8. Stationary Sets

In this chapter we develop the theory of closed unbounded and stationary subsets of a regular uncountable cardinal, and its generalizations.

Closed Unbounded Sets

If X is a set of ordinals and $\alpha > 0$ is a limit ordinal then α is a *limit point* of X if $\sup(X \cap \alpha) = \alpha$.

Definition 8.1. Let κ be a regular uncountable cardinal. A set $C \subset \kappa$ is a *closed unbounded* subset of κ if C is unbounded in κ and if it contains all its limit points less than κ.

A set $S \subset \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for every closed unbounded subset C of κ.

An unbounded set $C \subset \kappa$ is closed if and only if for every sequence $\alpha_0 < \alpha_1 < \ldots < \alpha_\xi < \ldots$ $(\xi < \gamma)$ of elements of C, of length $\gamma < \kappa$, we have $\lim_{\xi \to \gamma} \alpha_\xi \in C$.

Lemma 8.2. *If C and D are closed unbounded, then $C \cap D$ is closed unbounded.*

Proof. It is immediate that $C \cap D$ is closed. To show that $C \cap D$ is unbounded, let $\alpha < \kappa$. Since C is unbounded, there exists an $\alpha_1 > \alpha$ with $\alpha_1 \in C$. Similarly there exists an $\alpha_2 > \alpha_1$ with $\alpha_2 \in D$. In this fashion, we construct an increasing sequence

$$\alpha < \alpha_1 < \alpha_2 < \ldots < \alpha_n < \ldots \tag{8.1}$$

such that $\alpha_1, \alpha_3, \alpha_5, \ldots \in C$, $\alpha_2, \alpha_4, \alpha_6, \ldots \in D$. If we let β be the limit of the sequence (8.1), then $\beta < \kappa$, and $\beta \in C$ and $\beta \in D$. $\square$

The collection of all closed unbounded subsets of κ has the finite intersection property. The filter generated by the closed unbounded sets consists of all $X \subset \kappa$ that contain a closed unbounded subset. We call this filter the *closed unbounded filter* on κ.

The set of all limit ordinals $\alpha < \kappa$ is closed unbounded in κ. If A is an unbounded subset of κ, then the set of all limit points $\alpha < \kappa$ of A is closed unbounded.

A function $f : \kappa \to \kappa$ is *normal* if it is increasing and continuous ($f(\alpha) = \lim_{\xi\to\alpha} f(\xi)$ for every nonzero limit $\alpha < \kappa$). The range of a normal function is a closed unbounded set. Conversely, if C is closed unbounded, there is a unique normal function that enumerates C.

The closed unbounded filter on κ is κ-complete:

Theorem 8.3. *The intersection of fewer than κ closed unbounded subsets of κ is closed unbounded.*

Proof. We prove, by induction on $\gamma < \kappa$, that the intersection of a sequence $\langle C_\alpha : \alpha < \gamma\rangle$ of closed unbounded subsets of κ is closed unbounded. The induction step works at successor ordinals because of Lemma 8.2. If γ is a limit ordinal, we assume that the assertion is true for every $\alpha < \gamma$; then we can replace each C_α by $\bigcap_{\xi\le\alpha} C_\xi$ and obtain a decreasing sequence with the same intersection. Thus assume that

$$C_0 \supset C_1 \supset \ldots \supset C_\alpha \supset \ldots \qquad (\alpha < \gamma)$$

are closed unbounded, and let $C = \bigcap_{\alpha<\gamma} C_\alpha$.

It is easy to see that C is closed. To show that C is unbounded, let $\alpha < \kappa$. We construct a γ-sequence

$$\beta_0 < \beta_1 < \ldots \beta_\xi < \ldots \qquad (\xi < \gamma) \tag{8.2}$$

as follows: We let $\beta_0 \in C_0$ be such that $\beta_0 > \alpha$, and for each $\xi < \gamma$, let $\beta_\xi \in C_\xi$ be such that $\beta_\xi > \sup\{\beta_\nu : \nu < \xi\}$. Since κ is regular and $\gamma < \kappa$, such a sequence (8.2) exists and its limit β is less than κ. For each $\eta < \gamma$, β is the limit of a sequence $\langle\beta_\xi : \eta \le \xi < \gamma\rangle$ in C_η, and so $\beta \in C_\eta$. Hence $\beta \in C$. □

Let $\langle X_\alpha : \alpha < \kappa\rangle$ be a sequence of subsets of κ. The *diagonal intersection* of X_α, $\alpha < \kappa$, is defined as follows:

$$\mathop{\triangle}_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : \xi \in \bigcap_{\alpha<\xi} X_\alpha\}. \tag{8.3}$$

Note that $\triangle X_\alpha = \triangle Y_\alpha$ where $Y_\alpha = \{\xi \in X_\alpha : \xi > \alpha\}$. Note also that $\triangle X_\alpha = \bigcap_\alpha (X_\alpha \cup \{\xi : \xi \le \alpha\})$.

Lemma 8.4. *The diagonal intersection of a κ-sequence of closed unbounded sets is closed unbounded.*

Proof. Let $\langle C_\alpha : \alpha < \kappa\rangle$ be a sequence of closed unbounded sets. It is clear from the definition that if we replace each C_α by $\bigcap_{\xi\le\alpha} C_\xi$, the diagonal intersection is the same. In view of Theorem 8.3 we may thus assume that

$$C_0 \supset C_1 \supset \ldots \supset C_\alpha \supset \ldots \qquad (\alpha < \kappa).$$

Let $C = \triangle_{\alpha<\kappa} C_\alpha$. To show that C is closed, let α be a limit point of C. We want to show that $\alpha \in C$, or that $\alpha \in C_\xi$ for all $\xi < \alpha$. If $\xi < \alpha$, let $X = \{\nu \in C : \xi < \nu < \alpha\}$. Every $\nu \in X$ is in C_ξ, by (8.3). Hence $X \subset C_\xi$ and $\alpha = \sup X \in C_\xi$. Therefore $\alpha \in C$ and C is closed.

