

15. Applications of Forcing

In this chapter we present some important applications of the method of forcing. These applications establish several major consistency results and illustrate the techniques involved in use of forcing. Throughout we use V to denote the ground model, and $V[G]$ for the generic extension.

Cohen Reals

In (14.23) we described a notion of forcing that adjoins \aleph_2 real numbers to the ground model. In general, let κ be an infinite cardinal. The following notion of forcing adjoins κ real numbers, called *Cohen reals*.

Let P be the set of all functions p such that

- (15.1) (i) $\text{dom}(p)$ is a finite subset of $\kappa \times \omega$,
 (ii) $\text{ran}(p) \subset \{0, 1\}$,

and let p be stronger than q if and only if $p \supset q$.

Let G be a generic set of conditions and let $f = \bigcup G$. By a genericity argument, f is a function from $\kappa \times \omega$ into $\{0, 1\}$. For each $\alpha < \kappa$, we let f_α be the function on ω defined by $f_\alpha(n) = f(\alpha, n)$ and let $a_\alpha = \{n \in \omega : f_\alpha(n) = 1\}$. Each a_α is a real (a subset of ω), $a_\alpha \notin V$ and if $\alpha \neq \beta$, then $a_\alpha \neq a_\beta$. This is proved as in Theorem 14.32.

Also as in Theorem 14.32 one shows that P satisfies the countable chain condition. It follows that cardinals and cofinalities are *preserved* in the generic extension.

Since P adds κ distinct Cohen reals, the size of the continuum in $V[G]$ is at least κ . In fact, it is at least $(\kappa^{\aleph_0})^V$:

$$(2^{\aleph_0})^{V[G]} = ((2^{\aleph_0})^{\aleph_0})^{V[G]} \geq (\kappa^{\aleph_0})^{V[G]} \geq (\kappa^{\aleph_0})^V.$$

It turns out that there are precisely $(\kappa^{\aleph_0})^V$ reals in $V[G]$. The following is a general estimate of the number of new sets in a generic extension:

Lemma 15.1. *Let λ be a cardinal in V . If G is a V -generic ultrafilter on B , then*

$$(2^\lambda)^{V[G]} \leq (|B|^\lambda)^V.$$

Proof. Every subset $A \subset \lambda$ in $V[G]$ has a name $\dot{A} \in V^B$; every such \dot{A} determines a function $\alpha \mapsto \|\dot{\alpha} \in \dot{A}\|$ from λ into B . Different subsets correspond to different functions, and thus the number of all subsets of λ in $V[G]$ is not greater than the number of all functions from λ into B in V . \square

When P is the forcing (15.1) that adds κ Cohen reals, P satisfies c.c.c. and so every element of $B = B(P)$ is the Boolean sum of a countable antichain in P ; hence $|B| \leq |P|^{\aleph_0} = \kappa^{\aleph_0}$. By Exercise 7.32, $|B| = |B|^{\aleph_0}$ and it follows that $|B| = \kappa^{\aleph_0}$, and consequently, $(2^{\aleph_0})^{V[G]} = (\kappa^{\aleph_0})^V$.

If we start with a ground model that satisfies GCH, and if κ is (in V) a cardinal of uncountable cofinality, then $\kappa^{\aleph_0} = \kappa$ in V , and we get a model $V[G]$ in which $2^{\aleph_0} = \kappa$.

Adding Subsets of Regular Cardinals

The forcing that adds a Cohen real generalizes easily from ω to any regular cardinal κ . Let κ be, in V , a regular cardinal and assume that $2^{<\kappa} = \kappa$.

Let P be the set of all functions p such that

- (15.2) (i) $\text{dom}(p) \subset \kappa$ and $|\text{dom}(p)| < \kappa$;
(ii) $\text{ran}(p) \subset \{0, 1\}$.

A condition p be stronger than q if and only if $p \supset q$.

Let G be a set of conditions generic over V and let $f = \bigcup G$. As before, f is a function from κ into $\{0, 1\}$, and $X = \{\alpha < \kappa : f(\alpha) = 1\}$ is a subset of κ and $X \notin V$.

In order to add more new subsets of κ , we use a generalization of (15.1): Let κ be as above, and let λ be a cardinal greater than κ such that $\lambda^\kappa = \lambda$. Let P be the set of all functions p such that:

- (15.3) (i) $\text{dom}(p) \subset \lambda \times \kappa$ and $|\text{dom}(p)| < \kappa$,
(ii) $\text{ran}(p) \subset \{0, 1\}$,

and let p be stronger than q if and only if $p \supset q$.

Let G be a generic set of conditions and let $f = \bigcup G$. For each $\alpha < \lambda$, we let

$$a_\alpha = \{\xi < \kappa : f(\alpha, \xi) = 1\}.$$

Each a_α is a subset of κ , each $a_\alpha \notin V$ and $a_\alpha \neq a_\beta$ whenever $\alpha \neq \beta$.

We claim that in the generic extension, all cardinals are preserved, and $2^\kappa = \lambda$. But to show this, we need additional results in the theory of forcing, proved in the next two sections.

The κ -Chain Condition

Definition 15.2. A forcing notion P satisfies the κ -chain condition (κ -c.c.) if every antichain in P has cardinality less than κ .

The \aleph_1 -chain condition is the c.c.c. Note that P satisfies the κ -c.c. if and only if $B(P)$ satisfies the κ -c.c.

Theorem 14.34 generalizes as follows:

Theorem 15.3. *If κ is a regular cardinal and if P satisfies the κ -chain condition then κ remains a regular cardinal in the generic extension by P .*

Proof. The proof is exactly as the proof of Theorem 14.34. The only difference is that the set A_α is not necessarily countable but has cardinality less than κ . □

Consequently, all regular cardinals $\kappa \geq \text{sat}(B(P))$, and in particular all regular $\kappa \geq |P|^+$ are preserved in $V[G]$.

The following lemma generalizes Lemma 14.35, and implies that the forcing notion (15.3) satisfies the κ^+ -chain condition. We remark that Lemma 15.4 is related to (a generalization of) Theorem 9.18 on Δ -systems.

Lemma 15.4. *Let κ be a regular cardinal such that $2^{<\kappa} = \kappa$. Let S be an arbitrary set and let $|C| \leq \kappa$. Let P be the set of all functions p whose domains are subsets of S of size $< \kappa$, with values in C . Let $p < q$ if and only if $q \supset p$. Then P satisfies the κ^+ -chain condition.*

Proof. Let $W \subset P$ be an antichain. We construct sequences $A_0 \subset A_1 \subset \dots \subset A_\alpha \subset \dots$ ($\alpha < \kappa$) of subsets of S , and $W_0 \subset W_1 \subset \dots \subset W_\alpha \subset \dots$ ($\alpha < \kappa$) of subsets of W . If α is a limit ordinal, we let $W_\alpha = \bigcup_{\beta < \alpha} W_\beta$ and $A_\alpha = \bigcup_{\beta < \alpha} A_\beta$. Given A_α and W_α , we choose for each $p \in P$ with $\text{dom}(p) \subset A_\alpha$ some $q \in W$ (if there is one) such that $p = q \upharpoonright A_\alpha$. Then we let $W_{\alpha+1} = W_\alpha \cup \{\text{the chosen } q\text{'s}\}$ and $A_{\alpha+1} = \bigcup \{\text{dom}(q) : q \in W_{\alpha+1}\}$; finally, $A = \bigcup_{\alpha < \kappa} A_\alpha$.

Next we show that $W = \bigcup_{\alpha < \kappa} W_\alpha$: If $q \in W$, then there is an $\alpha < \kappa$ such that $\text{dom}(q) \cap A = \text{dom}(q) \cap A_\alpha$. Thus if $p = q \upharpoonright A_\alpha$, there exists some $q' \in W_{\alpha+1}$ such that $q' \upharpoonright A_\alpha = p$. Since $\text{dom}(q') \subset A$, it follows that q and q' are compatible; however, both are elements of W and thus $q = q'$. Hence $q \in W_{\alpha+1}$.

The proof is completed by showing that $|A_\alpha| \leq \kappa$ and $|W_\alpha| \leq \kappa$ for each $\alpha < \kappa$. This is proved by induction on α . If $|W_\alpha| \leq \kappa$, then $|A_\alpha| \leq \kappa$ because $A_\alpha = \bigcup \{\text{dom}(q) : q \in W_\alpha\}$. If α is a limit ordinal and $|W_\beta| \leq \kappa$ for all $\beta < \alpha$, then $|W_\alpha| = |\bigcup_{\beta < \alpha} W_\beta| \leq \kappa$. Thus let us assume that $|W_\alpha| \leq \kappa$ and let us show that $|W_{\alpha+1}| \leq \kappa$. The set $W_{\alpha+1}$ is obtained by adding to W_α at most one $q \in W$ for each $p \in P$ with $\text{dom}(p) \subset A_\alpha$. There are at most $\kappa^{<\kappa}$ subsets X of A_α of size $< \kappa$, and since κ is regular and $2^{<\kappa} = \kappa$, we have $\kappa^{<\kappa} = \kappa$. On each X there are $|C|^{|X|}$ functions with values in C , and therefore there are at most κ elements p of P with $\text{dom}(p) \subset A_\alpha$. Hence $|W_{\alpha+1}| \leq \kappa$. Then it follows that $|W| \leq \kappa$. □

Distributivity

In (7.28) we defined κ -distributivity of complete Boolean algebras. We now show that this concept plays a crucial role in the theory of forcing.

