6. The Axiom of Regularity

The Axiom of Regularity states that the relation $\in$ on any family of sets is well-founded:

Axiom of Regularity. *Every nonempty set has an $\in$-minimal element*:

$$\forall S\,(S \neq \emptyset \to (\exists x \in S)\, S \cap x = \emptyset).$$

As a consequence, there is no infinite sequence

$$x_0 \ni x_1 \ni x_2 \ni \dots .$$

(Consider the set $S = \{x_0, x_1, x_2, \dots\}$ and apply the axiom.) In particular, there is no set x such that

$$x \in x$$

and there are no "cycles"

$$x_0 \in x_1 \in \dots \in x_n \in x_0.$$

Thus the Axiom of Regularity postulates that sets of certain type do no exist. This restriction on the universe of sets is not contradictory (i.e., the axiom is consistent with the other axioms) and is irrelevant for the development of ordinal and cardinal numbers, natural and real numbers, and in fact of all ordinary mathematics. However, it is extremely useful in the metamathematics of set theory, in construction of models. In particular, all sets can be assigned ranks and can be arranged in a cumulative hierarchy.

We recall that a set T is *transitive* if $x \in T$ implies $x \subset T$.

Lemma 6.1. *For every set S there exists a transitive set $T \supset S$.*

Proof. We define by induction

$$S_0 = S, \qquad S_{n+1} = \bigcup S_n$$

and

$$T = \bigcup_{n=0}^{\infty} S_n. \tag{6.1}$$

Clearly, T is transitive and $T \supset S$. $\square$

Since every transitive set must satisfy $\bigcup T \subset T$, it follows that the set in (6.1) is the smallest transitive $T \supset S$; it is called *transitive closure* of S:

$$\mathrm{TC}(S) = \bigcap\{T : T \supset S \text{ and } T \text{ is transitive}\}.$$

Lemma 6.2. *Every nonempty class C has an $\in$-minimal element.*

Proof. Let $S \in C$ be arbitrary. If $S \cap C = \emptyset$, then S is a minimal element of C; if $S \cap C \neq \emptyset$, we let $X = T \cap C$ where $T = \mathrm{TC}(S)$. X is a nonempty set and by the Axiom of Regularity, there is $x \in X$ such that $x \cap X = \emptyset$. It follows that $x \cap C = \emptyset$; otherwise, if $y \in x$ and $y \in C$, then $y \in T$ since T is transitive, and so $y \in x \cap T \cap C = x \cap X$. Hence x is a minimal element of C. □

The Cumulative Hierarchy of Sets

We define, by transfinite induction,

$$\begin{aligned} &V_0 = \emptyset, \qquad V_{\alpha+1} = P(V_\alpha), \\ &V_\alpha = \bigcup_{\beta<\alpha} V_\beta \quad \text{if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

The sets V_α have the following properties (by induction):

(i) Each V_α is transitive.
(ii) If $\alpha < \beta$, then $V_\alpha \subset V_\beta$.
(iii) $\alpha \subset V_\alpha$.

The Axiom of Regularity implies that every set is in some V_α:

Lemma 6.3. *For every x there is α such that $x \in V_\alpha$:*

$$\bigcup_{\alpha \in Ord} V_\alpha = V. \tag{6.2}$$

Proof. Let C be the class of all x that are not in any V_α. If C is nonempty, then C has an $\in$-minimal element x. That is, $x \in C$, and $z \in \bigcup_\alpha V_\alpha$ for every $z \in x$. Hence $x \subset \bigcup_{\alpha\in Ord} V_\alpha$. By Replacement, there exists an ordinal γ such that $x \subset \bigcup_{\alpha<\gamma} V_\alpha$. Hence $x \subset V_\gamma$ and so $x \in V_{\gamma+1}$. Thus C is empty and we have (6.2). □

Since every x is in some V_α, we may define the *rank* of x:

$$\mathrm{rank}(x) = \text{the least } \alpha \text{ such that } x \in V_{\alpha+1}. \tag{6.3}$$

Thus each V_α is the collection of all sets of rank less than α, and we have

(i) If $x \in y$, then $\mathrm{rank}(x) < \mathrm{rank}(y)$.