To show that C is unbounded, let $\alpha < \kappa$. We construct a sequence $\langle \beta_n : n < \omega \rangle$ as follows: Let $\beta_0 > \alpha$ be such that $\beta_0 \in C_0$, and for each n, let $\beta_{n+1} > \beta_n$ be such that $\beta_{n+1} \in C_{\beta_n}$. Let us show that $\beta = \lim_n \beta_n$ is in C: If $\xi < \beta$, let us show that $\beta \in C_\xi$. Since $\xi < \beta$, there is an n such that $\xi < \beta_n$. Each β_k, $k > n$, belongs to C_{β_n} and so $\beta \in C_{\beta_n}$. Therefore $\beta \in C_\xi$. Thus $\beta \in C$, and C is unbounded. □

Corollary 8.5. *The closed unbounded filter on κ is closed under diagonal intersections.* □

The dual of the closed unbounded filter is the ideal of nonstationary sets, the *nonstationary ideal* I_{NS}. I_{NS} is κ-complete and is closed under *diagonal unions*:

$$\sum_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : \xi \in \bigcup_{\alpha<\xi} X_\alpha\}.$$

The quotient algebra $B = P(\kappa)/I_{\mathrm{NS}}$ is a κ-complete Boolean algebra, where the Boolean operations $\sum_{\alpha<\gamma}$ and $\prod_{\alpha<\gamma}$ for $\gamma < \kappa$ are induced by $\bigcup_{\alpha<\gamma}$ and $\bigcap_{\alpha<\gamma}$. As a consequence of Lemma 8.4, B is κ^+-complete: If $\{X_\alpha : \alpha < \kappa\}$ is a collection of subsets of κ then the equivalence classes of $\triangle_{\alpha<\kappa} X_\alpha$ and $\sum_{\alpha<\kappa} X_\alpha$ are, respectively, the greatest lower bound and the least upper bound of the equivalence classes $[X_\alpha]$ in B. It also follows that if $\langle X_\alpha : \alpha < \kappa \rangle$ and $\langle Y_\alpha : \alpha < \kappa \rangle$ are two enumerations of the same collection, then $\triangle_{\alpha<\kappa} X_\alpha$ and $\triangle_{\alpha<\kappa} Y_\alpha$ differ only by a nonstationary set.

Definition 8.6. An ordinal function f on a set S is *regressive* if $f(\alpha) < \alpha$ for every $\alpha \in S$, $\alpha > 0$.

Theorem 8.7 (Fodor). *If f is a regressive function on a stationary set $S \subset \kappa$, then there is a stationary set $T \subset S$ and some $\gamma < \kappa$ such that $f(\alpha) = \gamma$ for all $\alpha \in T$.*

Proof. Let us assume that for each $\gamma < \kappa$, the set $\{\alpha \in S : f(\alpha) = \gamma\}$ is nonstationary, and choose a closed unbounded set C_γ such that $f(\alpha) \neq \gamma$ for each $\alpha \in S \cap C_\gamma$. Let $C = \triangle_{\gamma<\kappa} C_\gamma$. The set $S \cap C$ is stationary and if $\alpha \in S \cap C$, we have $f(\alpha) \neq \gamma$ for every $\gamma < \alpha$; in other words, $f(\alpha) \geq \alpha$. This is a contradiction. □

For a regular uncountable cardinal κ and a regular $\lambda < \kappa$, let

$$E^\kappa_\lambda = \{\alpha < \kappa : \operatorname{cf}\alpha = \lambda\}. \tag{8.4}$$

It is easy to see that each E^κ_λ is a stationary subset of κ.

The closed unbounded filter on κ is not an ultrafilter. This is because there is a stationary subset of κ whose complement is stationary. If $\kappa > \omega_1$, this is clear: The sets E^κ_ω and $E^\kappa_{\omega_1}$ are disjoint. If $\kappa = \omega_1$, the decomposition of ω_1 into disjoint stationary sets uses the Axiom of Choice.

The use of AC is necessary: It is consistent (relative to large cardinals) that the closed unbounded filter on ω_1 is an ultrafilter.

In Theorem 8.10 below we show that every stationary subset of κ is the union of κ disjoint stationary sets. In the following lemma we prove a weaker result that illustrates a typical use of Fodor's Theorem.

Lemma 8.8. *Every stationary subset of E^κ_ω is the union of κ disjoint stationary sets.*

Proof. Let $W \subset \{\alpha < \kappa : \operatorname{cf}\alpha = \omega\}$ be stationary. For every $\alpha \in W$, we choose an increasing sequence $\langle a^\alpha_n : n \in \boldsymbol{N}\rangle$ such that $\lim_n a^\alpha_n = \alpha$. First we show that there is an n such that for all $\eta < \kappa$, the set

$$\{\alpha \in W : a^\alpha_n \geq \eta\} \tag{8.5}$$

is stationary. Otherwise there is η_n and a closed unbounded set C_n such that $a^\alpha_n < \eta_n$ for all $\alpha \in C_n \cap W$, for every n. If we let η be the supremum of the η_n and C the intersection of the C_n, we have $a^\alpha_n < \eta$ for all n and all $\alpha \in C \cap W$. This is a contradiction. Now let n be such that (8.5) is stationary for every $\eta < \kappa$. Let f be the following function on W: $f(\alpha) = a^\alpha_n$. The function f is regressive; and so for every $\eta < \kappa$, we find by Fodor's Theorem a stationary subset S_η of (8.5) and $\gamma_\eta \geq \eta$ such that $f(\alpha) = \gamma_\eta$ on S_η. If $\gamma_\eta \neq \gamma_{\eta'}$, then $S_\eta \cap S_{\eta'} = \emptyset$, and since κ is regular, we have $|\{S_\eta : \eta < \kappa\}| = |\{\gamma_\eta : \eta < \kappa\}| = \kappa$. □

The proof easily generalizes to the case when $\lambda > \omega$: Every stationary subset of E^κ_λ the union of κ stationary sets. From that it follows that every stationary subset W of the set $\{\alpha < \kappa : \operatorname{cf}\alpha < \alpha\}$ admits such a decomposition: By Fodor's Theorem, there exists some $\lambda < \kappa$ such that $W \cap E^\kappa_\lambda$ is stationary. The remaining case in Theorem 8.10 is when the set $\{\alpha < \kappa : \alpha$ is a regular cardinal$\}$ is stationary and the following lemma plays the key role.

