7. Filters, Ultrafilters and Boolean Algebras

Filters and Ultrafilters

Filters and ideals play an important role in several mathematical disciplines (algebra, topology, logic, measure theory). In this chapter we introduce the notion of filter (and ideal) on a given set. The notion of ideal extrapolates the notion of small sets: Given an ideal I on S, a set $X \subset S$ is considered small if it belongs to I.

Definition 7.1. A *filter* on a nonempty set S is a collection F of subsets of S such that

(7.1) (i) $S \in F$ and $\emptyset \notin F$,
(ii) if $X \in F$ and $Y \in F$, then $X \cap Y \in F$,
(iii) If $X, Y \subset S$, $X \in F$, and $X \subset Y$, then $Y \in F$.

An *ideal* on a nonempty set S is a collection I of subsets of S such that:

(7.2) (i) $\emptyset \in I$ and $S \notin I$,
(ii) if $X \in I$ and $Y \in I$, then $X \cup Y \in I$,
(iii) if $X, Y \subset S$, $X \in I$, and $Y \subset X$, then $Y \in I$.

If F is a filter on S, then the set $I = \{S - X : X \in F\}$ is an ideal on S; and conversely, if I is an ideal, then $F = \{S - X : X \in I\}$ is a filter. If this is the case we say that F and I are *dual* to each other.

Examples. 1. A *trivial* filter: $F = \{S\}$.

2. A *principal* filter. Let X_0 be a nonempty subset of S. The filter $F = \{X \subset S : X \supset X_0\}$ is a *principal* filter. Note that every filter on a finite set is principal.

The dual notions are a *trivial* ideal and a *principal* ideal.

3. The *Fréchet filter*. Let S be an infinite set, and let I be the ideal of all finite subsets of S. The dual filter $F = \{X \subset S : S - X \text{ is finite}\}$ is called the Fréchet filter on S. Note that the Fréchet filter is not principal.

4. Let A be an infinite set and let $S = [A]^{<\omega}$ be the set of all finite subsets of A. For each $P \in S$, let $\hat{P} = \{Q \in S : P \subset Q\}$. Let F be the set of all $X \subset S$ such that $X \supset \hat{P}$ for some $P \in S$. Then F is a nonprincipal filter on S.

5. A set $A \subset \boldsymbol{N}$ has *density* 0 if $\lim_{n\to\infty} |A \cap n|/n = 0$. The set of all A of density 0 is an ideal on $\boldsymbol{N}$.

A family G of sets has the *finite intersection property* if every finite $H = \{X_1, \ldots, X_n\} \subset G$ has a nonempty intersection $X_1 \cap \ldots \cap X_n \neq \emptyset$. Every filter has the finite intersection property.

Lemma 7.2.

(i) *If $\mathcal{F}$ is a nonempty family of filters on S, then $\bigcap \mathcal{F}$ is a filter on S.*
(ii) *If $\mathcal{C}$ is a $\subset$-chain of filters on S, then $\bigcup \mathcal{C}$ is a filter on S.*
(iii) *If $G \subset P(S)$ has the finite intersection property, then there is a filter F on S such that $G \subset F$.*

Proof. (i) and (ii) are easy to verify.

(iii) Let F be the set of all $X \subset S$ such that there is a finite $H = \{X_1, \ldots, X_n\} \subset G$ with $X_1 \cap \ldots \cap X_n \subset X$. Then F is a filter and $F \supset G$. □

Since every filter $F \supset G$ must contain all finite intersections of sets in G, it follows that the filter F constructed in the proof of Lemma 7.2(iii) is the smallest filter on S that extends G:

$$F = \bigcap \{D : D \text{ is a filter on } S \text{ and } G \subset D\}.$$

We say that the filter F is *generated* by G.

Definition 7.3. A filter U on a set S is an *ultrafilter* if

(7.3) $$\text{for every } X \subset S, \text{ either } X \in U \text{ or } S - X \in U.$$

The dual notion is a *prime ideal*: For every $X \subset S$, either $X \in I$ or $S - X \in I$. Note that $I = P(S) - U$.

A filter F on S is *maximal* if there is no filter F' on S such that $F \subset F'$ and $F \neq F'$.

Lemma 7.4. *A filter F on S is an ultrafilter if and only if it is maximal.*

Proof. (a) An ultrafilter U is clearly a maximal filter: Assume that $U \subset F$ and $X \in F - U$. Then $S - X \in U$, and so both $S - X \in F$ and $X \in F$, a contradiction.

(b) Let F be a filter that is not an ultrafilter. We will show that F is not maximal. Let $Y \subset S$ be such that neither Y nor $S - Y$ is in F. Consider the family $G = F \cup \{Y\}$; we claim that G has the finite intersection property. If $X \in F$, then $X \cap Y \neq \emptyset$, for otherwise we would have $S - Y \supset X$ and $S - Y \in F$. Thus, if $X_1, \ldots, X_n \in F$, we have $X_1 \cap \ldots \cap X_n \in F$ and so $Y \cap X_1 \cap \ldots \cap X_n \neq \emptyset$. Hence G has the finite intersection property, and by Lemma 7.2(iii) there is a filter $F' \supset G$. Since $Y \in F' - F$, F is not maximal. □

Theorem 7.5 (Tarski). *Every filter can be extended to an ultrafilter.*

Proof. Let F_0 be a filter on S. Let P be the set of all filters F on S such that $F \supset F_0$ and consider the partially ordered set $(P, \subset)$. If $\mathcal{C}$ is a chain in P, then by Lemma 7.2(ii), $\bigcup \mathcal{C}$ is a filter and hence an upper bound of $\mathcal{C}$ in P. By Zorn's Lemma there exists a maximal element U in P. This U is an ultrafilter by Lemma 7.4. □

For every $a \in S$, the principal filter $\{X \subset S : a \in X\}$ is an ultrafilter. If S is finite, then every ultrafilter on S is principal.

If S is infinite, then there is a nonprincipal ultrafilter on S: If U extends the Fréchet filter, then U is nonprincipal.

The proof of Theorem 7.5 uses the Axiom of Choice. We shall see later that the existence of nonprincipal ultrafilters cannot be proved without AC.