(ii) $\operatorname{rank}(\alpha) = \alpha$.

One of the uses of the rank function is a definition of equivalence classes for equivalence relations on a proper class. The basic trick is the following:

Given a class C, let

$$\hat{C} = \{x \in C : (\forall z \in C) \operatorname{rank} x \le \operatorname{rank} z\}. \tag{6.4}$$

$\hat{C}$ is always a set, and if C is nonempty, then $\hat{C}$ is nonempty. Moreover, (6.4) can be applied uniformly.

Thus, for example, if $\equiv$ is an equivalence on a proper class C, we apply (6.4) to each equivalence class of $\equiv$, and define

$$[x] = \{y \in C : y \equiv x \text{ and } \forall z \in C\,(z \equiv x \to \operatorname{rank} y \le \operatorname{rank} z)\}$$

and

$$C/{\equiv} = \{[x] : x \in C\}.$$

In particular, this trick enables us to define isomorphism types for a given isomorphism. For instance, one can define order-types of linearly ordered sets, or cardinal numbers (even without AC).

We use the same argument to prove the following.

Collection Principle.

$$\forall X\, \exists Y\, (\forall u \in X)[\exists v\, \varphi(u, v, p) \to (\exists v \in Y)\, \varphi(u, v, p)] \tag{6.5}$$

(p is a parameter).

The Collection Principle is a schema of formulas. We can formulate it as follows:

Given a "collection of classes" C_u, $u \in X$ (X is a set), then there is a set Y such that for every $u \in X$,

$$\text{if } C_u \ne \emptyset, \quad \text{then } C_u \cap Y \ne \emptyset.$$

To prove (6.5), we let

$$Y = \bigcup_{u \in X} \hat{C}_u$$

where $C_u = \{v : \varphi(u, v, p)\}$, i.e.,

$$v \in Y \leftrightarrow (\exists u \in X)(\varphi(u, v, p) \text{ and } \forall z\,(\varphi(u, z, p) \to \operatorname{rank} v \le \operatorname{rank} z)).$$

That Y is a set follows from the Replacement Schema.

Note that the Collection Principle implies the Replacement Schema: Given a function F, then for every set X we let Y be a set such that

$$(\forall u \in X)(\exists v \in Y)\, F(u) = v.$$

Then

$$F{\restriction}X = F \cap (X \times Y)$$

is a set by the Separation Schema.

$\in$-Induction

The method of transfinite induction can be extended to an arbitrary transitive class (instead of *Ord*), both for the proof and for the definition by induction:

Theorem 6.4 ($\in$-Induction). *Let T be a transitive class, let Φ be a property. Assume that*

(i) $\Phi(\emptyset)$;
(ii) *if $x \in T$ and $\Phi(z)$ holds for every $z \in x$, then $\Phi(x)$.*

Then every $x \in T$ has property Φ.

Proof. Let C be the class of all $x \in T$ that do not have the property Φ. If C is nonempty, then it has an $\in$-minimal element x; apply (i) or (ii). □

Theorem 6.5 ($\in$-Recursion). *Let T be a transitive class and let G be a function (defined for all x). Then there is a function F on T such that*

$$F(x) = G(F{\restriction}x) \tag{6.6}$$

for every $x \in T$.

Moreover, F is the unique function that satisfies (6.6).