Definition 15.5. A forcing notion P is κ -distributive if the intersection of κ open dense sets is open dense. P is $<\kappa$ -distributive if it is λ -distributive for all $\lambda < \kappa$.

Note that if P is dense in B then P is κ -distributive if and only if B is.

Theorem 15.6. Let κ be an infinite cardinal and assume that $(P, <)$ is κ -distributive. Then if $f \in V[G]$ is a function from κ into V , then $f \in V$. In particular, κ has no new subsets in $V[G]$.

Proof. Let $f : \kappa \rightarrow V$ and $f \in V[G]$, let \dot{f} be a name for f . There exist some $A \in V$ and a condition $p_0 \in G$ such that p_0 forces

$$\dot{f} \text{ is a function from } \check{\kappa} \text{ into } \check{A}.$$

For each $\alpha < \kappa$, the set

$$D_\alpha = \{p \leq p_0 : (\exists x \in A) p \Vdash \dot{f}(\check{\alpha}) = \check{x}\}$$

is open dense below p_0 . Thus $D = \bigcap_{\alpha < \kappa} D_\alpha$ is dense below p_0 and therefore there is some $p \in D \cap G$. Now we argue in V : For each $\alpha < \kappa$ there is some x_α such that $p \Vdash \dot{f}(\check{\alpha}) = \check{x}_\alpha$; let $g : \kappa \rightarrow A$ be the function defined by $g(\alpha) = x_\alpha$. However, it is easy to see that $f(\alpha) = x_\alpha = g(\alpha)$, for every $\alpha < \kappa$, and thus $f \in V$. □

See Exercise 15.5 for the converse.

The following property, stronger than distributivity, is often easy to verify:

Definition 15.7. P is κ -closed if for every $\lambda \leq \kappa$, every descending sequence $p_0 \geq p_1 \geq \dots \geq p_\alpha \geq \dots$ ($\alpha < \lambda$) has a lower bound. P is $<\kappa$ -closed if it is λ -closed for all $\lambda < \kappa$.

Lemma 15.8. If P is κ -closed then it is κ -distributive.

Proof. Let $\{D_\alpha : \alpha < \kappa\}$ be a collection of open dense sets. The intersection $D = \bigcap_{\alpha < \kappa} D_\alpha$ is clearly open; to show that D is dense, let $p \in P$ be arbitrary. By induction on $\alpha < \kappa$, we construct a descending κ -sequence of conditions $p \geq p_0 \geq p_1 \geq \dots$. We let p_α be a condition stronger than all p_ξ , $\xi < \alpha$, and such that $p_\alpha \in D_\alpha$. Finally, we let q be a condition stronger than all p_α , $\alpha < \kappa$. Clearly, $q \in D$. □

Now we can prove the claim about the generic extension by the forcing in (15.3). The forcing P is $<\kappa$ -closed, and therefore κ has no new bounded subsets; hence κ is preserved. The cardinals above κ are preserved because P satisfies the κ^+ -chain condition, by Lemma 15.4. We have $|P| = \lambda$ and therefore $|B| = |P|^\kappa = \lambda$, and so, by Lemma 15.1, $(2^\kappa)^{V[G]} = \lambda^\kappa = \lambda$.

Product Forcing

Let P and Q be two notions of forcing. The *product* $P \times Q$ is the coordinate-wise partially ordered set product of P and Q :

$$(15.4) \quad (p_1, q_1) \leq (p_2, q_2) \quad \text{if and only if} \quad p_1 \leq p_2 \text{ and } q_1 \leq q_2.$$

If G is a generic filter on $P \times Q$, let

$$(15.5) \quad G_1 = \{p \in P : \exists q (p, q) \in G\}, \quad G_2 = \{q \in Q : \exists p (p, q) \in G\}.$$

The sets G_1 and G_2 are generic on P and Q respectively, and $G = G_1 \times G_2$. The following lemma describes genericity on products:

Lemma 15.9 (The Product Lemma). *Let P and Q be two notions of forcing in M . In order that $G \subset P \times Q$ be generic over M , it is necessary and sufficient that $G = G_1 \times G_2$ where $G_1 \subset P$ is generic over M and $G_2 \subset Q$ is generic over $M[G_1]$. Moreover, $M[G] = M[G_1][G_2]$.*

As a corollary, if G_1 is generic over M and G_2 is generic over $M[G_1]$, then G_1 is generic over $M[G_2]$, and $M[G_1][G_2] = M[G_2][G_1]$.

Proof. First let G be an M -generic filter on $P \times Q$. We define G_1 and G_2 by (15.5). Clearly, G_1 and G_2 are filters, and $G \subset G_1 \times G_2$. If $(p_1, p_2) \in G_1 \times G_2$, then there are $p'_1 \in G_1$ and $p'_2 \in G_2$ such that $(p'_1, p_2) \in G$ and $(p_1, p'_2) \in G$. Since G is a filter, there exist $q_1 \leq p_1, p'_1$ and $q_2 \leq p_2, p'_2$ such $(q_1, q_2) \in G$. Hence $(p_1, p_2) \in G$ and we have $G = G_1 \times G_2$.

It is easy to see that G_1 is generic over M : If $D_1 \in M$ is dense in P , then $D_1 \times Q$ is dense in $P \times Q$; and since $(D_1 \times Q) \cap G \neq \emptyset$, we have $D_1 \cap G_1 \neq \emptyset$. To show that G_2 is generic over $M[G_1]$, let $D_2 \in M[G_1]$ be dense in Q . Let \Vdash be the forcing relation corresponding to P . Let \dot{D}_2 be a name for D_2 and let $p_1 \in G_1$ be such that p_1 forces “ \dot{D}_2 is dense in Q .” Let $p_2 \in G_2$ be arbitrary. For every $q_1 \leq p_1$ and every $q_2 \leq p_2$ there exist $r_1 \leq q_1$ and $r_2 \leq q_2$ such that $r_1 \Vdash r_2 \in \dot{D}_2$; thus

$$D = \{(r_1, r_2) : r_1 \leq p_1 \text{ and } r_1 \Vdash r_2 \in \dot{D}_2\}$$

is dense in $P \times Q$ below (p_1, p_2) and so there exist r_1, r_2 such that $r_1 \in G_1$ and $r_1 \Vdash r_2 \in \dot{D}_2$. Hence $r_2 \in D_2 \cap G_2$.

Conversely, let $G_1 \subset P$ be M -generic and let $G_2 \subset Q$ be $M[G_1]$ -generic. We let $G = G_1 \times G_2$. Clearly G is a filter on $P \times Q$. To show that G is M -generic, let $D \in M$ be dense in $P \times Q$. We let

$$D_2 = \{p_2 : (p_1, p_2) \in D \text{ for some } p_1 \in G_1\}.$$

The set D_2 is in $M[G_1]$; we shall show that D_2 is dense in Q and thus $D \cap (G_1 \times G_2) \neq \emptyset$.

Let $q_2 \in Q$ be arbitrary. Since D is dense in $P \times Q$, it follows that the set

$$D_1 = \{p_1 : (\exists p_2 \leq q_2) (p_1, p_2) \in D\}$$

is dense in P . Hence there is $p_1 \in G_1 \cap D_1$ and so D_2 is dense in Q . Since $G_1 \times G_2 \in M[G_1][G_2]$, it is obvious that $M[G_1 \times G_2] = M[G_1][G_2]$. \square

We shall now define products of infinitely many notions of forcing. In order to simplify the notation, we will assume that every notion of forcing has a greatest element, denoted 1. In practice, the *empty condition* \emptyset is often the greatest element of $(P, <)$.

Definition 15.10. Let $\{P_i : i \in I\}$ be a collection of partially ordered sets, each having a greatest element 1. The *product* $P = \prod_{i \in I} P_i$ consists of all functions p on I with values $p(i) \in P_i$, such that $p(i) = 1$ for all but finitely many $i \in I$. P is partially ordered by

$$(15.6) \quad p \leq q \quad \text{if and only if} \quad p(i) \leq q(i) \quad \text{for all } i \in I.$$

For each $p \in \prod_i P_i$, the finite set $s(p) = \{i \in I : p(i) \neq 1\}$ is called the *support* of p .

If G is a generic filter on $\prod_i P_i$, then for each $i \in I$, the set $G_i = \{p(i) : p \in G\}$, the *projection* of G on P_i , is a generic filter on P_i .

A natural generalization of a product is κ -product:

Definition 15.11. Let κ be a regular cardinal. The κ -*product* (the product with $<\kappa$ -support) of P_i is the set of all functions p on I with $p(i) \in P_i$ such that $|s(p)| < \kappa$; the ordering is coordinatewise (15.6).

As usual, λ -*support* means $<\lambda^+$ -support, *countable* support means $<\aleph_1$ -support, etc.

The following lemma is immediate:

Lemma 15.12. *If P and Q are λ -closed then $P \times Q$ is λ -closed. More generally, if each P_i is λ -closed and P is the κ -product of the P_i , with $\lambda < \kappa$, then P is λ -closed.*

Proof. Let $\alpha \leq \lambda$ and let $p^\xi = \langle p_i^\xi : i \in I \rangle$, $\xi < \alpha$, be a descending α -sequence of conditions in P . If we let $s = \bigcup_{\xi < \alpha} s(p^\xi)$, then $|s| < \kappa$, and since each P_i is λ -closed, it is easy to find $p = \langle p_i : i \in I \rangle$ such that $s(p) = s$ and that $p_i \leq p_i^\xi$ for each $i \in I$ and each $\xi < \alpha$. \square

Chain conditions are generally not preserved by products. While it is consistent that c.c.c. is preserved by products (we return to this in Chapter 16), it is also consistent to have a forcing P that satisfies c.c.c. but $P \times P$ does not (see Exercise 15.28).