If S is an infinite set of cardinality κ, then because every ultrafilter on S is a subset of $P(S)$, there are at most 2^{2^κ} ultrafilters on S. The next theorem shows that the number of ultrafilters on κ is exactly 2^{2^κ}. To get a slightly stronger result, let us call an ultrafilter D on κ *uniform* if $|X| = \kappa$ for all $X \in D$.

Theorem 7.6 (Pospíšil). *For every infinite cardinal κ, there exist 2^{2^κ} uniform ultrafilters on κ.*

We prove first the following lemma. Let us call a family $\mathcal{A}$ of subsets of κ *independent* if for any distinct sets $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ in $\mathcal{A}$, the intersection

$$X_1 \cap \ldots \cap X_n \cap (\kappa - Y_1) \cap \ldots \cap (\kappa - Y_m) \tag{7.4}$$

has cardinality κ.

Lemma 7.7. *There exists an independent family of subsets of κ of cardinality 2^κ.*

Proof. Let us consider the set P of all pairs $(F, \mathcal{F})$ where F is a finite subset of κ and $\mathcal{F}$ is a finite set of finite subsets of κ. Since $|P| = \kappa$, it suffices to find an independent family $\mathcal{A}$ of subsets of P, of size 2^κ.

For each $u \subset \kappa$, let

$$X_u = \{(F, \mathcal{F}) \in P : F \cap u \in \mathcal{F}\}$$

and let $\mathcal{A} = \{X_u : u \subset \kappa\}$. If u and v are distinct subsets of κ, then $X_u \neq X_v$: For example, if $\alpha \in u$ but $\alpha \notin v$, then let $F = \{\alpha\}$, $\mathcal{F} = \{F\}$, and $(F, \mathcal{F}) \in X_u$ while $(F, \mathcal{F}) \notin X_v$. Hence $|\mathcal{A}| = 2^\kappa$.

To show that $\mathcal{A}$ is independent, let $u_1, \ldots, u_n, v_1, \ldots, v_m$ be distinct subsets of κ. For each $i \le n$ and each $j \le m$, let $\alpha_{i,j}$ be some element of κ such that either $\alpha_{i,j} \in u_i - v_j$ or $\alpha_{i,j} \in v_j - u_i$. Now let F be any finite subset of κ such that $F \supset \{\alpha_{i,j} : i \le n,\ j \le m\}$ (note that there are κ many such finite sets). Clearly, we have $F \cap u_i \neq F \cap v_j$ for any $i \le n$ and $j \le m$. Thus if we let $\mathcal{F} = \{F \cap u_i : i \le n\}$, we have $(F, \mathcal{F}) \in X_{u_i}$ for all $i \le n$ and $(F, \mathcal{F}) \notin X_{v_j}$ for all $j \le m$. Consequently, the intersection

$$X_{u_1} \cap \ldots \cap X_{u_n} \cap (P - X_{v_1}) \cap \ldots \cap (P - X_{v_m})$$

has cardinality κ. □

Proof of Theorem 7.6. Let $\mathcal{A}$ be an independent family of subsets of κ. For every function $f : \mathcal{A} \to \{0,1\}$, consider this family of subsets of κ:

(7.5) $$G_f = \{X : |\kappa - X| < \kappa\} \cup \{X : f(X) = 1\} \cup \{\kappa - X : f(X) = 0\}.$$

By (7.4), the family G_f has the finite intersection property, and so there exists an ultrafilter D_f such that $D_f \supset G_f$. If follows from (7.5) that D_f is uniform. If $f \neq g$, then for some $X \in \mathcal{A}$, $f(X) \neq g(X)$; e.g., $f(X) = 1$ and $g(X) = 0$ and then $X \in D_f$, while $\kappa - X \in D_g$. Thus we obtain 2^{2^κ} distinct uniform ultrafilters on κ. □

Ultrafilters on ω

We present two properties of ultrafilters on ω that are frequently used in set-theoretic topology.

Let D be a nonprincipal ultrafilter on ω. D is called a *p-point* if for every partition $\{A_n : n \in \omega\}$ of ω into $\aleph_0$ pieces such that $A_n \notin D$ for all n, there exists $X \in D$ such that $X \cap A_n$ is finite, for all $n \in \omega$.

First we notice that it is easy to find a nonprincipal ultrafilter that is not a p-point: Let $\{A_n : n \in \omega\}$ be any partition of ω into $\aleph_0$ infinite pieces, and let F be the following filter on ω:

(7.6) $X \in F$ if and only if except for finitely many n, $X \cap A_n$ contains all but finitely many elements of A_n.

If D is any ultrafilter extending F, then D is not a p-point.

Theorem 7.8 below shows that existence of p-points follows from the Continuum Hypothesis. By a result of Shelah there exists a model of ZFC in which there are no p-points.

A nonprincipal ultrafilter D on ω is a *Ramsey* ultrafilter if for every partition $\{A_n : n \in \omega\}$ of ω into $\aleph_0$ pieces such that $A_n \notin D$ for all n, there exists $X \in D$ such that $X \cap A_n$ has one element for all $n \in \omega$.

Every Ramsey ultrafilter is a p-point.

Theorem 7.8. *If* $2^{\aleph_0} = \aleph_1$, *then a Ramsey ultrafilter exists.*

Proof. Let $\mathcal{A}_\alpha$, $\alpha < \omega_1$, enumerate all partitions of ω and let us construct an ω_1-sequence of infinite subsets of ω as follows: Given X_α, let $X_{\alpha+1} \subset X_\alpha$ be such that either $X_{\alpha+1} \subset A$ for some $A \in \mathcal{A}_\alpha$, or that $|X_{\alpha+1} \cap A| \leq 1$ for all $A \in \mathcal{A}_\alpha$. If α is a limit ordinal, let X_α be such that $X_\alpha - X_\beta$ is finite for all $\beta < \alpha$. (Such a set X_α exists because α is countable.) Then $D = \{X : X \supset X_\alpha \text{ for some } \alpha < \omega_1\}$ is a Ramsey ultrafilter. □

κ-Complete Filters and Ideals

A filter F on S is *countably complete* (σ-*complete*) if whenever $\{X_n : n \in \boldsymbol{N}\}$ is a countable family of subsets of S and $X_n \in F$ for every n, then

$$\bigcap_{n=0}^{\infty} X_n \in F. \tag{7.7}$$

A *countably complete* ideal (a σ-*ideal*) is such that if $X_n \in I$ for every n, then

$$\bigcup_{n=0}^{\infty} X_n \in I.$$

More generally, if κ is a regular uncountable cardinal, and F is a filter on S, then F is called κ-*complete* if F is closed under intersection of less than κ sets, i.e., if whenever $\{X_\alpha : \alpha < \gamma\}$ is a family of subsets of S, $\gamma < \kappa$, and $X_\alpha \in F$ for every $\alpha < \gamma$, then

$$\bigcap_{\alpha<\gamma} X_\alpha \in F. \tag{7.8}$$

The dual notion is a κ-complete ideal.