Proof. We let, for every $x \in T$,

$$\begin{aligned} F(x) = y \leftrightarrow {} & \text{there exists a function } f \text{ such that} \\ & \operatorname{dom}(f) \text{ is a transitive subset of } T \text{ and:} \\ & \quad \text{(i)} \ (\forall z \in \operatorname{dom}(f))\, f(z) = G(f{\restriction}z), \\ & \quad \text{(ii)} \ f(x) = y. \end{aligned}$$

That F is a (unique) function on T satisfying (6.6) is proved by $\in$-induction. □

Corollary 6.6. *Let A be a class. There is a unique class B such that*

$$B = \{x \in A : x \subset B\}. \tag{6.7}$$

Proof. Let

$$F(x) = \begin{cases} 1 & \text{if } x \in A \text{ and } F(z) = 1 \text{ for all } z \in x, \\ 0 & \text{otherwise.} \end{cases}$$

Let $B = \{x : F(x) = 1\}$. The uniqueness of B is proved by $\in$-induction. □

We say that each $x \in B$ is *hereditarily* in A.

One consequence of the Axiom of Regularity is that the universe does not admit nontrivial $\in$-automorphisms. More generally:

Theorem 6.7. *Let T_1, T_2 be transitive classes and let π be an $\in$-isomorphism of T_1 onto T_2; i.e., π is one-to-one and*

$$u \in v \leftrightarrow \pi u \in \pi v. \tag{6.8}$$

Then $T_1 = T_2$ and $\pi u = u$ for every $u \in T_1$.

Proof. We show, by $\in$-induction, that $\pi x = x$ for every $x \in T_1$. Assume that $\pi z = z$ for each $z \in x$ and let $y = \pi x$.

We have $x \subset y$ because if $z \in x$, then $z = \pi z \in \pi x = y$.

We also have $y \subset x$: Let $t \in y$. Since $y \subset T_2$, there is $z \in T_1$ such that $\pi z = t$. Since $\pi z \in y$, we have $z \in x$, and so $t = \pi z = z$. Thus $t \in x$.

Therefore $\pi x = x$ for all $x \in T_1$, and $T_2 = T_1$. □

Well-Founded Relations

The notion of well-founded relations that was introduced in Chapter 2 can be generalized to relations on proper classes, and one can extend the method of induction to well-founded relations.

Let E be a binary relation on a class P. For each $x \in P$, we let

$$\mathrm{ext}_E(x) = \{z \in P : z \mathrel{E} x\}$$

the *extension* of x.

Definition 6.8. A relation E on P is well-founded, if:

(6.9) (i) every nonempty set $x \subset P$ has an E-minimal element;
(ii) $\mathrm{ext}_E(x)$ is a set, for every $x \in P$.

(Condition (ii) is vacuous if P is a set.) Note that the relation $\in$ is well-founded on any class, by the Axiom of Regularity.

Lemma 6.9. *If E is a well-founded relation on P, then every nonempty class $C \subset P$ has an E-minimal element.*

Proof. We follow the proof of Lemma 6.2; we are looking for $x \in C$ such that $\mathrm{ext}_E(x) \cap C = \emptyset$. Let $S \in C$ be arbitrary and assume that $\mathrm{ext}_E(S) \cap C \neq \emptyset$. We let $X = T \cap C$ where

$$T = \bigcup_{n=0}^{\infty} S_n$$

and

$$S_0 = \mathrm{ext}_E S, \qquad S_{n+1} = \bigcup\{\mathrm{ext}_E(z) : z \in S_n\}.$$

As in Lemma 6.2, it follows that an E-minimal element x of X is E-minimal in C. □

Theorem 6.10 (Well-Founded Induction). *Let E be a well-founded relation on P. Let Φ be a property. Assume that:*

(i) *every E-minimal element x has property Φ;*
(ii) *if $x \in P$ and if $\Phi(z)$ holds for every z such that $z\ E\ x$, then $\Phi(x)$.*

Then every $x \in P$ has property Φ.

Proof. A modification of the proof of Theorem 6.4. □

Theorem 6.11 (Well-Founded Recursion). *Let E be a well-founded relation on P. Let G be a function (on $V \times V$). Then there is a unique function F on P such that*

$$F(x) = G(x, F{\restriction}\operatorname{ext}_E(x)) \tag{6.10}$$

for every $x \in P$.