The following property (K for Knaster) is stronger than the countable chain condition:

Definition 15.13. A notion of forcing has *property (K)* if every uncountable set of conditions has an uncountable subset of pairwise compatible elements.

Lemma 15.14. *If P and Q both have property (K) then so does $P \times Q$.*

Proof. Let $W \subset P \times Q$ be uncountable. If there exists a $p \in P$ such that the set $X = \{q : (p, q) \in W\}$ is uncountable, then since Q has property (K) there exists an uncountable $Y \subset X$ of pairwise compatible elements, and $\{p\} \times Y$ is such a subset of W in $P \times Q$.

The proof is similar if for some $q \in Q$, the set $\{p : (p, q) \in W\}$ is uncountable. In the remaining case, there is an uncountable set of pairs $F \subset W$ that is a one-to-one function. Applying successively property (K) to P and Q , we get an uncountable $G \subset F$ such that for any two elements (p_1, q_1) and (p_2, q_2) of G , p_1 is compatible with p_2 in P and q_1 is compatible with q_2 in Q , hence (p_1, q_1) and (p_2, q_2) are compatible. \square

Theorem 15.15. *If for every $i \in I$, P_i has property (K) then $\prod_{i \in I} P_i$ has property (K).*

Proof. Let X be an uncountable subset of P , and let $W = \{s(p) : p \in X\}$. If W is countable, then there is a finite set $J \subset I$ such that $s(p) = J$ for uncountably many p . By Lemma 15.14, $\prod_{i \in J} P_i$ has property (K) and the theorem follows. If W is uncountable, there exist, by Theorem 9.18, an uncountable $Z \subset X$ and a finite set $J \subset I$ such that $s(p) \cap s(q) = J$ whenever $p, q \in Z$, $p \neq q$. Since $\prod_{i \in I} P_i$ has property (K), Z has an uncountable subset Y such that for any $p, q \in Y$, $p \upharpoonright J$ and $q \upharpoonright J$ are compatible. But such p and q are compatible in $\prod_{i \in I} P_i$. \square

Corollary 15.16. *The product of any collection of countable forcing notions has property (K) and so it satisfies the countable chain condition.* \square

The best one can say about the chain condition in products is this:

Theorem 15.17. (i) *If each P_i has size λ (infinite) then the product of the P_i satisfies the λ^+ -chain condition.*

(ii) *If κ is regular, $\lambda \geq \kappa$, $\lambda^{<\kappa} = \lambda$ and $|P_i| \leq \lambda$ for all $i \in I$, then the κ -product of the P_i satisfies the λ^+ -chain condition.*

(iii) *If λ is inaccessible, $\kappa < \lambda$ is regular, and $|P_i| < \lambda$ for each i , then the κ -product satisfies the λ -chain condition.*

Proof. (i) is a special case of (ii); thus consider κ -products. Let P be the κ -product, and let W be an antichain in P . If $p = \langle p_i : i \in I \rangle$ and $q = \langle q_i : i \in I \rangle$ are incompatible in P , then for some $i \in s(p) \cap s(q)$, p_i and q_i are incompatible in P_i , and in particular $p_i \neq q_i$. Thus we can regard elements of W as functions whose domain is a subset $s(p)$ of I of size $< \kappa$, with values in the P_i , and show that if W consists of pairwise incompatible functions then $|W|$ has the required bound.

We follow the proof of Lemma 15.4. As there we construct κ -sequences $A_0 \subset A_1 \subset \dots \subset A_\alpha \subset \dots$ ($\alpha < \kappa$) of subsets of I and $W_0 \subset W_1 \subset \dots \subset W_\alpha \subset \dots$ ($\alpha < \kappa$) of subsets of W such that $A_\alpha = \bigcup\{s(p) : p \in W_\alpha\}$ for each α . And as there we show that $W = \bigcup_{\alpha < \kappa} W_\alpha$. Thus it remains to show, by induction on α , that $|W_\alpha| \leq \lambda$ (in (ii)) or that $|W_\alpha| < \lambda$ (in (iii)). Let us prove (ii); (iii) is similar.

If $|W_\alpha| \leq \lambda$, then $|A_\alpha| \leq \kappa \cdot \lambda = \lambda$. If $\alpha < \kappa$ is limit and if $|W_\beta| \leq \lambda$ for each $\beta < \alpha$, then $|W_\alpha| \leq |\alpha| \cdot \lambda = \lambda$. Thus let us assume that $|W_\alpha| \leq \lambda$ and let us show that $|W_{\alpha+1}| \leq \lambda$. The set $W_{\alpha+1}$ is obtained by adding to W_α at most one q for each $p \in P$ with $s(p) \subset A_\alpha$. However, since $|A_\alpha| \leq \lambda$, there are at most $\lambda^{<\kappa}$ functions p with $s(p) \subset A_\alpha$, $|s(p)| < \kappa$, and λ possible values for each $i \in s(p)$. Thus $|W_{\alpha+1}| \leq \lambda^{<\kappa} = \lambda$. □

Easton's Theorem

The theorem that we are about to prove shows that in ZFC alone the continuum function 2^κ can behave in any prescribed way consistent with König's Theorem, for *regular cardinals* κ . As we have seen in Chapter 8 (Silver's Theorem) and shall see again in Chapter 24, this is not the case with singular cardinals.

Theorem 15.18 (Easton). *Let M be a transitive model of ZFC and assume that the Generalized Continuum Hypothesis holds in M . Let F be a function (in M) whose arguments are regular cardinals and whose values are cardinals, such that for all regular κ and λ :*

- (15.7) (i) $F(\kappa) > \kappa$;
- (ii) $F(\kappa) \leq F(\lambda)$ whenever $\kappa \leq \lambda$;
- (iii) cf $F(\kappa) > \kappa$.

Then there is a generic extension $M[G]$ of M such that M and $M[G]$ have the same cardinals and cofinalities, and for every regular κ ,

$$M[G] \models 2^\kappa = F(\kappa).$$

We have to point out that the generic extension is obtained by forcing with a *class* of conditions. By Lemma 15.1, a notion of forcing can only increase the size of 2^κ for $\kappa < |B(P)|$; thus we *have* to use a class of conditions. We shall describe the appropriate generalization of the forcing method.

Since the proof of Easton's Theorem involves forcing with a class of conditions, we shall first give a proof of the special case, when the "continuum function" F is prescribed for only a *set* of regular cardinals. Thus let us work in a ground model M that satisfies the GCH and let F be a function defined on a set A of regular cardinals and having the properties (15.7)(i)–(iii).

For each $\kappa \in \text{dom}(F)$, let (P_κ, \supset) be the notion of forcing that adjoins $F(\kappa)$ subsets of κ (cf. (15.3)):

$$(15.8) \quad \text{dom}(p) \subset \kappa \times F(\kappa), \quad |\text{dom}(p)| < \kappa, \quad \text{and} \quad \text{ran}(p) \subset \{0, 1\}.$$

We let $(P, <)$ be the *Easton product* of P_κ , $\kappa \in A$: A condition p is a function $p = \langle p_\kappa : \kappa \in A \rangle \in \prod_{\kappa \in A} P_\kappa$ such that if we denote $s(p) = \{\kappa \in A : p_\kappa \neq \emptyset\}$, the support of p , then

$$(15.9) \quad \text{for every regular cardinal } \gamma, \quad |s(p) \cap \gamma| < \gamma.$$

We can regard the conditions as functions with values 0 and 1, whose domain consists of triples (κ, α, β) where $\kappa \in A$, $\alpha < \kappa$, and $\beta < F(\kappa)$, and such that for every regular cardinal γ ,

$$(15.10) \quad |\{(\kappa, \alpha, \beta) \in \text{dom}(p) : \kappa \leq \gamma\}| < \gamma$$

(and p is stronger than q if and only if $p \supset q$). Note that (15.10) implies that for each $\kappa \in A$, $|\text{dom}(p_\kappa)| < \kappa$, where p_κ is defined by

$$p_\kappa(\alpha, \beta) = p(\kappa, \alpha, \beta).$$

Let G be a generic set of conditions, and let for each $\kappa \in A$, G_κ be the projection of G on P_κ . Each G_κ is a generic filter on P_κ and thus produces $F(\kappa)$ new subsets of κ :

$$a_\beta^\kappa = \{\alpha < \kappa : (\exists p \in G) p(\kappa, \alpha, \beta) = 1\} \quad (\beta < F(\kappa)).$$

We shall show that $(P, <)$ preserves cardinals and cofinalities, and that each $\kappa \in A$ has exactly $F(\kappa)$ subsets in $M[G]$. The condition (15.10) is instrumental in the proof.

Given a regular cardinal λ , we can decompose each condition $p \in P$ into two parts:

$$(15.11) \quad p^{\leq \lambda} = p \upharpoonright \{(\kappa, \alpha, \beta) : \kappa \leq \lambda\}, \quad p^{> \lambda} = p \upharpoonright \{(\kappa, \alpha, \beta) : \kappa > \lambda\}.$$

Clearly $p = p^{\leq \lambda} \cup p^{> \lambda}$. We let

$$(15.12) \quad P^{\leq \lambda} = \{p^{\leq \lambda} : p \in P\}, \quad P^{> \lambda} = \{p^{> \lambda} : p \in P\}.$$

Obviously, $P^{\leq \lambda}$ is the Easton product of P_κ , $\kappa \in A$ and $\kappa \leq \lambda$, and $P^{> \lambda}$ is the Easton product of P_κ , $\kappa \in A$ and $\kappa > \lambda$. Moreover, P is (isomorphic to) the product $P^{\leq \lambda} \times P^{> \lambda}$.