An example of a κ-complete ideal is $I = \{X \subset S : |X| < \kappa\}$, on any set S such that $|S| \geq \kappa$.

A σ-complete filter is the same as an $\aleph_1$-complete filter.

There is no nonprincipal σ-complete filter on a countable set S. If S is uncountable, then

$$\{X \subset S : |X| \leq \aleph_0\}$$

is a σ-ideal on S.

Similarly, if $\kappa > \omega$ is regular and $|S| \geq \kappa$, then

$$\{X \subset S : |X| < \kappa\}$$

is the smallest κ-complete ideal on S containing all singletons $\{a\}$.

The question whether a nonprincipal *ultrafilter* on a set can be σ-complete gives rise to deep investigations of the foundations of set theory. In particular, if such ultrafilters exist, then there exist large cardinals (inaccessible, etc.).

Boolean Algebras

An algebra of sets (see Definition 4.9) is a collection of subsets of a given nonempty set that is closed under unions, intersections and complements. These properties of algebras of sets are abstracted in the notion of Boolean algebra:

Definition 7.9. A *Boolean algebra* is a set B with at least two elements, 0 and 1, endowed with binary operations $+$ and $\cdot$ and a unary operation $-$.

The Boolean operations satisfy the following axioms:

$$\begin{aligned}
(7.9)\quad & u+v=v+u, & u\cdot v=v\cdot u, & \quad \text{(commutativity)}\\
& u+(v+w)=(u+v)+w, & u\cdot(v\cdot w)=(u\cdot v)\cdot w, & \quad \text{(associativity)}\\
& u\cdot(v+w)=u\cdot v+u\cdot w, & u+(v\cdot w)=(u+v)\cdot(u+w), & \quad \text{(distributivity)}\\
& u\cdot(u+v)=u, & u+(u\cdot v)=u, & \quad \text{(absorption)}\\
& u+(-u)=1, & u\cdot(-u)=0. & \quad \text{(complementation)}
\end{aligned}$$

An algebra of sets $\mathcal{S}$, with $\bigcup\mathcal{S}=S$, is a Boolean algebra, with Boolean operations $X\cup Y$, $X\cap Y$ and $S-X$, and with $\emptyset$ and S being 0 and 1. If follows from Stone's Representation Theorem below that every Boolean algebra is isomorphic to an algebra of sets.

From the axioms (7.9) one can derive additional Boolean algebraic rules that correspond to rules for the set operations $\cup$, $\cap$ and $-$. Among others, we have

$$u+u=u,\quad u\cdot u=u,\quad u+0=u,\quad u\cdot 0=0,\quad u+1=1,\quad u\cdot 1=u$$

and the De Morgan laws

$$-(u+v)=-u\cdot -v,\qquad -(u\cdot v)=-u+-v.$$

Two elements $u,v\in B$ are *disjoint* if $u\cdot v=0$. Let us define

$$u-v=u\cdot(-v),$$

and

$$(7.10)\qquad u\le v \quad\text{if and only if}\quad u-v=0.$$

It is easy to see that $\le$ is a partial ordering of B and that

$$u\le v\quad\text{if and only if}\quad u+v=v\quad\text{if and only if}\quad u\cdot v=u.$$

Moreover, 1 is the greatest element of B and 0 is the least element. Also, for any $u,v\in B$, $u+v$ is the least upper bound of $\{u,v\}$ and $u\cdot v$ is the greatest lower bound of $\{u, v\}$. Since $-u$ is the unique v such that $u + v = 1$ and $u \cdot v = 0$, it follows that all Boolean-algebraic operations can be defined in terms of the partial ordering of B.

We shall now give an example showing the relation between Boolean algebras and logic:

Let $\mathcal{L}$ be a first order language and let S be the set of all sentences of $\mathcal{L}$. We consider the equivalence relation $\vdash \varphi \leftrightarrow \psi$ on S. The set B of all equivalence classes $[\varphi]$ is a Boolean algebra under the following operations:

$$\begin{aligned} [\varphi] + [\psi] &= [\varphi \vee \psi], & 0 &= [\varphi \wedge \neg\varphi], \\ [\varphi] \cdot [\psi] &= [\varphi \wedge \psi], & 1 &= [\varphi \vee \neg\varphi]. \\ -[\varphi] &= [\neg\varphi], \end{aligned}$$

This algebra is called the *Lindenbaum algebra.*

A subset A of a Boolean algebra B is a *subalgebra* if it contains 0 and 1 and is closed under the Boolean operations:

(7.11) (i) $0 \in A$, $1 \in A$;
(ii) if $u, v \in A$, then $u + v \in A$, $u \cdot v \in A$, $-u \in A$.

If $X \subset B$, then there is a smallest subalgebra A of B that contains X; A can be described either as $\bigcap\{A : X \subset A \subset B$ and A is a subalgebra$\}$, or as the set of all Boolean combinations in B of elements of X. The subalgebra A is *generated* by X. If X is infinite, then $|A| = |X|$. See Exercises 7.18–7.20.