Lemma 8.9. *Let S be a stationary subset of κ and assume that every $\alpha \in S$ is a regular uncountable cardinal. Then the set $T = \{\alpha \in S : S \cap \alpha$ is not a stationary subset of $\alpha\}$ is stationary.*

Proof. We prove that T intersects every closed unbounded subset of κ. Let C be closed unbounded. The set C' of all limit points of C is also closed unbounded, and hence $S \cap C' \neq \emptyset$. Let α be the least element of $S \cap C'$. Since α is regular and a limit point of C, $C \cap \alpha$ is a closed unbounded subset of α, and so is $C' \cap \alpha$. As α is the least element of $S \cap C'$, $C' \cap \alpha$ is disjoint from $S \cap \alpha$ and so $S \cap \alpha$ is a nonstationary subset of α. Hence $\alpha \in T \cap C$. □

Theorem 8.10 (Solovay). *Let κ be a regular uncountable cardinal. Then every stationary subset of κ is the disjoint union of κ stationary subsets.*

Proof. We follow the proof of Lemma 8.8 as much as possible. Let A be a stationary subset of κ. By Lemma 8.8, by the subsequent discussion and by Lemma 8.9, we may assume that the set W of all $\alpha \in A$ such that α is a regular cardinal and $A \cap \alpha$ is not stationary, is stationary. There exists for each $\alpha \in W$ a continuous increasing sequence $\langle a_\xi^\alpha : \xi < \alpha \rangle$ such that $a_\xi^\alpha \notin W$, for all α and ξ, and $\alpha = \lim_{\xi \to \alpha} a_\xi^\alpha$.

First we show that there is ξ such that for all $\eta < \kappa$, the set

$$\{\alpha \in W : a_\xi^\alpha \geq \eta\} \tag{8.6}$$

is stationary. Otherwise, there is for each ξ some $\eta(\xi)$ and a closed unbounded set C_ξ such that $a_\xi^\alpha < \eta(\xi)$ for all $\alpha \in C_\xi \cap W$ if a_ξ^α is defined. Let C be the diagonal intersection of the C_ξ. Thus if $\alpha \in C \cap W$, then $a_\xi^\alpha < \eta(\xi)$ for all $\xi < \alpha$. Now let D be the closed unbounded set of all $\gamma \in C$ such that $\eta(\xi) < \gamma$ for all $\xi < \gamma$. Since W is stationary, $W \cap D$ is also stationary; let $\gamma < \alpha$ be two ordinals in $W \cap D$. Now if $\xi < \gamma$, then $a_\xi^\alpha < \eta(\xi) < \gamma$ and it follows that $a_\gamma^\alpha = \gamma$. This is a contradiction since $\gamma \in W$ and $a_\gamma^\alpha \notin W$.

Once we have found ξ such that (8.6) is stationary for all $\eta < \kappa$, we proceed as in Lemma 8.8. Let f be the function on W defined by $f(\alpha) = a_\xi^\alpha$. The function f is regressive; and so for every $\eta < \kappa$, we find by Fodor's Theorem a stationary subset S_η of (8.6) and $\gamma_\eta \geq \eta$ such that $f(\alpha) = \gamma_\eta$ on S_η. If $\gamma_\eta \neq \gamma_{\eta'}$, then $S_\eta \cap S_{\eta'} = \emptyset$; and since κ is regular, we have $|\{S_\eta : \eta < \kappa\}| = |\{\gamma_\eta : \eta < \kappa\}| = \kappa$. □

Mahlo Cardinals

Let κ be an inaccessible cardinal. The set of all cardinals below κ is a closed unbounded subset of κ, and so is the set of its limit points, the set of all limit cardinals. In fact, the set of all strong limit cardinals below κ is closed unbounded.

If κ is the least inaccessible cardinal, then all strong limit cardinals below κ are singular, and so the set of all singular strong limit cardinals below κ is closed unbounded. If κ is the αth inaccessible, where $\alpha < \kappa$, then still the set of all regular cardinals below κ is nonstationary.

An inaccessible cardinal κ is called a *Mahlo cardinal* if the set of all regular cardinals below κ is stationary.

(Then the set of all inaccessibles below κ is stationary, and κ is the κth inaccessible cardinal.)

Similarly, we define a *weakly Mahlo* cardinal as a cardinal κ that is weakly inaccessible and the set of all regular cardinals bellow κ is stationary (then the set of all weakly inaccessibles is stationary in κ).

Normal Filters

Let F be a filter on a cardinal κ; F is *normal* if it is closed under diagonal intersections:

(8.7) $$\text{if } X_\alpha \in F \text{ for all } \alpha < \kappa, \quad \text{then } \triangle_{\alpha<\kappa} X_\alpha \in F.$$

An ideal I on κ is *normal* if the dual filter is normal.

The closed unbounded filter is κ-complete and normal, and contains all complements of bounded sets. It is the smallest such filter on κ:

Lemma 8.11. *If κ is regular and uncountable and if F is a normal filter on κ that contains all final segments $\{\alpha : \alpha_0 < \alpha < \kappa\}$, then F contains all closed unbounded sets.*

Proof. First we note that the set C_0 of all limit ordinals is in F: C_0 is the diagonal intersection of the sets $X_\alpha = \{\xi : \alpha + 1 < \xi < \kappa\}$. Now let C be a closed unbounded set, and let $C = \{a_\alpha : \alpha < \kappa\}$ be its increasing enumeration. We let $X_\alpha = \{\xi : a_\alpha < \xi < \kappa\}$. Then $C \supset C_0 \cap \triangle_{\alpha<\kappa} X_\alpha$. □

Silver's Theorem

We shall now apply the techniques using ultrafilters and stationary sets to prove the following theorems.