First we notice that $P^{> \lambda}$ is λ -closed: If $C \subset P^{> \lambda}$ consists of pairwise compatible conditions and $|C| \leq \lambda$, then $p = \bigcup C$ is a condition in $P^{> \lambda}$; (15.10) holds for all regular $\gamma > \lambda$, and holds trivially for $\gamma \leq \lambda$ because if $(\kappa, \alpha, \beta) \in \text{dom}(p)$, then $\kappa > \lambda$.

Furthermore, $P^{\leq \lambda}$ satisfies the λ^+ -chain condition: If $W \subset P^{\leq \lambda}$ is an antichain, then $|W| \leq \lambda$. The proof given in Theorem 15.17 works in this case as well because $|\text{dom}(p)| < \lambda$ for each $p \in P^{\leq \lambda}$ (and because GCH holds). Thus $P = P^{> \lambda} \times P^{\leq \lambda}$ where $P^{> \lambda}$ is λ -closed and $P^{\leq \lambda}$ satisfies the λ^+ -chain condition.

Lemma 15.19. *Let $G \times H$ be an M -generic filter on $P \times Q$, where P is λ -closed and Q satisfies the λ^+ -chain condition. Then every function $f : \lambda \rightarrow M$ in $M[G \times H]$ is in $M[H]$. In particular,*

$$P^{M[G \times H]}(\lambda) = P^{M[H]}(\lambda).$$

Proof. Let \dot{f} be a name for f ; let us assume, without loss of generality, that for some A , every condition forces that \dot{f} is a function from λ into A . For each $\alpha < \lambda$, let $D_\alpha \subset P$ be defined as follows:

$$(15.13) \quad \begin{aligned} & p \in D_\alpha \text{ if and only if there exist a maximal antichain } W \subset Q \\ & \text{and a family } \{a_{p,q}^{(\alpha)} : q \in W\} \text{ such that for each } q \in W, \\ & (p, q) \Vdash \dot{f}(\alpha) = a_{p,q}^{(\alpha)}. \end{aligned}$$

We claim that each D_α is open dense in P . Clearly, D_α is open; thus let $p_0 \in P$ be arbitrary and let us find $p \in D_\alpha$ such that $p \leq p_0$. There exist $p_1 \leq p_0$, $q_1 \in Q$ and $a_1 \in A$ such that $(p_1, q_1) \Vdash \dot{f}(\alpha) = a_1$. By induction on $\gamma < \lambda^+$, we construct $p_\gamma \in P$, $q_\gamma \in Q$, and $a_\gamma \in A$ such that $p_0 \geq p_1 \geq \dots \geq p_\gamma \geq \dots$, that the q_γ are pairwise incompatible and that (p_γ, q_γ) forces $\dot{f}(\alpha) = a_\gamma$. If $\{q_\xi : \xi < \gamma\}$ is not maximal, we can find such p_γ , q_γ , and a_γ since P is λ -closed. By the λ^+ -chain condition, there is some $\beta < \lambda^+$ such that $W = \{q_\gamma : \gamma < \beta\}$ is a maximal antichain; then we find $p \in P$ stronger than all p_γ , $\gamma < \beta$. Thus D_α is open dense in P .

Since P is λ -closed, it follows that $\bigcap_{\alpha < \lambda} D_\alpha$ is open dense, and so there exists some $p \in G$ such that $p \in D_\alpha$ for all $\alpha < \lambda$. We pick (in M) for each $\alpha < \lambda$ a maximal antichain $W_\alpha \subset Q$ and a family $\{a_{p,q}^{(\alpha)} : q \in W_\alpha\}$ such that (15.13) holds for each $q \in W_\alpha$. By the genericity of H , for every α there is a unique $q \in W_\alpha$ such that $q \in H$, and we have, for every $\alpha < \lambda$,

$$(15.14) \quad f(\alpha) = a_{p,q}^{(\alpha)}, \quad \text{where } q \text{ is the unique } q \in W_\alpha \cap H.$$

However, (15.14) defines the function f in $M[H]$. □

Now we can finish the proof of Easton's Theorem, that is, at least in the case when F is defined on a set A of regular cardinals.

Let κ be a regular cardinal in M ; we shall show that κ is a regular cardinal in $M[G]$. If κ fails to be a regular cardinal, then there exists a function f that maps some $\lambda < \kappa$, regular in M , cofinally into κ . We consider P as the product: $P = P^{>\lambda} \times P^{\leq\lambda}$. Then $G = G^{>\lambda} \times G^{\leq\lambda}$ and $M[G] = M[G^{>\lambda}][G^{\leq\lambda}] = M[G^{\leq\lambda}][G^{>\lambda}]$. By Lemma 15.19, f is in $M[G^{\leq\lambda}]$ and so κ is not a regular cardinal in $M[G^{\leq\lambda}]$. However, this is a contradiction since $P^{\leq\lambda}$ satisfies the κ -chain condition and hence κ is regular in $M[G^{\leq\lambda}]$.

It remains to prove that $(2^\lambda)^{M[G]} = F(\lambda)$, for each $\lambda \in A$. Again, we regard P as the product $P^{>\lambda} \times P^{\leq\lambda}$ and $G = G^{>\lambda} \times G^{\leq\lambda}$. By Lemma 15.19, every subset of λ in $M[G]$ is in $M[G^{\leq\lambda}]$ and we have $(2^\lambda)^{M[G]} = (2^\lambda)^{M[G^{\leq\lambda}]}$. However, an easy computation shows that $|P^{\leq\lambda}| = F(\lambda)$ and $|B(P^{\leq\lambda})| = F(\lambda)$, and hence $(2^\lambda)^{M[G]} \leq F(\lambda)$. On the other hand, we have exhibited $F(\lambda)$ subsets of λ for each $\lambda \in A$, and so $M[G] \models 2^\lambda = F(\lambda)$. □

Forcing with a Class of Conditions

We shall now show how to generalize the preceding construction to prove Easton’s Theorem in full generality, when the function F is defined for all regular cardinals. This generalization involves forcing with a proper class of conditions. Although it is possible to give a general method of forcing with a class, we shall concentrate only on the particular example.

Thus let M be a transitive model of $ZFC + GCH$. Moreover, we assume that M has a well-ordering of the universe (e.g., if M satisfies $V = L$). Let F be a function (in M) defined on all regular cardinals and having the properties (15.7)(i)–(iii). We define a class P of forcing conditions as follows: P is the class of all functions p with values 0 and 1, whose domain consists of triples (κ, α, β) where κ is a regular cardinal, $\alpha < \kappa$ and $\beta < F(\kappa)$, and such that for every regular cardinal γ , (15.10) holds, i.e.,

$$|\{(\kappa, \alpha, \beta) \in \text{dom}(p) : \kappa \leq \gamma\}| < \gamma$$

(and p is stronger than q if and only if $p \supset q$).

As before, we define $P^{\leq \lambda}$ and $P^{> \lambda}$ for every regular cardinal λ . Note that $P^{\leq \lambda}$ is a set. To define the Boolean-valued model M^B and the forcing relation, we use the fact that P is the Easton product of P_κ , κ a regular cardinal. For each regular λ , we let $B_\lambda = B(P^{\leq \lambda})$. If $\lambda < \mu$ then the inclusion $P^{\leq \lambda} \subset P^{\leq \mu}$ defines an obvious embedding of B_λ into B_μ ; thus we arrange the definition of the B_λ so that B_λ is a complete subalgebra of B_μ whenever $\lambda < \mu$. Then we let $B = \bigcup_\lambda B_\lambda$. B is a proper class; otherwise it has all the features of a complete Boolean algebra. In particular, $\sum X$ exists for every set $X \subset B$. Also, P is dense in B .

To define M^B , we cannot quite use the inductive definition (14.15) since B is not a set. However, we simply let $M^B = \bigcup_\lambda M^{B_\lambda}$; the formal definition of M^B does not present any problem. Similarly, to define $\|x \in y\|$ and $\|x = y\|$, we first notice that if $x, y \in M^{B_\lambda}$ and $\lambda \leq \mu$, then $\|x \in y\|_{B^\lambda} = \|x \in y\|_{B_\mu}$ and so we let $\|x \in y\| = \|x \in y\|_{B_\lambda}$ where λ is such that $x, y \in M^{B_\lambda}$. The same for $\|x = y\|$.

As for the forcing relation in general, we cannot define $\|\varphi\|$ unless φ is Δ_0 ; this is because $\sum X$ does not generally exist if $X \subset B$ is a class. However, we can still define $p \Vdash \varphi$ using the formulas from Theorem 14.7.

Now, we call $G \subset P$ generic over M if (i) $p \supset q$ and $p \in G$ implies $q \in G$, (ii) $p, q \in G$ implies $p \cup q \in G$, and (iii) if D is a class in M and D is dense in P , then $D \cap G \neq \emptyset$.

The question of existence of a generic filter can be settled in a more or less the same way as in the case when P is a set. One possible way is to assume that M is a countable transitive model. Then there are only countably many classes in M and G exists. Another possible way is to use the canonical generic ultrafilter. It is the class \dot{G} in M^B defined by $\dot{G}(\dot{p}) = p$ for all $p \in P$ (here we need the assumption that M is a class in M^B).