If B is a Boolean algebra, let $B^+ = B - \{0\}$ denote the set of all nonzero elements of B. If $a \in B^+$, the set $B{\restriction}a = \{u \in B : u \leq a\}$ with the partial order inherited from B, is a Boolean algebra; its $+$ and $\cdot$ are the same as in B, and the complement of u is $a - u$. An element $a \in B$ is called an *atom* if it is a minimal element of B^+; equivalently, if there is no x such that $0 < x < a$. A Boolean algebra is *atomic* if for every $u \in B^+$ there is an atom $a \leq u$; B is *atomless* if it has no atoms.

Let B and C be two Boolean algebras. A mapping $h : B \to C$ is a *homomorphism* if it preserves the operations:

(7.12) (i) $h(0) = 0$, $h(1) = 1$,
(ii) $h(u + v) = h(u) + h(v)$, $h(u \cdot v) = h(u) \cdot h(v)$, $h(-u) = -h(u)$.

Note that the range of a homomorphism is a subalgebra of C and that $h(u) \leq h(v)$ whenever $u \leq v$. A one-to-one homomorphism of B onto C is called an *isomorphism.* An *embedding* of B in C is an isomorphism of B onto a subalgebra of C. Note that if $h : B \to C$ is a one-to-one mapping such that $u \leq v$ if and only if $h(u) \leq h(v)$, then h is an isomorphism. An isomorphism of a Boolean algebra onto itself is called an *automorphism.*

Ideals and Filters on Boolean Algebras

The definition of filter (and ideal) given earlier in this chapter generalizes to arbitrary Boolean algebras. Let B be a Boolean algebra. An *ideal* on B is a subset I of B such that:

(7.13) (i) $0 \in I$, $1 \notin I$;
(ii) if $u \in I$ and $v \in I$, then $u + v \in I$;
(iii) if $u, v \in B$, $u \in I$ and $v \le u$, then $v \in I$.

A *filter* on B is a subset F of B such that:

(7.14) (i) $1 \in F$, $0 \notin F$;
(ii) if $u \in F$ and $v \in F$, then $u \cdot v \in F$;
(iii) if $u, v \in B$, $u \in F$ and $u \le v$, then $v \in F$.

The *trivial* ideal is the ideal $\{0\}$; an ideal is *principal* if $I = \{u \in B : u \le u_0\}$ for some $u_0 \ne 1$. Similarly for filters.

A subset G of $B - \{0\}$ has the *finite intersection property* if for every finite $\{u_1, \ldots, u_n\} \subset G$, $u_1 \cdot \ldots \cdot u_n \ne 0$. Every $G \subset B$ that has the finite intersection property generates a filter on B; this and the other two clauses of Lemma 7.2 hold also for Boolean algebras.

There is a relation between ideals and homomorphisms. If $h : B \to C$ is a homomorphism, then

$$I = \{u \in B : h(u) = 0\} \tag{7.15}$$

is an ideal on B (the *kernel* of the homomorphism). On the other hand, let I be an ideal on B. Let us consider the following equivalence relation on B:

$$u \sim v \quad \text{if and only if} \quad u \bigtriangleup v \in I \tag{7.16}$$

where

$$u \bigtriangleup v = (u - v) + (v - u).$$

Let C be the set of all equivalence classes, $C = B/{\sim}$, and endow C with the following operations:

$$\begin{aligned} [u] + [v] &= [u + v], & 0 &= [0], \\ [u] \cdot [v] &= [u \cdot v], & 1 &= [1]. \\ -[u] &= [-u], \end{aligned} \tag{7.17}$$

Then C is a Boolean algebra, the *quotient* of B mod I, and is a homomorphic image of B under the homomorphism

$$h(u) = [u]. \tag{7.18}$$

The quotient algebra is denoted B/I.

An ideal I on B is a *prime ideal* if

$$\text{for every } u \in B, \text{ either } u \in I \text{ or } -u \in I. \tag{7.19}$$

The dual of a prime ideal is an *ultrafilter*.

Lemma 7.4 holds in general: An ideal is a prime ideal (and a filter is an ultrafilter) if and only if it is maximal. Also, an ideal I on B is prime if and only if the quotient of B mod I is the trivial algebra $\{0, 1\}$.

Tarski's Theorem 7.5 easily generalizes to Boolean algebras:

Theorem 7.10 (The Prime Ideal Theorem). *Every ideal on B can be extended to a prime ideal.* □

The proof of the Prime Ideal Theorem uses the Axiom of Choice. It is known that the theorem cannot be proved without using the Axiom of Choice. However, it is also known that the Prime Ideal Theorem is weaker than the Axiom of Choice.

Theorem 7.11 (Stone's Representation Theorem). *Every Boolean algebra is isomorphic to an algebra of sets.*

Proof. Let B be a Boolean algebra. We let

$$S = \{p : p \text{ is an ultrafilter on } B\}. \tag{7.20}$$

For every $u \in B$, let X_u be the set of all $p \in S$ such that $u \in p$. Let

$$\mathcal{S} = \{X_u : u \in B\}. \tag{7.21}$$

Let us consider the mapping $\pi(u) = X_u$ from B onto $\mathcal{S}$. Clearly, $\pi(1) = S$ and $\pi(0) = \emptyset$. It follows from the definition of ultrafilter that

$$\pi(u \cdot v) = \pi(u) \cap \pi(v), \qquad \pi(u + v) = \pi(u) \cup \pi(v), \qquad \pi(-u) = S - \pi(u).$$

Thus π is a homomorphism of B onto the algebra of sets $\mathcal{S}$. It remains to show that π is one-to-one.

If $u \neq v$, then using the Prime Ideal Theorem, one can find an ultrafilter p on B containing one of these two elements but not the other. Thus π is an isomorphism. □

Complete Boolean Algebras

The partial ordering $\leq$ of a Boolean algebra can be used to define infinitary operations on B, generalizing $+$ and $\cdot$. Let us recall that $u + v = \sup\{u, v\}$

and $u \cdot v = \inf\{u, v\}$ in the partial ordering of B. Thus for any nonempty $X \subset B$, we define

$$\sum\{u : u \in X\} = \sup X \quad \text{and} \quad \prod\{u : u \in X\} = \inf X, \tag{7.22}$$

provided that the least upper bound (the greatest lower bound) exists. We also define $\sum\emptyset = 0$ and $\prod\emptyset = 1$.