Theorem 8.12 (Silver). *Let κ be a singular cardinal such that $\operatorname{cf}\kappa > \omega$. If $2^\alpha = \alpha^+$ for all cardinals $\alpha < \kappa$, then $2^\kappa = \kappa^+$.*

Theorem 8.13 (Silver). *If the Singular Cardinals Hypothesis holds for all singular cardinals of cofinality ω, then it holds for all singular cardinals.*

The proofs of both theorems use the following lemma:

Lemma 8.14. *Let κ be a singular cardinal, let $\operatorname{cf}\kappa > \omega$, and assume that $\lambda^{\operatorname{cf}\kappa} < \kappa$ for all $\lambda < \kappa$. If $\langle \kappa_\alpha : \alpha < \operatorname{cf}\kappa \rangle$ is a normal sequence of cardinals such that $\lim \kappa_\alpha = \kappa$, and if the set $\{\alpha < \operatorname{cf}\kappa : \kappa_\alpha^{\operatorname{cf}\kappa_\alpha} = \kappa_\alpha^+\}$ is stationary in $\operatorname{cf}\kappa$, then $\kappa^{\operatorname{cf}\kappa} = \kappa^+$.*

If GCH holds below κ then the assumptions of Lemma 8.14 are satisfied, and $2^\kappa = \kappa^{\operatorname{cf}\kappa}$. Thus Theorem 8.12 follows from Lemma 8.14.

Proof of Theorem 8.13. We prove by induction on the cofinality of κ that $2^{\operatorname{cf}\kappa} < \kappa$ implies $\kappa^{\operatorname{cf}\kappa} = \kappa^+$. The assumption of the theorem is that this holds for each κ of cofinality ω. Thus let κ be of uncountable cofinality and let $2^{\operatorname{cf}\kappa} < \kappa$. Using the induction hypothesis and the proof of Theorem 5.22(ii) one verifies, by induction on λ, that $\lambda^{\operatorname{cf}\kappa} < \kappa$ for all $\lambda < \kappa$.

Let $\langle \kappa_\alpha : \alpha < \operatorname{cf}\kappa \rangle$ be any normal sequence of cardinals such that $\lim \kappa_\alpha = \kappa$. The set $S = \{\alpha < \operatorname{cf}\kappa : \operatorname{cf}\kappa_\alpha = \omega \text{ and } 2^{\aleph_0} < \kappa_\alpha\}$ is clearly stationary in $\operatorname{cf}\kappa$, and for every $\alpha \in S$, $\kappa_\alpha^{\operatorname{cf}\kappa_\alpha} = \kappa_\alpha^+$ by the assumption. Hence $\kappa^{\operatorname{cf}\kappa} = \kappa^+$. □

We now proceed toward a proof of Lemma 8.14. To simplify the notation, we shall consider the special case when

$$\kappa = \aleph_{\omega_1}.$$

The general case is proved in a similar way.

Let f and g be two functions on ω_1. We say that f and g are *almost disjoint* if there is $\alpha_0 < \omega_1$ such that $f(\alpha) \neq g(\alpha)$ for all $\alpha \geq \alpha_0$. A family F of functions on ω_1 is an *almost disjoint family* if any two distinct $f, g \in F$ are almost disjoint.

Lemma 8.14 follows from

Lemma 8.15. *Assume that $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ for all $\alpha < \omega_1$. Let F be an almost disjoint family of functions*

$$F \subset \prod_{\alpha<\omega_1} A_\alpha,$$

such that the set

$$\{\alpha < \omega_1 : |A_\alpha| \leq \aleph_{\alpha+1}\} \tag{8.8}$$

is stationary. Then $|F| \leq \aleph_{\omega_1+1}$.

[In the general case, we consider almost disjoint functions on $\operatorname{cf}\kappa$.]

Proof of Lemma 8.14 *from Lemma* 8.15. We assume that $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ and that $\aleph_\alpha^{\operatorname{cf}\aleph_\alpha} = \aleph_{\alpha+1}$ for a stationary set of α's; we want to show that $\aleph_{\omega_1}^{\aleph_1} = \aleph_{\omega_1+1}$. For every $h : \omega_1 \to \aleph_{\omega_1}$, we let $f_h = \langle h_\alpha : \alpha < \omega_1 \rangle$, where $\operatorname{dom} h_\alpha = \omega_1$ and

$$h_\alpha(\xi) = \begin{cases} h(\xi) & \text{if } h(\xi) < \aleph_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

and let $F = \{f_h : h \in {\aleph_{\omega_1}}^{\omega_1}\}$. If $h \neq g$, then f_h and f_g are almost disjoint. Moreover,

$$F \subset \prod_{\alpha<\omega_1} {\aleph_\alpha}^{\omega_1}.$$

Since for a stationary set of α's, $\aleph_\alpha^{\aleph_1} = \aleph_{\alpha+1}$ (namely for all α such that $\aleph_\alpha > 2^{\aleph_1}$ and $\aleph_\alpha^{\aleph_0} = \aleph_{\alpha+1}$), we have $|F| \le \aleph_{\omega_1+1}$, and so $|{\aleph_{\omega_1}}^{\omega_1}| = \aleph_{\omega_1+1}$.

[In the general case of Lemma 8.14 we have to show that

$$\{\alpha < \operatorname{cf}\kappa : \kappa_\alpha^{\operatorname{cf}\kappa_\alpha} = \kappa_\alpha^+\}$$

is stationary. Note that the set

$$C = \{\alpha : \alpha \text{ is a limit ordinal and } (\forall\lambda < \kappa_\alpha)\, \lambda^{\operatorname{cf}\kappa} < \kappa_\alpha\}$$

is closed unbounded in $\operatorname{cf}\kappa$; if $\alpha \in C$, then $\operatorname{cf}\kappa_\alpha < \operatorname{cf}\kappa$ and we have $\kappa_\alpha^{\operatorname{cf}\kappa} = \kappa_\alpha^{\operatorname{cf}\alpha}$.] □

The first step in the proof of Lemma 8.15 is

Lemma 8.16. *Assume that $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ for all $\alpha < \omega_1$. Let F be an almost disjoint family of functions*

$$F \subset \prod_{\alpha<\omega_1} A_\alpha$$

such that the set

$$\{\alpha < \omega_1 : |A_\alpha| \le \aleph_\alpha\} \tag{8.9}$$

is stationary. Then $|F| \le \aleph_{\omega_1}$.