Thus let G be an M -generic filter on P . For every regular λ , $G_\lambda = G \cap P^{\leq \lambda}$ is generic on $P^{\leq \lambda}$. If $\dot{x} \in M^{B^\lambda}$ and $\lambda \leq \mu$, then $\dot{x}^{G_\lambda} = \dot{x}^{G_\mu}$, and so we define $\dot{x}^G = \dot{x}^{G_\lambda}$ where λ is such that $\dot{x} \in M^{B^\lambda}$. Then we define $M[G] = \bigcup_\lambda M[G_\lambda]$.

Using the genericity of G and properties of the forcing relation, we get the Forcing Theorem,

$$(15.15) \quad M[G] \models \varphi(x_1, \dots, x_n) \quad \text{if and only if} \quad (\exists p \in G) p \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n)$$

where $\dot{x}_1, \dots, \dot{x}_n \in M^B$ are names for x_1, \dots, x_n .

The formula (15.15) is proved first for atomic formulas and then by induction on φ ; in the induction step involving the quantifiers, we use the fact that G intersects every dense class of M .

We shall now show that $M[G]$ is a model of ZFC. The proofs of all axioms of ZFC except Power Set and Replacement go through as when we forced with a set. (Separation also needs some extra work which we leave to the reader.) It is no surprise that the Power Set and Replacement Axioms present problems. It is easy to construct either a class of forcing conditions adding a proper class of Cohen reals, or a class of conditions collapsing Ord onto ω (as in the following section). The present proof of the Power Set and Replacement Axioms uses the fact that for every regular λ (or at least for arbitrarily large regular λ), $P = P^{>\lambda} \times P^{\leq \lambda}$ where $P^{>\lambda}$ is λ -closed and $P^{\leq \lambda}$ is a set and satisfies the λ^+ -chain condition.

Power Set. Let λ be a regular cardinal. Lemma 15.19 remains true even when applied to $P^{>\lambda} \times P^{\leq \lambda}$. It does not matter that each D_α is a class. The “sequence” of classes $\langle D_\alpha : \alpha < \lambda \rangle$ can be defined (e.g., as a class of pairs $\{(p, \alpha) : p \in D_\alpha\}$) and since $P^{>\lambda}$ is λ -closed, the intersection $\bigcap_{\alpha < \lambda} D_\alpha$ is dense, and there exists $p \in G \cap P^{>\lambda}$ such that $p \in D_\alpha$ for all $\alpha < \lambda$. The rest of the proof of Lemma 15.19 remains unchanged, and thus we have proved that every subset of λ in $M[G]$ is in $M[G]$. Since $P^{\leq \lambda}$ is a set, it follows that the Power Set Axiom holds in $M[G]$.

Replacement. To show that the Axioms of Replacement hold in $M[G]$, we combine the proof for ordinary generic extension with Lemma 15.19. It suffices to prove that if in $M[G]$, $\varphi(\alpha, v)$ defines a function $K : Ord \rightarrow M[G]$, then $\{K(\alpha) : \alpha < \lambda\}$ is a set in $M[G]$ for every regular cardinal λ . Without loss of generality, let us assume that for every $p \in P$

$$(15.16) \quad p \Vdash \text{for every } \alpha \text{ there is a unique } v \text{ such that } \varphi(\alpha, v).$$

Let λ be a regular cardinal, and let us consider again $P = P^{>\lambda} \times P^{\leq \lambda}$, and $G = (G \cap P^{>\lambda}) \times G_\lambda$. As in Lemma 15.19, let us define, for each $\alpha < \lambda$, a class $D_\alpha \subset P^{>\lambda}$:

$$(15.17) \quad \begin{aligned} p \in D_\alpha & \text{ if and only if there is a maximal antichain } W \subset P^{\leq \lambda} \\ & \text{ and a family } \{\dot{a}_{p,q}^{(\alpha)} : q \in W\} \text{ such that for each } q \in W, \\ & p \cup q \Vdash \varphi(\alpha, \dot{a}_{p,q}^{(\alpha)}). \end{aligned}$$

As in Lemma 15.19, each D_α , $\alpha < \lambda$, is open dense; since $P^{>\lambda}$ is λ -closed, $\bigcap_{\alpha < \lambda} D_\alpha$ is dense and there exists $p \in G \cap P^{>\lambda}$ such that $p \in D_\alpha$ for all $\alpha < \lambda$. We pick (in M) for each $\alpha < \lambda$ a maximal antichain $W_\alpha \subset P^{\leq \lambda}$ and a family $\{\dot{a}_{p,q}^{(\alpha)} : q \in W_\alpha\}$ such that (15.17) holds for each $q \in W_\alpha$. Now, if we let $S = \{\dot{a}_{p,q}^{(\alpha)} : \alpha < \lambda \text{ and } q \in W_\alpha\}$, then it follows that $\{K(\alpha) : \alpha < \lambda\} \subset \{\dot{a}^G : \dot{a} \in S\}$. However, the latter is a set in $M[G]$: There is a γ such that $S \subset M^{B_\gamma}$, and we have $\{\dot{a}^G : \dot{a} \in S\} = \{\dot{a}^{G_\gamma} : \dot{a} \in S\} \in M[G_\gamma]$.

Thus $M[G]$ is a model of ZFC and it remains to show that $M[G]$ has the same cardinals and cofinalities as M , and that in $M[G]$, $2^\kappa = F(\kappa)$ for every regular cardinal κ . However, this is proved exactly the same way as when we forced with a set of Easton conditions. \square

We conclude the section with a remark on the Bernays-Gödel axiomatic set theory. If a sentence involving only set variables is provable in $BGC = BG + \text{Axiom E}$, then it is provable in $BG + AC$. This is a consequence of the following: If M is a transitive model of $BG + AC$, then there is a generic extension $M[G]$ that has the same sets and has a choice function F defined for all nonempty sets. The forcing conditions $p \in P$ used in the proof are choice functions whose domain is a set of nonempty sets (and $p < q$ means $p \supset q$). The proof that $M[G]$ is a model of BG is rather easy since no new sets are added (P is κ -closed for all κ). The generic filter on P defines a choice function $F = \bigcup G$, and F is defined for all nonempty sets $X \in M[G]$.

The Lévy Collapse

One of the most useful techniques provided by forcing is *collapsing cardinals*. We start with the simplest example:

Example 15.20. Let λ be an uncountable cardinal. Let P be the set of all finite sequences $\langle p(0), \dots, p(n-1) \rangle$ of ordinals less than λ ; p is stronger than q if $p \supset q$.

Let G be a generic filter on P and let $f = \bigcup G$; f is a function with domain ω and range λ . Thus P *collapses* λ : Its cardinality in $V[G]$ is \aleph_0 .

As $|P| = \lambda$, P satisfies the λ^+ -chain condition and so all cardinals greater than λ are preserved (as are all cofinalities greater than λ). \square

This construction generalizes to collapsing λ to κ :

Lemma 15.21. *Let κ be a regular cardinal and let $\lambda > \kappa$ be a cardinal. There is a notion of forcing $(P, <)$ that collapses λ onto κ , i.e., λ has cardinality κ in the generic extension. Moreover,*

- (i) every cardinal $\alpha \leq \kappa$ in V remains a cardinal in $V[G]$; and
- (ii) if $\lambda^{<\kappa} = \lambda$, then every cardinal $\alpha > \lambda$ remains a cardinal in the extension.

[The condition in (ii) is satisfied if GCH holds and $\text{cf } \lambda \geq \kappa$.]

Proof. Let P be the set of all functions p such that:

- (15.18) (i) $\text{dom}(p) \subset \kappa$ and $|\text{dom}(p)| < \kappa$,
 (ii) $\text{ran}(p) \subset \lambda$,

and let $p < q$ if and only if $p \supset q$.

Let G be a generic set of conditions and let $f = \bigcup G$. Clearly, f is a function, and it maps κ onto λ .

$(P, <)$ is $<\kappa$ -closed and therefore all cardinals $\leq \kappa$ are preserved. If $\lambda^{<\kappa} = \lambda$, then $|P| = \lambda$ and it follows that all cardinals $\geq \lambda^+$ are preserved. \square

The following technique collapses all cardinals below an inaccessible cardinal λ while preserving λ , thus making λ a successor cardinal in the generic extension. The forcing notion P defined in (15.19) is called the *Lévy collapse*; we denote $B(P) = \text{Col}(\kappa, <\lambda)$.

Theorem 15.22 (Lévy). *Let κ be a regular cardinal and let $\lambda > \kappa$ be an inaccessible cardinal. There is a notion of forcing $(P, <)$ such that:*

- (i) every α such that $\kappa \leq \alpha < \lambda$ has cardinality κ in $V[G]$; and
 (ii) every cardinal $\leq \kappa$ and every cardinal $\geq \lambda$ remains a cardinal in $V[G]$.

Hence $V[G] \models \lambda = \kappa^+$.

Proof. For each $\alpha < \lambda$, let P_α be the set of all functions p_α such that $\text{dom}(p_\alpha) \subset \kappa$, $|\text{dom}(p_\alpha)| < \kappa$, and $\text{ran}(p_\alpha) \subset \alpha$; let $p_\alpha < q_\alpha$ if and only if $p_\alpha \supset q_\alpha$.

Let $(P, <)$ be the κ -product of the P_α , $\alpha < \lambda$. Equivalently, the conditions $p \in P$ are functions on subsets of $\lambda \times \kappa$ such that

- (15.19) (i) $|\text{dom}(p)| < \kappa$;
 (ii) $p(\alpha, \xi) < \alpha$ for each $(\alpha, \xi) \in \text{dom}(p)$.