If the infinitary sum and product is defined for all $X \subset B$, the Boolean algebra is called *complete*. Similarly, we call B *κ-complete* (where κ is a regular uncountable cardinal) if sums and products exist for all X of cardinality $< \kappa$. An $\aleph_1$-complete Boolean algebra is called *σ-complete* or *countably complete*.

An algebra of sets $\mathcal{S}$ is *κ-complete* if it is closed under unions and intersections of $< \kappa$ sets. A κ-complete algebra of sets is a κ-complete Boolean algebra and for every $X \subset \mathcal{S}$ such that $|X| < \kappa$, $\sum X = \bigcup X$.

An ideal I on a κ-complete Boolean algebra is *κ-complete* if

$$\sum\{u : u \in X\} \in I$$

whenever $X \subset I$ and $|X| < \kappa$. A *κ-complete filter* is the dual notion.

If I is a κ-complete ideal on a κ-complete Boolean algebra B, then B/I is κ-complete, and

$$\sum\{[u] : u \in X\} = [\sum\{u : u \in X\}]$$

for every $X \subset B$, $|X| < \kappa$. Similarly for products.

An $\aleph_1$-complete ideal is called a *σ-ideal*.

There are two important examples of σ-ideals on the Boolean algebra of all Borel sets of reals: the σ-ideal of Borel sets of Lebesgue measure 0, and the σ-ideal of meager Borel sets. (Exercises 7.14 and 7.15.)

Let A be a subalgebra of a Boolean algebra B. A is a *dense* subalgebra of B if for every $u \in B^+$ there is a $v \in A^+$ such that $v \le u$.

A *completion* of a Boolean algebra B is a complete Boolean algebra C such that B is a dense subalgebra of C.

Lemma 7.12. *The completion of a Boolean algebra B is unique up to isomorphism.*

Proof. Let C and D be completions of B. We define an isomorphism $\pi : C \to D$ by

$$\pi(c) = \sum\nolimits^D\{u \in B : u \le c\}. \tag{7.23}$$

To verify that π is an isomorphism, one uses the fact that B is a dense subalgebra of both C and D. For example, to show that $\pi(c) \ne 0$ whenever $c \ne 0$: There is $u \in B$ such that $0 < u \le c$, and we have $0 < u \le \pi(c)$. $\square$

Theorem 7.13. *Every Boolean algebra has a completion.*

Proof. We use a construction similar to the method of Dedekind cuts. Let A be a Boolean algebra. Let us call a set $U \subset A^+$ a *cut* if

(7.24) $$p \le q \text{ and } q \in U \text{ implies } p \in U.$$

For every $p \in A^+$, let U_p denote the cut $\{x : x \le p\}$.

A cut U is *regular* if

(7.25) $$\text{whenever } p \notin U, \text{ then there exists } q \le p \text{ such that } U_q \cap U = \emptyset.$$

Note that every U_p is regular, and that every cut includes some U_p.

We let B be the set of all regular cuts in A^+. We claim that B, under the partial ordering by inclusion, is a complete Boolean algebra. Note that the intersection of any collection of regular cuts is a regular cut, and hence each cut U is included in a least regular cut $\overline{U}$. In fact,

$$\overline{U} = \{p : (\forall q \le p)\, U \cap U_q \neq \emptyset\}.$$

Thus for $u, v \in B$ we have

$$u \cdot v = u \cap v, \qquad u + v = \overline{u \cup v}.$$

The complement of $u \in B$ is the regular cut

$$-u = \{p : U_p \cap u = \emptyset\}.$$

And, of course, $\emptyset$ and A^+ are the zero and the unit of B. It is not difficult to verify that B is a complete Boolean algebra, and we leave the verification to the reader.

Furthermore, for all $p, q \in A^+$ we have $U_p + U_q = U_{p+q}$, $U_p \cdot U_q = U_{p \cdot q}$ and $-U_p = U_{-p}$. Thus A embeds in B as a dense subalgebra. □

Complete and Regular Subalgebras

Let B be a complete Boolean algebra. A subalgebra A of B is a *complete subalgebra* if $\sum X \in A$ and $\prod X \in A$ for all $X \subset A$. (Caution: A subalgebra A of B that is itself complete is not necessarily a complete subalgebra of B.) Similarly, a *complete homomorphism* is a homomorphism h of B into C such that for all $X \subset B$,

(7.26) $$h(\textstyle\sum X) = \textstyle\sum h(X), \qquad h(\textstyle\prod X) = \textstyle\prod h(X).$$

A *complete embedding* is an embedding that satisfies (7.26). Note that every isomorphism is complete.

Since the intersection of any collection of complete subalgebras of B is a complete subalgebra, every $X \subset B$ is included in a smallest complete subalgebra of B. This algebra is called the complete subalgebra of B *completely generated* by X.

Definition 7.14. A set $W \subset B^+$ is an *antichain* in a Boolean algebra B if $u \cdot v = 0$ for all distinct $u, v \in W$.

If W is an antichain and if $\sum W = u$ then we say that W is a *partition* of u. A partition of 1 is just a *partition*, or a *maximal antichain.*

If B is a Boolean algebra and A is a subalgebra of B then an antichain in A that is maximal in A need not be maximal in B. If every maximal antichain in A is also maximal in B, then A is called a *regular subalgebra* of B.

If A is a complete subalgebra of a complete Boolean algebra B then A is a regular subalgebra of B. Also, if A is a dense subalgebra of B then A is a regular subalgebra. See also Exercise 7.31.

Saturation

Let κ be an infinite cardinal. A Boolean algebra B is *κ-saturated* if there is no partition W of B such that $|W| = \kappa$, and

(7.27) $$\operatorname{sat}(B) = \text{the least } \kappa \text{ such that } B \text{ is } \kappa\text{-saturated.}$$

B is also said to satisfy the *κ-chain condition*; this is because if B is complete, B is κ-saturated if and only if there exists no descending κ-sequence $u_0 > u_1 > \ldots > u_\alpha > \ldots$, $\alpha < \kappa$, of elements of B. The $\aleph_1$-chain condition is called the *countable chain condition* (c.c.c.).

