

If f is an isolated point of a closed set F in $\mathcal{N}$, then there is $n \in \mathcal{N}$ such that there is no $g \in F$, $g \neq f$, such that $g{\restriction}n = f{\restriction}n$. Thus the following definition:

A nonempty sequential tree T is *perfect* if for every $t \in T$ there exist $s_1 \supset t$ and $s_2 \supset t$, both in T, that are *incomparable*, i.e., neither $s_1 \supset s_2$ nor $s_2 \supset s_1$.

Lemma 4.11. *A closed set $F \subset \mathcal{N}$ is perfect if and only if the tree T_F is a perfect tree.* □

The Cantor-Bendixson analysis for closed sets in the Baire space is carried out as follows: For each tree $T \subset \mathit{Seq}$, we let

$$T' = \{t \in T : \text{there exist incomparable } s_1 \supset t \text{ and } s_2 \supset t \text{ in } T\}. \tag{4.7}$$

(Thus T is perfect if and only if $\emptyset \neq T = T'$.)

The set $[T] - [T']$ is at most countable: For each $f \in [T]$ such that $f \notin [T']$, let $s_f = f{\restriction}n$ where n is the least number such that $f{\restriction}n \notin T'$. If $f, g \in [T] - [T']$, then $s_f \neq s_g$, by (4.7). Hence the mapping $f \mapsto s_f$ is one-to-one and $[T] - [T']$ is at most countable.

Now we let

$$T_0 = T, \qquad T_{\alpha+1} = T'_\alpha, \tag{4.8}$$
$$T_\alpha = \bigcap_{\beta<\alpha} T_\beta \quad \text{if } \alpha > 0 \text{ is a limit ordinal.}$$

Since $T_0 \supset T_1 \supset \ldots \supset T_\alpha \supset \ldots$, and T_0 is at most countable, there is an ordinal $\theta < \omega_1$ such that $T_{\theta+1} = T_\theta$. If $T_\theta \neq \emptyset$, then it is perfect.

Now it is easy to see that $\left[\bigcap_{\beta<\alpha} T_\beta\right] = \bigcap_{\beta<\alpha}[T_\beta]$, and so

$$[T] - [T_\theta] = \bigcup_{\alpha<\theta} ([T_\alpha] - [T'_\alpha]); \tag{4.9}$$

hence (4.9) is at most countable. Thus if $[T]$ is an uncountable closed set in $\mathcal{N}$, the sets $[T_\theta]$ and $[T] - [T_\theta]$ constitute the decomposition of $[T]$ into a perfect and an at most countable set.

In modern descriptive set theory one often speaks about the *Lebesgue measure* on $\mathcal{N}$. This measure is the extension of the product measure m on Borel sets in the Baire space induced by the probability measure on $\boldsymbol{N}$ that gives the singleton $\{n\}$ measure $1/2^{n+1}$. Thus for every sequence $s \in \mathit{Seq}$ of length $n \geq 1$ we have

$$m(O(s)) = \prod_{k=0}^{n-1} 1/2^{s(k)+1}. \tag{4.10}$$

Polish Spaces

Definition 4.12. A *Polish space* is a topological space that is homeomorphic to a separable complete metric space.

Examples of Polish spaces include $\boldsymbol{R}$, $\mathcal{N}$, the Cantor space, the unit interval $[0,1]$, the unit circle T, the Hilbert cube $[0,1]^\omega$, etc.

Every Polish space is a continuous image of the Baire space. In Chapter 11 we prove a somewhat more general statement.

Exercises

4.1. The set of all continuous functions $f : \boldsymbol{R} \to \boldsymbol{R}$ has cardinality $\mathfrak{c}$ (while the set of all functions has cardinality $2^{\mathfrak{c}}$).

[A continuous function on $\boldsymbol{R}$ is determined by its values at rational points.]