Proof. A modification of the proof of Theorem 6.5. □

(Note that if $F(x) = G(F{\restriction}\operatorname{ext}(x))$ for some G, then $F(x) = F(y)$ whenever $\operatorname{ext}(x) = \operatorname{ext}(y)$; in particular, $F(x)$ is the same for all minimal elements.)

Example 6.12 (The Rank Function). We define, by induction, for all $x \in P$:

$$\rho(x) = \sup\{\rho(z) + 1 : z\ E\ x\}$$

(compare with (2.7)). The range of ρ is either an ordinal or the class *Ord*. For all $x, y \in P$,

$$x\ E\ y \to \rho(x) < \rho(y).$$ □

Example 6.13 (The Transitive Collapse). By induction, let

$$\pi(x) = \{\pi(z) : z\ E\ x\}$$

for every $x \in P$. The range of π is a transitive class, and for all $x, y \in P$,

$$x\ E\ y \to \pi(x) \in \pi(y).$$ □

The transitive collapse of a well-founded relation is not necessarily a one-to-one function. It is one-to-one if E satisfies an additional condition, extensionality.

Definition 6.14. A well-founded relation E on a class P is *extensional* if

$$\operatorname{ext}_E(X) \neq \operatorname{ext}_E(Y) \tag{6.11}$$

whenever X and Y are distinct elements of P.

A class M is *extensional* if the relation $\in$ on M is extensional, i.e., if for any distinct X and $Y \in M$, $X \cap M \neq Y \cap M$.

The following theorem shows that the transitive collapse of an extensional well-founded relation is one-to-one, and that every extensional class is $\in$-isomorphic to a transitive class.

Theorem 6.15 (Mostowski's Collapsing Theorem).

(i) *If E is a well-founded and extensional relation on a class P, then there is a transitive class M and an isomorphism π between (P, E) and $(M, \in)$. The transitive class M and the isomorphism π are unique.*
(ii) *In particular, every extensional class P is isomorphic to a transitive class M. The transitive class M and the isomorphism π are unique.*
(iii) *In case* (ii), *if $T \subset P$ is transitive, then $\pi x = x$ for every $x \in T$.*

Proof. Since (ii) is a special case of (i) ($E = \in$ in case (ii)), we shall prove the existence of an isomorphism in the general case.

Since E is a well-founded relation, we can define π by well-founded induction (Theorem 6.11), i.e., $\pi(x)$ can be defined in terms of the $\pi(z)$'s, where $z \mathrel{E} x$. We let, for each $x \in P$

$$\pi(x) = \{\pi(z) : z \mathrel{E} x\}. \tag{6.12}$$

In particular, in the case $E = \in$, (6.12) becomes

$$\pi(x) = \{\pi(z) : z \in x \cap P\}. \tag{6.13}$$

The function π maps P onto a class $M = \pi(P)$, and it is immediate from the definition (6.12) that M is transitive.

We use the extensionality of E to show that π is one-to-one. Let $z \in M$ be of least rank such that $z = \pi(x) = \pi(y)$ for some $x \neq y$. Then $\mathrm{ext}_E(x) \neq \mathrm{ext}_E(y)$ and there is, e.g., some $u \in \mathrm{ext}_E(x)$ such that $u \notin \mathrm{ext}_E(y)$. Let $t = \pi(u)$. Since $t \in z = \pi(y)$, there is $v \in \mathrm{ext}_E(y)$ such that $t = \pi(v)$. Thus we have $t = \pi(u) = \pi(v)$, $u \neq v$, and t is of lesser rank than z (since $t \in z$). A contradiction.