Theorem 7.15. *If B is an infinite complete Boolean algebra, then* $\operatorname{sat}(B)$ *is a regular uncountable cardinal.*

Proof. Let $\kappa = \operatorname{sat}(B)$. It is clear that κ is uncountable. Let us assume that κ is singular; we shall obtain a contradiction by constructing a partition of size κ.

For $u \in B$, $u \neq 0$, let $\operatorname{sat}(u)$ denote $\operatorname{sat}(B_u)$. Let us call $u \in B$ *stable* if $\operatorname{sat}(v) = \operatorname{sat}(u)$ for every nonzero $v \leq u$. The set S of stable elements is dense in B; otherwise, there would be a descending sequence $u_0 > u_1 > u_2 > \ldots$ with decreasing cardinals $\operatorname{sat}(u_0) > \operatorname{sat}(u_1) > \ldots$. Let T be a maximal set of pairwise disjoint elements of S. Thus T is a partition of B, and $|T| < \kappa$.

First we show that $\sup\{\operatorname{sat}(u) : u \in T\} = \kappa$. For every regular $\lambda < \kappa$ such that $\lambda > |T|$, consider a partition W of B of size λ. Then at least one $u \in T$ is partitioned by W into λ pieces.

Thus we consider two cases:

Case I. There is $u \in T$ such that $\operatorname{sat}(u) = \kappa$. Since $\operatorname{cf} \kappa < \kappa$, there is a partition W of u of size $\operatorname{cf} \kappa$: $W = \{u_\alpha : \alpha < \operatorname{cf} \kappa\}$. Let κ_α, $\alpha < \operatorname{cf} \kappa$, be an increasing sequence with limit κ. For each α, $\operatorname{sat}(u_\alpha) = \operatorname{sat}(u) = \kappa$ and so let W_α be a partition of u_α of size κ_α. Then $\bigcup_{\alpha < \operatorname{cf} \kappa} W_\alpha$ is a partition of u of size κ.

Case II. For all $u \in T$, $\mathrm{sat}(u) < \kappa$, but $\sup\{\mathrm{sat}(u) : u \in T\} = \kappa$. Again, let $\kappa_\alpha \to \kappa$, $\alpha < \mathrm{cf}\,\kappa$. For each $\alpha < \mathrm{cf}\,\kappa$ (by induction), we find $u_\alpha \in T$, distinct from all u_β, $\beta < \alpha$, which admits a partition W_α of size κ_α. Then $\bigcup_{\alpha < \mathrm{cf}\,\kappa} W_\alpha$ is an antichain in B of size κ. □

Distributivity of Complete Boolean Algebras

The following distributive law holds for every complete Boolean algebra:

$$\sum_{i \in I} u_{0,i} \cdot \sum_{u \in J} u_{1,j} = \sum_{(i,j) \in I \times J} u_{0,i} \cdot u_{1,j}.$$

To formulate a general distributive law, let κ be a cardinal, and let us call B *κ-distributive* if

$$\prod_{\alpha<\kappa} \sum_{i \in I_\alpha} u_{\alpha,i} = \sum_{f \in \prod_{\alpha<\kappa} I_\alpha} \prod_{\alpha<\kappa} u_{\alpha,f(\alpha)}. \tag{7.28}$$

(Every complete algebra of sets satisfies (7.28).) We shall see later that distributivity plays an important role in generic models. For now, let us give two equivalent formulations of κ-distributivity.

If W and Z are partitions of B, then W is a *refinement* of Z if for every $w \in W$ there is $z \in Z$ such that $w \le z$. A set $D \subset B$ is *open dense* if it is dense in B and $0 \ne u \le v \in D$ implies $u \in D$.

Lemma 7.16. *The following are equivalent, for any complete Boolean algebra B:*

(i) *B is κ-distributive.*
(ii) *The intersection of κ open dense subsets of B is open dense.*
(iii) *Every collection of κ partitions of B has a common refinement.*

Proof. (i) → (ii). Let D_α, $\alpha < \kappa$, be open dense, $D = \bigcap_{\alpha<\kappa} D_\alpha$. D is certainly open; thus let $u \ne 0$. If we let $\{u_{\alpha,i} : i \in I_\alpha\} = \{u \cdot v : v \in D_\alpha\}$, then $\sum_i u_{\alpha,i} = u$ for every α and the left-hand side of (7.28) is u. For each $f \in \prod_\alpha I_\alpha$, let $u_f = \prod_\alpha u_{\alpha,f(\alpha)}$; clearly, each nonzero u_f is in D. However, $\sum_f u_f = u$, by (7.28), and so some u_f is nonzero.

(ii) → (iii). Let W_α, $\alpha < \kappa$ be partions of B. For each α, let $D_\alpha = \{u : u \le v \text{ for some } v \in W_\alpha\}$; each D_α is open dense. Let $D = \bigcap_{\alpha<\kappa} D_\alpha$, and let W be a maximal set of pairwise disjoint elements of D. Since D is dense, W is a partition of B, and clearly, W is a refinement of each W_α.

(iii) → (i). Let $\{u_{\alpha,i} : \alpha < \kappa,\ i \in I_\alpha\}$ be a collection of elements of B. First we show that the right-hand side of (7.28) is always $\le$ the left-hand side. For each $f \in \prod_{\alpha<\kappa} I_\alpha$, let $u_f = \prod_{\alpha<\kappa} u_{\alpha,f(\alpha)}$; we have $u_f \le u_{\alpha,f(\alpha)}$ and so $u_f \le \sum_{i \in I_\alpha} u_{\alpha,i}$ for each α. Thus, for each α,

$$\sum_f u_f \le \sum_i u_{\alpha,i}$$

and so

$$\sum_f \prod_\alpha u_{\alpha,f(\alpha)} = \sum_f u_f \le \prod_\alpha \sum_i u_{\alpha,i}.$$