(The assumption (8.8) is replaced by (8.9) and the bound for $|F|$ is $\aleph_{\omega_1}$ rather than $\aleph_{\omega_1+1}$.)

Proof. We may as well assume that each A_α is a set of ordinals and that $A_\alpha \subset \omega_\alpha$ for all α in some stationary subset of $\aleph_1$. Let

$$S_0 = \{\alpha < \omega_1 : \alpha \text{ is a limit ordinal and } A_\alpha \subset \omega_\alpha\}.$$

Thus if $f \in F$, then $f(\alpha) < \omega_\alpha$ for all $\alpha \in S_0$. Given $f \in F$, we can find for each $\alpha > 0$ in S_0 some $\beta < \alpha$ such that $f(\alpha) < \omega_\beta$; call this $\beta = g(\alpha)$. The function g is regressive on S, and by Fodor's Theorem there is a stationary $S \subset S_0$ such that g is constant on S. In other words, the function f is bounded on S, by some $\omega_\gamma < \omega_{\omega_1}$.

We assign to each f a pair $(S, f{\restriction}S)$ where $S \subset S_0$ is a stationary set and $f{\restriction}S$ is a bounded function. For any S, if $f{\restriction}S = g{\restriction}S$, then $f = g$ since any two distinct functions in F are almost disjoint. Thus the correspondence

$$f \mapsto (S, f{\restriction}S)$$

is one-to-one.

For a given S, the number of bounded functions on S is at most

$$\sum_{\gamma<\omega_1} \aleph_\gamma^{|S|} = \sup\nolimits_{\gamma<\omega_1} \aleph_\gamma^{\aleph_1} = \aleph_{\omega_1}.$$

Since $|P(\omega_1)| = 2^{\aleph_1} < \aleph_{\omega_1}$, the number of pairs $(S, f{\restriction}S)$ is at most $\aleph_{\omega_1}$. Hence $|F| \le \aleph_{\omega_1}$. □

Proof of Lemma 8.15. Let U be an ultrafilter on ω_1 that extends the closed unbounded filter. Every $S \in U$ is stationary.

We may assume that each A_α is a subset of $\omega_{\alpha+1}$. For every $f, g \in F$, let

$$f < g \text{ if and only if } \{\alpha < \omega_1 : f(\alpha) < g(\alpha)\} \in U. \tag{8.10}$$

Since U is a filter, the relation $f < g$ is transitive. Since U is an ultrafilter, and $\{\alpha : f(\alpha) = g(\alpha)\} \notin U$ for distinict $f, g \in F$, the relation $f < g$ is a linear ordering of F. For every $f \in F$, let $F_f = \{g \in F : \text{for some stationary set } T, g(\alpha) < f(\alpha) \text{ for all } \alpha \in T\}$. By Lemma 8.16, $|F_f| \le \aleph_{\omega_1}$. If $g < f$, then $g \in F_f$, and so $|\{g \in F : g < f\}| \le \aleph_{\omega_1}$. It follows that $|F| \le \aleph_{\omega_1+1}$. □

A Hierarchy of Stationary Sets

If α is a limit ordinal of uncountable cofinality, it still makes sense to talk about closed unbounded and stationary subsets of α. Since $\operatorname{cf} \alpha > \omega$, Lemma 8.2 holds, and the closed unbounded sets generate a filter on α. The closed unbounded filter is $\operatorname{cf} \alpha$-complete. A set $S \subset \alpha$ is stationary if and only if for some (or for any) normal function $f : \operatorname{cf} \alpha \to \alpha$, $f_{-1}(S)$ is a stationary subset of $\operatorname{cf} \alpha$.

Let κ be a regular uncountable cardinal, and let us consider the following operation (the *Mahlo operation*) on stationary sets:

Definition 8.17. If $S \subset \kappa$ is stationary, the *trace* of S is the set

$$\operatorname{Tr}(S) = \{\alpha < \kappa : \operatorname{cf} \alpha > \omega \text{ and } S \cap \alpha \text{ is stationary}\}.$$

The Mahlo operation is invariant under equivalence mod I_{NS} and can thus be considered as an operation on the Boolean algebra $P(\kappa)/I_{\text{NS}}$ (see Exercise 8.11).

In the context of closed unbounded and stationary sets we use the phrase *for almost all* $\alpha \in S$ to mean that the set of all contrary $\alpha \in S$ is nonstationary.

Definition 8.18. Let S and T be stationary subsets of κ.

$$S < T \quad \text{if and only if} \quad S \cap \alpha \text{ is stationary for almost all } \alpha \in T.$$

(It is implicit in the definition that almost all $\alpha \in T$ have uncountable cofinality.)

As an example, if $\lambda < \mu$ are regular, then $E^\kappa_\lambda < E^\kappa_\mu$. The following properties are easily verified:

Lemma 8.19.

(i) $A < \operatorname{Tr}(A)$,

(ii) *if* $A < B$ *and* $B < C$ *then* $A < C$,
(iii) *if* $A < B$, $A \simeq A' \bmod I_{\mathrm{NS}}$ *and* $B \simeq B' \bmod I_{\mathrm{NS}}$ *then* $A' < B'$. □

Thus $<$ is a transitive relation on $P(\kappa)/I_{\mathrm{NS}}$. The next theorem shows that it is a well-founded partial ordering:

Theorem 8.20 (Jech). *The relation* $<$ *is well-founded.*

Proof. Assume to the contrary that there exist stationary sets such that $A_1 > A_2 > A_3 \ldots$. Therefore there exist closed unbounded sets C_n such that $A_n \cap C_n \subset \mathrm{Tr}(A_{n+1})$ for $n = 1, 2, 3, \ldots$. For each n, let

$$B_n = A_n \cap C_n \cap \mathrm{Lim}(C_{n+1}) \cap \mathrm{Lim}(\mathrm{Lim}(C_{n+2})) \cap \ldots$$

where $\mathrm{Lim}(C)$ is the set of all limit points of C.