Let G be a generic set of conditions; for each $\alpha < \lambda$, let G_α be the projection of G on P_α . Then G_α is a generic filter on P_α ; and as in Lemma 15.21, the set $f_\alpha = \bigcup G_\alpha$ is a function that maps κ onto α . Thus $V[G] \models |\alpha| \leq |\kappa|$, for every $\alpha < \lambda$.

The notion of forcing $(P, <)$ is $<\kappa$ -closed and hence it preserves all cardinals and cofinalities $\leq \kappa$. In particular, κ is a cardinal in $V[G]$.

By Theorem 15.17(iii), $(P, <)$ satisfies the λ -chain condition. Hence λ remains a cardinal in $V[G]$, and so do all cardinals greater than λ . It follows that in $V[G]$, λ is the cardinal successor of κ . \square

Suslin Trees

One of the earliest applications of forcing was the solution of Suslin's Problem: The existence of a Suslin line is independent of ZFC. In this section we show how to construct a Suslin tree by forcing and in L ; in Chapter 16 we will construct a generic model in which there are no Suslin trees.

Theorem 15.23. *There is a generic extension in which there exists a Suslin tree.*

Proof. Let P be the collection of all countable normal trees, i.e., all T such that for some $\alpha < \omega_1$,

- (15.20) (i) each $t \in T$ is a function $t : \beta \rightarrow \omega$ for some $\beta < \alpha$;
 (ii) if $t \in T$ and s is an initial segment of t then $s \in T$;
 (iii) if $\beta + 1 < \alpha$ and $t : \beta \rightarrow \omega$ is in T , then $t \restriction n \in T$ for all $n \in \omega$;
 (iv) if $\beta < \alpha$ and $t : \beta \rightarrow \omega$ is in T , then for every γ such that $\beta \leq \gamma < \alpha$ there exists an $s : \gamma \rightarrow \omega$ in T such that $t \subset s$;
 (v) $T \cap \omega^\beta$ is at most countable for all $\beta < \alpha$.

(See (9.9) and Exercise 9.6.) T_1 is stronger than T_2 if T_1 is an extension of T_2 , i.e.,

$$(15.21) \quad T_1 < T_2 \quad \text{if and only if} \quad \exists \alpha < \text{height}(T_1) \quad T_2 = \{t \restriction \alpha : t \in T_1\}.$$

Let G be a generic set of conditions and let $\mathcal{T} = \bigcup \{T : T \in G\}$. We shall show that in $V[G]$, \mathcal{T} is a normal Suslin tree.

First we note that if T_1 and T_2 are two conditions, then either one is an extension of the other, or T_1 and T_2 are incompatible. Thus G consists of pairwise comparable trees and one can easily verify that \mathcal{T} is a normal tree (of height $\leq \omega_1$).

If $T_0, T_1, \dots, T_n, \dots$ is a sequence of conditions such that for each n , T_{n+1} is an extension of T_n , then $\bigcup_{n=0}^\infty T_n$ is a normal countable tree (and extends each T_n). Hence P is \aleph_0 -closed, and consequently, the cardinal \aleph_1 is preserved (and $V[G]$ has the same countable sequences in V as V).

To show that the height of \mathcal{T} is ω_1 , we verify that for every $\alpha < \omega_1$, G contains a condition T of height at least α . We show that the set $\{T \in P : \text{height}(T) \geq \alpha\}$ is dense in P , for any $\alpha < \omega_1$. In other words, we show that for each $T_0 \in P$ and each $\alpha < \omega_1$, there is an extension $T \in P$ of T_0 , of height at least α . It suffices to show that each $T_0 \in P$ has an extension $T \in P$ that has one more level; for then we can proceed by induction and take unions at limit steps.

If $\text{height}(T_0)$ is a successor ordinal, then an extension of T_0 is easily obtained. If $\text{height}(T_0)$ is a limit ordinal, then we first observe that for each $t \in T_0$ there exists a branch b of length α in T_0 such that $t \in b$: Using an increasing sequence $\alpha_0 < \alpha_1 < \dots < \alpha_n \dots$ with limit α , we use the normality

condition (15.20)(iv) to obtain such a branch. Now we construct an extension T of T_0 , of height $\alpha + 1$, as follows: For each $t \in T_0$, we pick a branch b_t of length α in T_0 such that $t \in b_t$, and let $T = T_0 \cup \{s : s = \bigcup b_t \text{ for some } t \in T_0\}$ (we extend all the branches b_t , $t \in T_0$, of T_0). Since T_0 is countable the added level is countable, and one can verify that $T \in P$.

It remains to show that \mathcal{T} has no uncountable antichain. If T is a tree and A is an antichain in T , then A is called a *maximal antichain* if there is no antichain A' in T such that $A' \supset A$: Each $t \in T$ is comparable with some $a \in A$. If A is a maximal antichain in T and if T' is an extension of T , then A is not necessarily maximal in T' . Let us call a set $S \subset T'$ *bounded* in T' if there is some $\alpha < \text{height}(T')$ such that all elements of S are at levels $\leq \alpha$. (If the height of T' is a successor ordinal, then every $S \subset T'$ is bounded.)

Lemma 15.24. *If A is a maximal antichain in a normal tree T and if A is bounded in T (in particular, if the height of T is a successor ordinal), then A is maximal in every extension of T .*

Proof. Let T' be an extension of T . Let $\alpha < \text{height}(T')$ be such that each $a \in A$ is at level $\leq \alpha$. If $t' \in T' - T$, then there exists $t \in T$ at level α such that $t \subset t'$; in turn, there exists $a \in A$ such that $a \subset t$. Hence t' is comparable with some $a \in A$. □

Lemma 15.25. *Let α be a countable limit ordinal, let $T \in P$ be a normal α -tree and let A be a maximal antichain in T . Then there exists an extension $T' \in P$ of T of height $\alpha + 1$ such that A is a maximal antichain in T' (and hence A is a bounded maximal antichain in T').*

Proof. For each $t \in T$ there exists $a \in A$ such that either $t \subset a$ or $a \subset t$. In either case, there exists a branch $b = b_t$ of length α in T such that $t \in b$ and $a \in b$. Let T' be the extension of T obtained by extending the branches b_t , for all $t \in T$: $T' = T \cup \{\bigcup b_t : t \in T\}$. The tree T' is a normal $(\alpha + 1)$ -tree and extends T ; moreover, since every $s \in T'$ is comparable with some $a \in A$, A is maximal in T' . □

Now we finish the proof of Theorem 15.23 by showing that in $V[G]$, every antichain in \mathcal{T} is countable. Since every antichain can be extended to a maximal antichain, it suffices to show that every maximal antichain is countable. Thus let A be a maximal antichain in \mathcal{T} . There is a name \dot{A} for A and a condition $T \in G$ such that

$$T \Vdash \dot{A} \text{ is a maximal antichain in } \mathcal{T}.$$

We will show that the following set of conditions is dense below T :

$$D = \{T' \leq T : \text{there is a bounded maximal antichain } A' \text{ in } T' \\ \text{such that } T' \Vdash A' \subset \dot{A}\}.$$

Then some $T' \in D$ is in G and there is a bounded maximal antichain A' in T' such that $A' \subset A$. However, \mathcal{T} is an extension of T' , and by Lemma 15.24,

A' is maximal in \mathcal{T} . Consequently, $A = A'$, and since A' is countable, we are done.

To show that D is dense below T let $T_0 \leq T$ be arbitrary. We shall construct a tree $T' \leq T_0$ such that $T' \in D$. Since $T_0 \Vdash (\dot{A}$ is a maximal antichain in \mathcal{T} and \mathcal{T} is an extension of T_0), there exist for each $s \in T_0$ an extension T'_0 of T_0 and some $t_s \in T'_0$ such that

$$(15.22) \quad s \text{ and } t_s \text{ are comparable} \quad \text{and} \quad T'_0 \Vdash t_s \in \dot{A}.$$

Since T_0 is countable, we repeat this countably many times and obtain an extension $T'_0 < T_0$ such that (15.22) holds for every $s \in T_0$. Let $T_1 = T'_0$. Then we proceed by induction and construct a sequence of trees $T_0 \geq T_1 \geq \dots \geq T_n \geq \dots$ such that for each n , T_{n+1} extends T_n and

$$(15.23) \quad (\forall s \in T_n)(\exists t_s \in T_{n+1}) \text{ } s \text{ and } t_s \text{ are comparable and } T_{n+1} \Vdash t_s \in \dot{A}.$$

We let $T_\infty = \bigcup_{n=0}^\infty T_n$, and $A' = \{t_s : s \in T_\infty\}$. By (15.23), A' is a maximal antichain in T_∞ , and $T_\infty \Vdash A' \subset \dot{A}$. Now we apply Lemma 15.25 and get an extension T' of T such that A' is a bounded maximal antichain in T' . Clearly, $T' \Vdash A' \subset \dot{A}$, and hence $T' \in D$. □

In the Exercises (15.21 and 15.22) we present another forcing notion (with finite conditions) that produces a Suslin tree. Later in the book we show that the forcing that adds a Cohen real also adds a Suslin tree.

The following theorem shows that a Suslin tree exists in L .

Theorem 15.26 (Jensen). *If $V = L$ then there exists a Suslin tree.*

Proof. We shall prove that the Diamond Principle \diamond implies that a Suslin tree exists. First we make the following observation. If T is a normal ω_1 -tree, let $T_\alpha = \{x \in T : o(x) < \alpha\}$.