To prove (7.28), assume that (iii) holds, and let $u = \prod_\alpha \sum_i u_{\alpha,i}$; we want to show that $\sum_f \prod_\alpha u_{\alpha,f(\alpha)} = u$. Without loss of generality, we can assume that $u = 1$ (otherwise we argue in the algebra $B{\restriction}u$). For each α, let us replace $\{u_{\alpha,i} : i \in I_\alpha\}$ by pairwise disjoint $\{v_{\alpha,i} : i \in I_\alpha\} = W_\alpha$ such that $v_{\alpha,i} \le u_{\alpha,i}$ and $\sum_i v_{\alpha,i} = \sum_i u_{\alpha,i}$ (some of the $v_{\alpha,i}$ may be 0). Clearly $\sum_f \prod_\alpha v_{\alpha,f(\alpha)} \le \sum_f \prod_\alpha u_{\alpha,f(\alpha)}$. Each W_α is a partition of B and so there is a partition W that is a refinement of each W_α. Now for each $w \in W$ there exists f such that $w \le \prod_\alpha v_{\alpha,f(\alpha)}$, and so $\sum_f \prod_\alpha v_{\alpha,f(\alpha)} = 1$. □

Exercises

7.1. If F is a filter and $X \in F$, then $P(X) \cap F$ is a filter on X.

7.2. The filter in Example 4 is generated by the sets $\{a\}^\wedge$, $a \in A$.

7.3. If U is an ultrafilter and $X \cup Y \in U$, then either $X \in U$ or $Y \in U$.

7.4. Let U be an ultrafilter on S. Then the set of all $X \subset S \times S$ such that $\{a \in S : \{b \in S : (a,b) \in X\} \in U\} \in U$ is an ultrafilter on $S \times S$.

7.5. Let U be an ultrafilter on S and let $f : S \to T$. Then the set $f_*(U) = \{X \subset T : f_{-1}(X) \in U\}$ is an ultrafilter on T.

7.6. Let U be an ultrafilter on $\boldsymbol{N}$ and let $\langle a_n \rangle_{n=0}^\infty$ be a bounded sequence of real numbers. Prove that there exists a unique *U-limit* $a = \lim_U a_n$ such that for every $\varepsilon > 0$, $\{n : |a_n - a| < \varepsilon\} \in U$.

7.7. A nonprincipal ultrafilter D on ω is a *p*-point if and only if it satisfies the following: If $A_0 \supset A_1 \supset \ldots \supset A_n \supset \ldots$ is a decreasing sequence of elements of D, then there exists $X \in D$ such that for each n, $X - A_n$ is finite.

7.8. If $(P, <)$ is a countable linearly ordered set and if D is a *p*-point on P, then there exists $X \in D$ such that the order-type of X is either ω or ω^*. (X has order-type ω^* if and only if $X = \{x_n\}_{n=0}^\infty$ and $x_0 > x_1 \ldots > x_n > \ldots$.)

7.9. An ultrafilter D on ω is Ramsey if and only if every function $f : \omega \to \omega$ is either one-to-one on a set in D, or constant on a set in D.

If D and E are ultrafilters on ω, then $D \le E$ means that for some function $f : \omega \to \omega$, $D = f_*(E)$ (the Rudin-Keisler ordering, see Exercise 7.5).

$D \equiv E$ means that there is a one-to-one function of ω onto ω such that $E = f_*(D)$.

7.10. If $D = f_*(D)$, then $\{n : f(n) = n\} \in D$.

[Let $X = \{n : f(n) < n\}$, $Y = \{n : f(n) > n\}$. For each $n \in X$, let $l(n)$ be the length of the maximal sequence such that $n > f(n) > f(f(n)) > \ldots$. Let $X_0 = \{n \in X : l(n) \text{ is even}\}$ and $X_1 = \{n \in X : l(n) \text{ is odd}\}$. Neither X_0 nor X_1 can be in D since, e.g., $X_0 \cap f_{-1}(X_0) = \emptyset$. The set Y is handled similarly, except that it remains to show that the set Z of all n such that the sequence $n < f(n) < f^2(n) < f^3(n) < \ldots$ is infinite cannot be in D. For $x, y \in Z$ let $x \equiv y$ if $f^k(x) = f^m(y)$ for some k and m. For each $x \in Z$, let a_x be a fixed representative of the class $\{y : y \equiv x\}$; let $l(x)$ be the least $k + m$ such that $f^k(x) = f^m(a_x)$. Let $Z_0 = \{x \in Z : l(x) \text{ is even}\}$ and $Z_1 = \{x \in Z : l(x) \text{ is odd}\}$. Clearly $f_{-1}(Z_1) \cap Z = Z_0$.]

7.11. If $D \le E$ and $E \le D$, then $D \equiv E$.
[Use Exercise 7.10.]

Thus $\le$ is a partial ordering of ultrafilters on ω. A nonprincipal ultrafilter D is *minimal* if there is no nonprincipal E such that $E \le D$ and $E \not\equiv D$.

7.12. An ultrafilter D on ω is minimal if and only if it is Ramsey.
[If D is Ramsey and $E = f_*(D)$ is nonprincipal, then f is unbounded mod D, hence one-to-one mod D and consequently, $E \equiv D$. If D is minimal and f is unbounded mod D, then $D \le f_*(D)$ and hence $D = g_*(f_*(D))$ for some g. It follows, by Exercise 7.10, that f is one-to-one mod D.]

7.13. If ω_α is singular, then there is no nonprincipal ω_α-complete ideal on ω_α.

7.14. The set of all sets $X \subset \boldsymbol{R}$ that have Lebesgue measure 0 is a σ-ideal.

A set $X \subset \boldsymbol{R}$ is *meager* if it is the union of a countable collection of nowhere dense sets.

7.15. The set of all meager sets $X \subset \boldsymbol{R}$ is a σ-ideal.
[By the Baire Category Theorem, $\boldsymbol{R}$ is not meager.]

7.16. Let κ be a regular uncountable cardinal, let $|A| \ge \kappa$ and let $S = P_\kappa(A)$. Let F be the set of all $X \subset S$ such that $X \supset \hat{P}$ for some $P \in S$, where $\hat{P} = \{Q \in S : P \subset Q\}$. Then F is a κ-complete filter on S.

7.17. Let B be a Boolean algebra and define

$$u \oplus v = (u - v) + (v - u).$$

Then B with operations $\oplus$ and $\cdot$ is a ring (with zero 0 and unit 1).