Each B_n is stationary, and for every n, $B_n \subset \mathrm{Tr}(B_{n+1})$. Let $\alpha_n = \min B_n$. Since $B_{n+1} \cap \alpha_n$ is stationary, we have $\alpha_{n+1} < \alpha_n$ and therefore, a decreasing sequence $\alpha_1 > \alpha_2 > \ldots$. A contradiction. □

The rank of a stationary set $A \subset \kappa$ in the well-founded relation $<$ is called the *order* of the set A, and the height of $<$ is the *order* of the cardinal κ:

$$o(A) = \sup\{o(X) + 1 : X < A\},$$
$$o(\kappa) = \sup\{o(A) + 1 : A \subset \kappa \text{ is stationary}\}.$$

We also define $o(\aleph_0) = 0$, and $o(\alpha) = o(\mathrm{cf}\,\alpha)$ for every limit ordinal α. Note that $o(E^\kappa_\omega) = 0$, $o(E^\kappa_{\omega_1}) = 1$, $o(\aleph_1) = 1$, $o(\aleph_2) = 2$, etc. See Exercises 8.13 and 8.14.

The Closed Unbounded Filter on $P_\kappa(\lambda)$

We shall now consider a generalization of closed unbounded and stationary sets, to the space $P_\kappa(\lambda)$. This generalization replaces $(\kappa, <)$ with the structure $(P_\kappa(\lambda), \subset)$.

Let κ be a regular uncountable cardinal and let A be a set of cardinality at least κ.

Definition 8.21. A set $X \subset P_\kappa(A)$ is *unbounded* if for every $x \in P_\kappa(A)$ there exists a $y \supset x$ such that $y \in X$.

A set $X \subset P_\kappa(A)$ is *closed* if for any chain $x_0 \subset x_1 \subset \ldots \subset x_\xi \subset \ldots$, $\xi < \alpha$, of sets in X, with $\alpha < \kappa$, the union $\bigcup_{\xi<\alpha} x_\xi$ is in X.

A set $C \subset P_\kappa(A)$ is *closed unbounded* if it is closed and unbounded.

A set $S \subset P_\kappa(A)$ is *stationary* if $S \cap C \neq \emptyset$ for every closed unbounded $C \subset P_\kappa(A)$.

The *closed unbounded filter* on $P_\kappa(A)$ is the filter generated by the closed unbounded sets.

When $|A| = |B|$ then $P_\kappa(A)$ and $P_\kappa(B)$ are isomorphic, with closed unbounded and stationary sets corresponding to closed unbounded and stationary sets, and so it often suffices to consider such sets in $P_\kappa(\lambda)$ where λ is a cardinal $\geq \kappa$.

When $|A| = \kappa$, then the set $\kappa \subset P_\kappa(\kappa)$ is closed unbounded, and the closed unbounded filter on κ is the restriction to κ of the closed unbounded filter on $P_\kappa(\kappa)$.

Theorem 8.22 (Jech). *The closed unbounded filter on $P_\kappa(A)$ is κ-complete.*

Proof. This is a generalization of Theorem 8.3. First we proceed as in Lemma 8.2 and show that if C and D are closed unbounded then $C \cap D$ is closed unbounded. Both proofs have straightforward generalizations from $(\kappa, <)$ to $(P_\kappa(A), \subset)$. □

Fodor's Theorem also generalizes to $P_\kappa(A)$; with regressive functions replaced by choice functions. The *diagonal intersection* of subsets of $P_\kappa(A)$ is defined as follows

$$\mathop{\triangle}_{a\in A} X_a = \{x \in P_\kappa(A) : x \in \bigcap_{a\in x} X_a\}.$$

Lemma 8.23. *If $\{C_a : a \in A\}$ is a collection of closed unbounded subsets of $P_\kappa(A)$ then its diagonal intersection is closed unbounded.*

Proof. Let $C = \triangle_{a\in A} C_a$. First we show that C is closed. Let $x_0 \subset x_1 \subset \ldots \subset x_\xi \subset \ldots$, $\xi < \alpha$, be a chain in C, with $\alpha < \kappa$, and let x be its union. To show that $x \in C$, let $a \in x$ and let us show that $x \in C_a$. There is some $\eta < \alpha$ such that $a \in x_\xi$ for all $\xi \geq \eta$; hence $x_\xi \in C_a$ for all $\xi \geq \eta$, and so $x \in C_a$.

Now we show that C is unbounded. Let $x_0 \in P_\kappa(A)$, we shall find an $x \in C$ such that $x \supset x_0$. By induction, we find $x_0 \subset x_1 \subset \ldots \subset x_n \subset \ldots$, $n \in \boldsymbol{N}$, such that $x_{n+1} \in \bigcap_{a\in x_n} C_a$; this is possible because each $\bigcap_{a\in x_n} C_a$ is closed unbounded. Then we let $x = \bigcup_{n=0}^\infty x_n$ and show that $x \in C_a$ for all $a \in x$. But if $a \in x$ then $a \in x_k$ for some k, and then $x_n \in C_a$ for all $n \geq k+1$. Hence $x \in C_a$. □

Theorem 8.24 (Jech). *If f is a function on a stationary set $S \subset P_\kappa(\lambda)$ and if $f(x) \in x$ for every nonempty $x \in S$, then there exist a stationary set $T \subset S$ and some $a \in A$ such that $f(x) = a$ for all $a \in T$.*

Proof. The proof uses Lemma 8.23 and generalizes the proof of Theorem 8.7. □

Let us call a set $D \subset P_\kappa(A)$ *directed* if for all x and y in D there is a $z \in D$ such that $x \cup y \subset z$.