Lemma 15.27. *If A is a maximal antichain in T , then the set*

$$C = \{\alpha : A \cap T_\alpha \text{ is a maximal antichain in } T_\alpha\}$$

is closed unbounded.

Proof. It is easy to see that C is closed. To show that C is unbounded, let $\alpha_0 < \omega_1$ be arbitrary. Since T_{α_0} is countable, there exists a countable ordinal $\alpha_1 > \alpha_0$ such that every $t \in T_{\alpha_0}$ is compatible with some $a \in A \cap T_{\alpha_1}$. Then there is $\alpha_2 > \alpha_1$ such that each $t \in T_{\alpha_1}$ is comparable with some $a \in A \cap T_{\alpha_2}$, etc. If $\alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < \dots$ is constructed in this way and if $\alpha = \lim_n \alpha_n$, then $A \cap T_\alpha$ is a maximal antichain in T_α . □

We now use \diamond to construct a normal Suslin tree $(T, <_T)$. We proceed by induction on levels. To facilitate the use of \diamond , we let points of T be countable

ordinals, $T = \omega_1$, and in fact each T_α (the first α levels of T) is an initial segment of ω_1 .

We construct T_α , $\alpha < \omega_1$, such that each T_α is a normal α -tree and such that T_β extends T_α whenever $\beta > \alpha$. T_1 consists of one point. If α is a limit ordinal, then $(T_\alpha, <_T)$ is the union of the trees $(T_\beta, <_T)$, $\beta < \alpha$. If α is a successor ordinal, then $(T_{\alpha+1}, <_T)$ is an extension of $(T_\alpha, <_T)$ obtained by adjoining infinitely immediate successors to each x at the top level of T_α .

It remains to describe the construction of $T_{\alpha+1}$ if α is a limit ordinal. Let $\langle S_\alpha : \alpha < \omega_1 \rangle$ be a \diamond -sequence. If S_α is a maximal antichain in $(T_\alpha, <_T)$, then we use Lemma 15.25 and find an extension $(T_{\alpha+1}, <_T)$ of T_α such that S_α is maximal in $T_{\alpha+1}$. Otherwise, we let $T_{\alpha+1}$ be any extension of T_α that is a normal $(\alpha + 1)$ -tree. (In either case, we let the set $T_{\alpha+1}$ be an initial segment of countable ordinals.)

We shall now show that the tree $T = \bigcup_{\alpha < \omega_1} T_\alpha$ is a normal Suslin tree. It suffices to verify that T has no uncountable antichain. If $A \subset T$ ($= \omega_1$) is a maximal antichain in T , then by Lemma 15.27, $A \cap T_\alpha$ is a maximal antichain in T_α , for a closed unbounded set of α 's. It follows that easily from the construction that for a closed unbounded set of α 's, $T_\alpha = \alpha$. Thus using the Diamond Principle, we find a limit ordinal α such that $A \cap \alpha = S_\alpha$ and $A \cap \alpha$ is a maximal antichain in T_α . However, we constructed $T_{\alpha+1}$ in such a way that $A \cap \alpha$ is maximal in $T_{\alpha+1}$, and therefore in T . It follows that $A = A \cap \alpha$ and so A is countable. □

Suslin trees are a fruitful source of counterexamples in set-theoretic topology as well as in the theory of Boolean algebras. As an example, let $(T, <)$ be a Suslin tree, and consider the partial ordering $(P_T, <) = (T, >)$. Any two elements of T are incomparable in T if and only if they are incompatible in P_T . Thus P_T satisfies the countable chain condition.

Lemma 15.28. *If T is a normal Suslin tree, then P_T is \aleph_0 -distributive.*

Proof. Let D_n , $n = 0, 1, 2 \dots$, be open dense subsets of P_T . We shall prove that $\bigcap_{n=0}^\infty D_n$ is dense in P_T . First we claim that if $D \subset P_T$ is open dense, then there is an $\alpha < \omega_1$ such that D contains all levels of T above α . To prove this, let A be a maximal antichain in D . A is an antichain in T and hence countable. Thus let $\alpha < \omega_1$ be such that all $a \in A$ are below level α . Now if $x \in T$ is at level $\geq \alpha$, x is comparable with some $a \in A$ (by maximality of A), and hence $a \leq_T x$. Since D is open, we have $x \in D$.

Now if D_n , $n = 0, 1, \dots$, are open dense, we pick countable ordinals α_n such that D_n contains all levels of T above α_n ; and since T is normal, this implies that $\bigcap_{n=0}^\infty D_n$ is dense in P_T . □

Corollary 15.29. *If T is a normal Suslin tree, then $B = B(P_T)$ is an \aleph_0 -distributive, c.c.c., atomless, complete Boolean algebra.* □

Random Reals

Consider the notion of forcing where forcing conditions are Borel sets of reals of positive Lebesgue measure; a condition p is stronger than q if $p \subset q$. The corresponding complete Boolean algebra is \mathcal{B}/I_μ where \mathcal{B} is the σ -algebra of all Borel sets of reals and I_μ is the σ -ideal of all null sets. As I_μ is σ -saturated, \mathcal{B}/I_μ satisfies the countable chain condition, and hence the forcing preserves cardinals.

The generic extension $V[G]$ is determined by a single real, called a *random real*. Let $a \in \mathbf{R}^{V[G]}$ be the unique member of each rational interval $[r_1, r_2]^{V[G]}$ such that $[r_1, r_2]^V \in G$. Conversely, G can be defined from a , and so $V[G] = V[a]$. (see Exercise 13.34 for the meaning of $V[a]$.)

The following lemma illustrates one of the differences between random and generic reals. If f and g are functions from ω to ω we say that g *dominates* f if $f(n) < g(n)$ for all n .

Lemma 15.30. (i) *In the random real extension $V[G]$, every $f : \omega \rightarrow \omega$ is dominated by some $g \in V$.*

(ii) *In the Cohen real extension $V[G]$, there exists a function $f : \omega \rightarrow \omega$ that is not dominated by any $g \in V$.*

Proof. (i) Forcing conditions are Borel sets of positive measure, and we freely confuse them with their equivalence classes in \mathcal{B}/I_μ .

Let $p \Vdash \dot{f} : \omega \rightarrow \omega$; we shall find a $q < p$ and some $g : \omega \rightarrow \omega$ such that q forces that g dominates \dot{f} . For each n , let $g(n)$ be sufficiently large, so that

$$\mu(p - \|\dot{f}(n) < g(n)\|) < \frac{1}{2^n} \cdot \frac{1}{4} \cdot \mu(p).$$

The Borel set $q = p \cap \bigcap_{n=0}^\infty \|\dot{f}(n) < g(n)\|$ has measure at least $\mu(p)/2$, and forces $\forall n \dot{f}(n) < g(n)$.

(ii) We use the following variant of Cohen forcing: Forcing conditions are finite sequences $\langle p(0), \dots, p(n-1) \rangle$ of natural numbers, and $p < q$ if and only if $p \supset q$. (This forcing produces the same generic extension—and has the same $B(P)$ —as the forcing from Example 14.2).

Let \dot{f} be the name for the function $f = \bigcup G$. If p is any condition and $g : \omega \rightarrow \omega$ is in V , then there exist a stronger $q \supset p$ and some $n \in \text{dom}(q)$ such that $q(n) > g(n)$. It follows that q forces $g(n) > \dot{f}(n)$ (because $q \Vdash \dot{f}(n) = q(n)$). \square

To add a large number of random reals, we use *product measure*:

Example 15.31. Let κ be an infinite cardinal and let $I = \kappa \times \omega$. Let $\Omega = \{0, 1\}^I$. Let T be the set of all finite 0–1 functions with $\text{dom}(t) \subset I$. Let \mathcal{S} be the σ -algebra generated by the sets S_t , $t \in T$, where $S_t = \{f \in \Omega : t \subset f\}$. The product measure on \mathcal{S} is the unique σ -additive measure such that each S_t has measure $1/2^{|\text{dom}(t)|}$. Let $B = \mathcal{S}/I$ where I is the ideal of measure 0 sets.

If G is a generic ultrafilter on B then $f = \bigcup\{t : S_t \in G\}$ is a 0–1 function on I , and for each $\alpha < \kappa$, we define $f_\alpha(n) = f(\alpha, n)$, for all $n < \omega$. The f_α , $\alpha < \kappa$, are κ -distinct random reals, and the continuum in $V[G]$ has size at least κ . But since $|B|^{\aleph_0} = \kappa^{\aleph_0}$, we have $(2^{\aleph_0})^{V[G]} = \kappa^{\aleph_0}$. \square

Forcing with Perfect Trees

This section describes forcing with perfect trees (due to Gerald Sacks) that produces a real of *minimal degree* of constructibility. If forced over L , the generic filter yields a real a such that $a \notin L$ and such that for every real $x \in L[a]$, either $x \in L$ or $a \in L[x]$.

Let $Seq(\{0, 1\})$ denote the set of all finite 0–1 sequences. A *tree* is a set $T \subset Seq(\{0, 1\})$ that satisfies

$$(15.24) \quad \text{if } t \in T \text{ and } s = t \upharpoonright n \text{ for some } n, \text{ then } s \in T.$$

A nonempty tree T is *perfect* if for every $t \in T$ there exists an $s \supset t$ such that both $s \frown 0$ and $s \frown 1$ are in T . (Compare with (4.4) and Lemma 4.11.) The set of all paths in a perfect tree is a perfect set in the Cantor space $\{0, 1\}^\omega$.

Definition 15.32 (Forcing with Perfect Trees). Let P be the set of all perfect trees $p \subset Seq(\{0, 1\})$; p is stronger than q if and only if $p \subset q$.