7.18. Every element of the subalgebra generated by X is equal to $u_1 + \ldots + u_n$ where each u_s is of the form $u_s = \pm x_1 \cdot \pm x_2 \cdot \ldots \cdot \pm x_k$ with $x_i \in X$.

7.19. If A is a subalgebra of B and $u \in B$, then the subalgebra generated by $A \cup \{u\}$ is equal to $\{a \cdot u + (b - u) : a, b \in A\}$.

7.20. A finitely generated Boolean algebra is finite. If A has k generators, then $|A| \le 2^{2^k}$.

7.21. Every finite Boolean algebra is atomic. If $A = \{a_1, \ldots, a_n\}$ are the atoms of B, then B is isomorphic to the field of sets $P(A)$. Hence B has 2^n elements.

7.22. Any two countable atomless Boolean algebras are isomorphic.

7.23. $B{\restriction}a$ is isomorphic to B/I where I is the principal ideal $\{u : u \le -a\}$.

7.24. Let A be a subalgebra of a Boolean algebra B and let $u \in B - A$. Then there exist ultrafilters F, G on B such that $u \in F$, $u \notin G$, and $F \cap A = G \cap A$.

7.25. Let B be an infinite Boolean algebra, $|B| = \kappa$. There are at least κ ultrafilters on B.

[Assume otherwise. For each pair $(F, G) \in S \times S$ pick $u \in F - G$, and let these u's generate a subalgebra A. Since $|A| \leq |S| < \kappa$, let $u \in B - A$. Use Exercise 7.24 to get a contradiction.]

7.26. For B to be complete it is sufficient that all the sums $\sum X$ exist.

[$\prod X = \sum\{u : u \leq x \text{ for all } x \in X\}$.]

7.27. Let B be a complete Boolean algebra.

(i) Verify the distributive laws:

$$a \cdot \sum\{u : u \in X\} = \sum\{a \cdot u : u \in X\},$$
$$a + \prod\{u : u \in X\} = \prod\{a + u : u \in X\}.$$

(ii) Verify the De Morgan laws:

$$-\sum\{u : u \in X\} = \prod\{-u : u \in X\},$$
$$-\prod\{u : u \in X\} = \sum\{-u : u \in X\}.$$

7.28. Let A and B be σ-complete Boolean algebras. If A is isomorphic to $B{\restriction}b$ and B is isomorphic to $A{\restriction}a$, then A and B are isomorphic.

[Follow the proof of the Cantor-Bernstein Theorem.]

7.29. Let A be a subalgebra of a Boolean algebra B, let $u \in B$ and let $A(u)$ be the algebra generated by $A \cup \{u\}$. If h is a homomorphism from A into a complete Boolean algebra C then h extends to a homomorphism from $A(u)$ into C.

[Let $v \in C$ be such that $\sum\{h(a) : a \in A,\ a \leq u\} \leq v \leq \sum\{h(b) : b \in A,\ u \leq b\}$. Define $h(a \cdot u + b \cdot (-u)) = h(a) \cdot v + h(b) \cdot (-v)$.]

7.30 (Sikorski's Extension Theorem). Let A be a subalgebra of a Boolean algebra B and let h be a homomorphism from A into a complete Boolean algebra C. Then h can be extended to a homomorphism from B into C.

[Use Exercise 7.29 and Zorn's Lemma.]

7.31. If B is a Boolean algebra and A is a regular subalgebra of B then the inclusion mapping extends to a (unique) complete embedding of the completion of A into the completion of B.

[Use Sikorski's Extension Theorem.]

7.32. If B is an infinite complete Boolean algebra, then $|B|^{\aleph_0} = |B|$.

[First consider the case when $|B{\restriction}a| = |B|$ for all $a \neq 0$: There is a partition W such that $|W| = \aleph_0$, and $|B| = \prod\{|B{\restriction}a| : a \in W\} = |B|^{\aleph_0}$. In general, call $a \neq 0$ *stable* if $|B{\restriction}x| = |B{\restriction}a|$ for all $x \leq a$, $x \neq 0$. The set of all stable $a \in B$ is dense, and $|B{\restriction}a| = 2$ or $|B{\restriction}a|^{\aleph_0} = |B{\restriction}a|$ if a is stable. Let W be a partition of B such that each $a \in W$ is stable; we have $|B| = \prod\{|B{\restriction}a| : a \in W\}$ and the theorem follows.]

7.33. If B is a κ-complete, κ-saturated Boolean algebra, then B is complete.

[It suffices to show that $\sum X$ exists for every *open* X (i.e., $u \leq v \in X$ implies $u \in X$). If $X \subset B$ is open, show that $\sum X = \sum W$ where W is a maximal subset of X that is an antichain.]

Historical Notes

The notion of filter is, according to Kuratowski's book [1966], due to H. Cartan. Theorem 7.5 was first proved by Tarski in [1930].

Theorem 7.6 is due to Pospíšil [1937]; the present proof uses independent sets (Lemma 7.7); cf. Fichtenholz and Kantorovich [1935] ($\kappa = \omega$) and Hausdorff [1936b].

W. Rudin [1956] proved that p-points exist if $2^{\aleph_0} = \aleph_1$, a recent result of Shelah shows that existence of p-points is unprovable in ZFC. Galvin showed that $2^{\aleph_0} = \aleph_1$ implies the existence of Ramsey ultrafilters.

Facts about Boolean algebras can be found in Handbook of Boolean algebras [1989] which also contains an extensive bibliography. The Representation Theorem for Boolean algebras as well as the existence of the completion (Theorems 7.11 and 7.13) are due to Stone [1936]. Theorem 7.15 on saturation was proved by Erdős and Tarski [1943].

Exercise 7.8: Booth [1970/71].

Exercise 7.10: Frolík [1968], M. E. Rudin [1971].

The Rudin-Keisler equivalence was first studied by W. Rudin in [1956]; the study of the Rudin-Keisler ordering was initiated by M. E. Rudin [1966].

Exercise 7.25: Makinson [1969].

Exercises 7.29 and 7.30: Sikorski [1964].

Exercise 7.32: Pierce [1958]. The assumption can be weakened to "σ-complete," see Comfort and Hager [1972].