Lemma 8.25. *If C is a closed subset of $P_\kappa(A)$ then for every directed set $D \subset C$ with $|D| < \kappa$, $\bigcup D \in C$.*

Proof. By induction on $|D|$. Let $|D| = \gamma$, $D = \{x_\alpha : \alpha < \gamma\}$, and assume the lemma holds for every directed set of cardinality $< \gamma$. By induction on $\alpha < \gamma$, let D_α be a smallest directed subset of D such that $x_\alpha \in D_\alpha$ and $D_\alpha \supset \bigcup_{\beta<\alpha} D_\beta$. Letting $y_\alpha = \bigcup D_\alpha$, we have $y_\alpha \in C$ for all $\alpha < \gamma$, and $y_\beta \subset y_\alpha$ if $\beta < \alpha$. It follows that $\bigcup D = \bigcup_{\alpha<\gamma} y_\alpha \in C$. □

Consider a function $f : [A]^{<\omega} \to P_\kappa(A)$; a set $x \in P_\kappa(A)$ is a *closure point* of f if $f(e) \subset x$ whenever $e \subset x$. The set C_f of all closure points $x \in P_\kappa(A)$ is a closed unbounded set. Moreover, the sets C_f generate the closed unbounded filter:

Lemma 8.26. *For every closed unbounded set C in $P_\kappa(A)$ there exists a function $f : [A]^{<\omega} \to P_\kappa(A)$ such that $C_f \subset C$.*

Proof. By induction on $|e|$ we find for each $e \in [A]^{<\omega}$ an infinite set $f(e) \in C$ such that $e \subset f(e)$ and that $f(e_1) \subset f(e_2)$ whenever $e_1 \subset e_2$. We will show that $C_f \subset C$. Let x be a closure point of f. As $x = \bigcup\{f(e) : e \in [x]^{<\omega}\}$ is the union of *a* directed subset of C (of cardinality $< \kappa$), by Lemma 8.25 we have $x \in C$. □

Let $A \subset B$ (and $|A| \geq \kappa$). For $X \in P_\kappa(B)$, the *projection* of X to A is the set

$$X \restriction A = \{x \cap A : x \in X\}.$$

For $Y \in P_\kappa(A)$, the *lifting* of Y to B is the set

$$Y^B = \{x \in P_\kappa(B) : x \cap A \in Y\}.$$

Theorem 8.27 (Menas). *Let $A \subset B$.*

(i) *If S is stationary in $P_\kappa(B)$, then $S \restriction A$ is stationary in $P_\kappa(A)$.*
(ii) *If S is stationary in $P_\kappa(A)$, then S^B is stationary in $P_\kappa(B)$.*

Proof. (i) holds because if C is a closed unbounded set in $P_\kappa(A)$, then C^B is closed unbounded in $P_\kappa(B)$. For (ii), it suffices to prove that if C is closed unbounded in $P_\kappa(B)$, then $C \restriction A$ contains a closed unbounded set.

If $C \subset P_\kappa(B)$ is closed unbounded, then by Lemma 8.26, $C \supset C_f$ for some $f : [B]^{<\omega} \to P_\kappa(B)$. Let $g : [A]^{<\omega} \to P_\kappa(A)$ be the following function: For $e \in [A]^{<\omega}$, let x be the smallest closure point of f such that $x \supset e$, and let $g(e) = x \cap A$. Then $C_f \restriction A = C_g$ (where C_f is defined in $P_\kappa(B)$ and C_g in $P_\kappa(A)$), and we have $C_g \subset C \restriction A$. □

When $\kappa = \omega_1$, Lemma 8.26 can be improved to give the following basis theorem for $[A]^\omega = \{x \subset A : |x| = \aleph_0\}$. An *operation* on A is a function $F : [A]^{<\omega} \to A$. A set x is *closed under* F if $f(e) \in x$ for all $e \in [x]^{<\omega}$.

Theorem 8.28 (Kueker). *For every closed unbounded set $C \subset [A]^\omega$ there is an operation F on A such that $C \supset C_F = \{x \in [A]^\omega : x$ is closed under $F\}$.*

Proof. We may assume that $A = \lambda$ is an infinite cardinal, and let C be a closed unbounded subset of $[\lambda]^\omega$. As in the proof of Lemma 8.26 there exists a function $f : [\lambda]^{<\omega} \to C$ such that $e \subset f(e)$ and $f(e_1) \subset f(e_2)$ if $e_1 \subset e_2$. As each $f(e)$ is countable, there exist functions f_k, $k \in \boldsymbol{N}$, such that $f(e) = \{f_k(e) : k \in \boldsymbol{N}\}$ for all e. Let $n \mapsto (k_n, m_n)$ be a pairing function.

Now we define an operation F on λ as follows: Let $F(\{\alpha\}) = \alpha + 1$, and if $\alpha_1 < \ldots < \alpha_n$, let $F(\{\alpha_1, \ldots, \alpha_n\}) = f_{k_n}(\{\alpha_1, \ldots, \alpha_{m_n}\})$. It is enough to show that if $x \in [\lambda]^\omega$ is closed under F then x is a closure point of f, and so $C_F \subset C_f \subset C$.

Let x be closed under F, let $k \in \boldsymbol{N}$ and let $e \in [x]^{<\omega}$; we want to show that $f_k(e) \in x$. If $e = \{\alpha_1, \ldots, \alpha_m\}$ with $\alpha_1 < \ldots < \alpha_m$, let $n \geq m$ be such that $k = k_n$ and $m = m_n$. As x does not have a greatest element (because $F(\{\alpha\}) = \alpha + 1$), there are $\alpha_{m+1}, \ldots, \alpha_n \in x$ such that $f_k(\{\alpha_1, \ldots, \alpha_m\}) = F(\{\alpha_1, \ldots, \alpha_n\}) \in x$. □

Theorem 8.28 does not generalize outright to $P_\kappa(A)$ for $\kappa > \omega_1$ (see Exercise 8.18); we shall return to the subject in Part III.

Exercises

8.1. The set of all fixed points (i.e., $f(\alpha) = \alpha$) of a normal function is closed unbounded.

8.2. If $f : \kappa \to \kappa$, then the set of all $\alpha < \kappa$ such that $f(\xi) < \alpha$ for all $\xi < \alpha$ is closed unbounded.

8.3. If S is stationary and C is closed unbounded, then $S \cap C$ is stationary.

8.4. If $X \subset \kappa$ is nonstationary, then there exists a regressive function f on X such that $\{\alpha : f(\alpha) \leq \gamma\}$ is bounded, for every $\gamma < \kappa$.