4.2. There are at least $\mathfrak{c}$ countable order-types of linearly ordered sets.

[For every sequence $a = \langle a_n : n \in \boldsymbol{N} \rangle$ of natural numbers consider the order-type

$$\tau_a = a_0 + \xi + a_1 + \xi + a_2 + \ldots$$

where ξ is the order-type of the integers. Show that if $a \neq b$, then $\tau_a \neq \tau_b$.]

A real number is *algebraic* if it is a root of a polynomial whose coefficients are integers. Otherwise, it is *transcendental.*

4.3. The set of all algebraic reals is countable.

4.4. If S is a countable set of reals, then $|\boldsymbol{R} - S| = \mathfrak{c}$.

[Use $\boldsymbol{R} \times \boldsymbol{R}$ rather than $\boldsymbol{R}$ (because $|\boldsymbol{R} \times \boldsymbol{R}| = 2^{\aleph_0}$).]

4.5. (i) The set of all irrational numbers has cardinality $\mathfrak{c}$.
(ii) The set of all transcendental numbers has cardinality $\mathfrak{c}$.

4.6. The set of all open sets of reals has cardinality $\mathfrak{c}$.

4.7. The Cantor set is perfect.

4.8. If P is a perfect set and (a, b) is an open interval such that $P \cap (a, b) \neq \emptyset$, then $|P \cap (a, b)| = \mathfrak{c}$.

4.9. If $P_2 \not\subset P_1$ are perfect sets, then $|P_2 - P_1| = \mathfrak{c}$.
[Use Exercise 4.8.]

If A is a set of reals, a real number a is called a *condensation point* of A if every neighborhood of a contains uncountably many elements of A. Let A^* denote the set of all condensation points of A.

4.10. If P is perfect then $P^* = P$.
[Use Exercise 4.8.]

4.11. If F is closed and $P \subset F$ is perfect, then $P \subset F^*$.
[$P = P^* \subset F^*$.]

4.12. If F is an uncountable closed set and P is the perfect set constructed in Theorem 4.6, then $F^* \subset P$; thus $F^* = P$.
[Every $a \in F^*$ is a limit point of P since $|F - P| \leq \aleph_0$.]

4.13. If F is an uncountable closed set, then $F = F^* \cup (F - F^*)$ is the unique partition of F into a perfect set and an at most countable set.
[Use Exercise 4.9.]

4.14. $\boldsymbol{Q}$ is not the intersection of a countable collection of open sets.
[Use the Baire Category Theorem.]

4.15. If B is Borel and f is a continuous function then $f_{-1}(B)$ is Borel.

4.16. Let $f : \boldsymbol{R} \to \boldsymbol{R}$. Show that the set of all x at which f is continuous is a G_δ set.

4.17. (i) $\mathcal{N} \times \mathcal{N}$ is homeomorphic to $\mathcal{N}$.
(ii) $\mathcal{N}^\omega$ is homeomorphic to $\mathcal{N}$.

4.18. The tree T_F in (4.6) has no maximal node, i.e., $s \in T$ such that there is no $t \in T$ with $s \subset t$. The map $F \mapsto T_F$ is a one-to-one correspondence between closed sets in $\mathcal{N}$ and sequential trees without maximal nodes.

4.19. Every perfect Polish space has a closed subset homeomorphic to the Cantor space.

4.20. Every Polish space is homeomorphic to a G_δ subspace of the Hilbert cube.
[Let $\{x_n : n \in \boldsymbol{N}\}$ be a dense set, and define $f(x) = \langle d(x, x_n) : n \in \boldsymbol{N} \rangle$.]

Historical Notes

Theorems 4.1, 4.3 and 4.5 are due to Cantor. The construction of real numbers by completion of the rationals is due to Dedekind [1872].

Suslin's Problem: Suslin [1920].

Theorem 4.6: Cantor, Bendixson [1883].

Theorem 4.8: Baire [1899].

Exercise 4.5: Cantor.