Now it follows easily that

$$x \mathrel{E} y \leftrightarrow \pi(x) \in \pi(y). \tag{6.14}$$

If $x \mathrel{E} y$, then $\pi(x) \in \pi(y)$ by definition (6.12). On the other hand, if $\pi(x) \in \pi(y)$, then by (6.12), $\pi(x) = \pi(z)$ for some $z \mathrel{E} y$. Since π is one-to-one, we have $x = z$ and so $x \mathrel{E} y$.

The uniqueness of the isomorphism π, and the transitive class $M = \pi(P)$, follows from Theorem 6.7. If π_1 and π_2 are two isomorphisms of P and M_1, M_2, respectively, then $\pi_2\pi_1^{-1}$ is an isomorphism between M_1 and M_2, and therefore the identity mapping. Hence $\pi_1 = \pi_2$.

It remains to prove (iii). If $T \subset P$ is transitive, then we first observe that $x \subset P$ for every $x \in T$ and so $x \cap P = x$, and we have

$$\pi(x) = \{\pi(z) : z \in x\}$$

for all $x \in T$. It follows easily by $\in$-induction that $\pi(x) = x$ for all $x \in T$. $\square$

The Bernays-Gödel Axiomatic Set Theory

There is an alternative axiomatization of set theory. We consider two types of objects: *sets* (for which we use lower case letters) and *classes* (denoted by capital letters).

A. 1. Extensionality: $\forall u\,(u \in X \leftrightarrow y \in Y) \rightarrow X = Y$.
   2. Every set is a class.
   3. If $X \in Y$, then X is a set.
   4. Pairing: For any sets x and y there is a set $\{x, y\}$.

B. Comprehension:

$$\forall X_1 \ldots \forall X_n\, \exists Y\;\; Y = \{x : \varphi(x, X_1, \ldots, X_n)\}$$

where φ is a formula in which only set variables are quantified.

C. 1. Infinity: There is an infinite set.
   2. Union: For every set x the set $\bigcup x$ exists.
   3. Power Set: For every set x the power set $P(x)$ of x exists.
   4. Replacement: If a class F is a function and x is a set, then $\{F(z) : z \in x\}$ is a set.

D. Regularity.

E. Choice: There is a function F such that $F(x) \in x$ for every nonempty set x.

Let BG denote the axiomatic theory A–D and let BGC denote BG + Choice.

If a set-theoretical statement is provable in ZF (ZFC), then it is provable in BG (BGC).

On the other hand, a theorem of Shoenfield (using proof-theoretic methods) states that if a sentence involving only set variables is provable in BG, then it is provable in ZF. This result can be extended to BGC/ZFC using the method of forcing.

Exercises

6.1. $\operatorname{rank}(x) = \sup\{\operatorname{rank}(z) + 1 : z \in x\}$.

6.2. $|V_\omega| = \aleph_0$, $|V_{\omega+\alpha}| = \beth_\alpha$.

6.3. If κ is inaccessible, then $|V_\kappa| = \kappa$.

6.4. If x and y have rank $\le \alpha$ then $\{x, y\}$, $\langle x, y\rangle$, $x \cup y$, $\bigcup x$, $P(x)$, and x^y have rank $< \alpha + \omega$

6.5. The sets $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ are in $V_{\omega+\omega}$.

6.6. Let B be the class of all x that are hereditarily in the class A. Show that

(i) $x \in B$ if and only if $\operatorname{TC}(x) \subset A$,
(ii) B is the largest transitive class $B \subset A$.

Historical Notes

The Axiom of Regularity was introduced by von Neumann in [1925], although a similar principle had been considered previously by Skolem (see [1970], pp. 137–152). The concept of rank appears first in Mirimanov [1917]. The transitive collapse is defined in Mostowski [1949]. Induction on well-founded relations (Theorems 6.10, 6.11) was formulated by Montague in [1955].

The axiomatic system BG was introduced by Bernays in [1937]. Shoenfield's result was published in [1954].

For more references on the history of axioms of set theory consult Fraenkel *et al.* [1973].