[Let $C \cap X = \emptyset$, and let $f(\alpha) = \sup(C \cap \alpha)$.]

8.5. For every stationary $S \subset \omega_1$ and every $\alpha < \omega_1$ there is a closed set of ordinals A of length α such that $A \subset S$.

[By induction on α: $\forall\gamma\ \exists$closed $A \subset S$ of length α such that $\gamma < \min A$. The nontrivial step: If true for a limit α, find a closed $A \subset S$ of length α such that $\sup A \in S$. Let A_ξ, $\xi < \omega_1$, be closed subsets of S, of length α, such that $\lambda_\xi = \sup \bigcup_{\nu<\xi} A_\nu < \min A_\xi$. There is ξ such that $\lambda_\xi \in S$. Let $\xi = \lim_n \xi_n$. Pick initial segments $B_{\xi_n} \subset A_{\xi_n}$ of length $\alpha_n + 1$ where $\lim_n \alpha_n = \alpha$. Let $A = \bigcup_{n=0}^\infty B_{\xi_n}$.]

Exercise 8.5 does not generalize to closed sets of uncountable length. It is not provable in ZFC that given $X \subset \omega_2$, either X or $\omega_2 - X$ contains a closed set of length ω_1. On the other hand, this statement is consistent, relative to large cardinals.

8.6. Let κ be the least inaccessible cardinal such that κ is the κth inaccessible cardinal. Then κ is not Mahlo.

[Use $f(\lambda) = \alpha$ where λ is the αth inaccessible.]

8.7. If κ is a limit (weakly inaccessible, weakly Mahlo) cardinal and the set of all strong limit cardinals below κ is unbounded in κ, then κ is a strong limit (inaccessible, Mahlo) cardinal.

8.8. A κ-complete ideal I on κ is normal if and only if for every $S_0 \notin I$ and any regressive f on S_0 there is $S \subset S_0$, $S \notin I$, such that f is constant on S.

[One direction is like Fodor's Theorem. For the other direction, let $X_\alpha \in F$ for each $\alpha < \kappa$. If $\triangle X_\alpha \notin F$, let $S_0 = \kappa - \triangle X_\alpha$ and let $f(\alpha) =$ some $\xi < \alpha$ such that $\alpha \notin X_\xi$. If $f(\alpha) = \gamma$ for all $\alpha \in S$, then $X_\gamma \cap S = \emptyset$, a contradiction.]

8.9. There is no normal nonprincipal filter on ω.

[Use the regressive function $f(n+1) = n$.]

8.10. If κ is singular, then there is no normal ideal on κ that contains all bounded subsets of κ.

8.11. (i) If $S \subset T$ then $\mathrm{Tr}(S) \subset \mathrm{Tr}(T)$,

(ii) $\mathrm{Tr}(S \cup T) = \mathrm{Tr}(S) \cup \mathrm{Tr}(T)$,

(iii) $\mathrm{Tr}(\mathrm{Tr}(S)) \subset \mathrm{Tr}(S)$,

(iv) if $S \simeq T \bmod I_{\mathrm{NS}}$ then $\mathrm{Tr}(S) \simeq \mathrm{Tr}(T) \bmod I_{\mathrm{NS}}$.

8.12. Show that $\mathrm{Tr}(E^\kappa_\lambda) = \{\alpha < \kappa : \mathrm{cf}\,\alpha \geq \lambda^+\}$.

8.13. If $\lambda < \kappa$ is the αth regular cardinal cardinal, then $o(E^\kappa_\lambda) = \alpha$.

8.14. $o(\kappa) \geq \kappa$ if and only if κ is weakly inaccessible; $o(\kappa) \geq \kappa + 1$ if and only if κ is weakly Mahlo.

8.15. For each $a \in P_\kappa(A)$, the set $\{x \in P_\kappa(A) : x \supset a\}$ is closed unbounded.

A κ-complete filter F on $P_\kappa(A)$ is *normal* if for every $a \in A$, $\{x \in P_\kappa(A) : a \in x\} \in F$, and if F is closed under diagonal intersections. A set $X \subset P_\kappa(A)$ is *F-positive* if its complement is not in F.

8.16. Let F be a normal κ-complete filter on $P_\kappa(A)$. If g is a function on an F-positive set such that $g(x) \in [x]^{<\omega}$ for all x, then g is constant on an F-positive set.

8.17. If F is a normal κ-complete filter on $P_\kappa(A)$ then F contains all closed unbounded sets.

[Use Lemma 8.26 and Exercise 8.16.]

8.18. If $\kappa > \omega_1$ then the set $\{x \in P_\kappa(A) : |x| \geq \aleph_1\}$ is closed unbounded.

Contrast this with the fact that for every $F : [A]^{<\omega} \to A$ there exists a countable x closed under A.

8.19. The set $\{x \in P_\kappa(\lambda) : x \cap \kappa \in \kappa\}$ is closed unbounded.

Historical Notes

The definition of stationary set is due to Bloch [1953], and the fundamental theorem (Theorem 8.7) was proved by Fodor [1956]. (A precursor of Fodor's Theorem appeared in Aleksandrov-Urysohn [1929].) The concept of stationary sets is implicit in Mahlo [1911].

Theorem 8.10 was proved by Solovay [1971] using the technique of saturated ideals.

Mahlo cardinals are named after P. Mahlo, who in 1911–1913 investigated what is now called weakly Mahlo cardinals. Theorems 8.12 and 8.13 are due to Silver [1975]. Silver's proof uses generic ultrapowers; the elementary proof given here is as in Baumgartner-Prikry [1976, 1977]. Lemma 8.16: Erdös, Hajnal, and Milner [1968].

Definition 8.18 and Theorem 8.20 are due to Jech [1984]. The generalization of closed unbounded and stationary sets (Definition 8.21 and Theorems 8.22 and 8.24) was given by Jech [1971b] and [1972/73]; Kueker [1972, 1977] also formulated these concepts for $\kappa = \omega_1$ and proved Theorem 8.28. Theorem 8.27 is due to Menas [1974/75].

Exercise 8.5: Friedman [1974].

Exercise 8.17: Carr [1982].