If G is a generic set of perfect trees, let

$$(15.25) \quad f = \bigcup\{s : (\forall p \in G) s \in p\}.$$

The function $f : \omega \rightarrow \{0, 1\}$ is called a *Sacks real*. Note that $V[G] = V[f]$. Since $|P| = 2^{\aleph_0}$, if we assume CH in the ground model, P satisfies the \aleph_2 -chain condition and all cardinals $\geq \aleph_2$ are preserved. We prove below that \aleph_1 is preserved as well.

Definition 15.33. A generic filter G is *minimal* over the ground model M if for every set of ordinals X in $M[G]$, either $X \in M$ or $G \in M[X]$.

Theorem 15.34 (Sacks). *When forcing with perfect trees, the generic filter is minimal over the ground model.*

The proof uses the technique of *fusion*. Let p be a perfect tree. A node $s \in p$ is a *splitting node* if both $s \frown 0 \in p$ and $s \frown 1 \in p$; a splitting node s is an n th splitting node if there are exactly n splitting nodes t such that $t \subset s$. (A perfect tree has 2^{n-1} n th splitting nodes.) For each $n \geq 1$, let

$$(15.26) \quad p \leq_n q \text{ if and only if } p \leq q \text{ and every } n\text{th splitting node of } q \text{ is an } n\text{th splitting node of } p.$$

A *fusion sequence* is a sequence of conditions $\{p_n\}_{n=0}^\infty$ such that $p_n \leq_n p_{n-1}$ for all $n \geq 1$. The following is the key property of fusion sequences:

Lemma 15.35. *If $\{p_n\}_{n=0}^\infty$ is a fusion sequence then $\bigcap_{n=0}^\infty p_n$ is a perfect tree. \square*

If s is a node in p , let $p \upharpoonright s$ denote the tree $\{t \in p : t \subset s \text{ or } t \supset s\}$. If A is a set of incompatible nodes of p and for each $s \in A$, q_s is a perfect tree such that $q_s \subset p \upharpoonright s$, then the *amalgamation* of $\{q_s : s \in A\}$ into p is the perfect tree

$$(15.27) \quad \{t \in p : \text{if } t \supset s \text{ for some } s \in A \text{ then } t \in q_s\}.$$

(Replace in p each $p \upharpoonright s$ by q_s .)

Proof of Theorem 15.34. Let \dot{X} be a name for a set of ordinals and let $p \in P$ be a condition that forces $\dot{X} \notin V$; no stronger condition forces $\dot{X} = A$, for any $A \in V$. We shall find a condition $q \leq p$ and a set of ordinals $\{\gamma_s : s$ is a splitting node of $q\}$ such that $q_{s \smallfrown 0}$ and $q_{s \smallfrown 1}$ decide $\gamma_s \in \dot{X}$, but in opposite ways. Then the generic branch (15.25) can be recovered from \dot{X}^G , and so $V[\dot{X}^G] = V[G]$.

To construct q and $\{\gamma_s\}_s$ we build a fusion sequence $\{p_n\}_{n=0}^\infty$ as follows: Let $p_0 = p$. For each $n \geq 1$, let S_n be the set of all n th splitting nodes of p_{n-1} . For each $s \in S_n$, let γ_s be an ordinal such that $p_{n-1} \upharpoonright s$ does not decide $\gamma_s \in \dot{X}$, and let $q_{s \smallfrown 0} \leq p_{n-1} \upharpoonright s \smallfrown 0$ and $q_{s \smallfrown 1} \leq p_{n-1} \upharpoonright s \smallfrown 1$ be conditions that decide $\gamma_s \in \dot{X}$ in opposite ways. Then let p_n be the amalgamation of $\{q_{s \smallfrown i} : s \in S_n$ and $i = 0, 1\}$ into p_{n-1} . Clearly, $p_n \leq_n p_{n-1}$, and so $\{p_n\}_{n=0}^\infty$ is a fusion sequence. Then we set $q = \bigcap_{n=0}^\infty p_n$. \square

A similar argument shows that forcing with perfect trees preserves \aleph_1 :

Lemma 15.36. *If X is a countable set of ordinals in $V[G]$ then there exists a set $A \in V$, countable in V , such that $X \subset A$.*

Proof. Let \dot{F} be a name and let $p \in P$ be such that p forces “ \dot{F} is a function from ω into the ordinals.” We build a fusion sequence $\{p_n\}_{n=0}^\infty$ with $p_0 = p$ as follows: For each $n \geq 1$, let S_n be the set of all n th splitting nodes of p_{n-1} . For each $s \in S_n$, let $q_{s \smallfrown 0}, q_{s \smallfrown 1}, a_{s \smallfrown 0}, a_{s \smallfrown 1}$ be such that (for $i = 0, 1$) $q_{s \smallfrown i} \leq p_{n-1} \upharpoonright s \smallfrown i$ and $q_{s \smallfrown i} \Vdash \dot{F}(n-1) = a_{s \smallfrown i}$. Let p_n be the amalgamation of $\{q_{s \smallfrown i} : s \in S_n$ and $i = 0, 1\}$. Then let $q = \bigcap_{n=0}^\infty p_n$, and

$$A = \bigcup_{n=0}^\infty \{a_{s \smallfrown i} : s \in S_n \text{ and } i = 0, 1\}.$$

It follows that $q \Vdash \text{ran}(\dot{F}) \subset A$. \square

More on Generic Extensions

Properties of a generic extensions are determined by properties of the forcing notion that constructs it. For instance, if P satisfies the countable chain condition then $V[G]$ preserves cardinals. Or, if P is ω -distributive then $V[G]$ has

no new countable sets of ordinals. But since the model $V[G]$ is determined by the complete Boolean algebra $B(P)$, its properties depend on properties of the algebra. Below we illustrate the correspondence between properties of a complete Boolean algebra B and truth in the model V^B .

The first example shows the importance of distributivity.

Let κ and λ be cardinals. A complete Boolean algebra B is (κ, λ) -distributive if

$$(15.28) \quad \prod_{\alpha < \kappa} \sum_{\beta < \lambda} u_{\alpha, \beta} = \sum_{f: \kappa \rightarrow \lambda} \prod_{\alpha < \kappa} u_{\alpha, f(\alpha)}.$$

Note that (15.28) is a special case of (7.28); B is κ -distributive if and only if it is (κ, λ) -distributive for all λ . As in Lemma 7.16 we can reformulate (κ, λ) -distributivity as follows:

Lemma 15.37. *B is (κ, λ) -distributive if and only if every collection of κ partitions of B of size at most λ has a common refinement.* \square

Theorem 15.6 and Exercise 15.5 yield the following equivalence:

Theorem 15.38. *B is (κ, λ) -distributive if and only if every $f : \kappa \rightarrow \lambda$ in the generic extension by B is in the ground model.*

Proof. If $\| \dot{f}$ is a function from κ to λ $\| = 1$, then $\{ \| \dot{f}(\alpha) = \beta \| : \beta < \lambda \}$ is a partition of B of size $\leq \lambda$. \square

Exercises 15.31 and 15.32 give short proofs of Boolean algebraic results using generic extensions.

A related concept is weak distributivity: B is called *weakly (κ, λ) -distributive*, if

$$(15.29) \quad \prod_{\alpha < \kappa} \sum_{\beta < \lambda} u_{\alpha, \beta} = \sum_{g: \kappa \rightarrow \lambda} \prod_{\alpha < \kappa} \sum_{\beta < g(\alpha)} u_{\alpha, \beta}.$$

A modification of Theorem 15.38 gives this:

Lemma 15.39. *B is weakly (κ, λ) -distributive if and only if every $f : \kappa \rightarrow \lambda$ in $V[G]$ is dominated by some $g : \kappa \rightarrow \lambda$ that is in V (i.e., $f(\alpha) < g(\alpha)$ for all $\alpha < \kappa$).* \square

Consequently, by Lemma 15.30(i), the measure algebra \mathcal{B}/I_μ is weakly (ω, ω) -distributive.

Let B be a complete Boolean algebra and let D be a complete subalgebra of B . If G is generic on B , then it is easy to see that $G \cap D$ is generic on D , and so $V[G \cap D]$ is a model of ZFC, and $V \subset V[G \cap D] \subset V[G]$. We shall prove that every model of ZFC between V and $V[G]$ is obtained this way, and that for every subset A of V in $V[G]$ there is a complete subalgebra D of B such that $V[G \cap D] = V[A]$.

We recall (cf. Chapter 7) that a complete subalgebra B of a complete Boolean algebra D is (completely) generated by a set $X \subset D$ if B is the smallest complete subalgebra of D such that $X \subset B$. Let κ be a cardinal. We say that a complete Boolean algebra B is κ -generated if there exists some $X \subset B$ of size at most κ such that the complete subalgebra of B generated by X is equal to B .

Lemma 15.40. *Let X be a subset of a complete Boolean algebra B such that B is completely generated by X . Then for every generic G on B , $V[G] = V[X \cap G]$.*

Proof. We want to show that $V[G]$ is the least model such that the set $A = X \cap G$ is in $V[G]$. It suffices to show that G can be defined in terms of A .

Since B is generated by X , every element of B can be obtained from the elements of X by successive (transfinite) application of the operation $-$ and \sum . Thus let X_α be subsets of B defined recursively as follows:

$$X_0 = X, \quad \overline{X}_\alpha = \{-a : a \in X_\alpha\}, \quad \text{and} \\ X_\alpha = \{a : a = \sum Z \text{ where } Z \subset \bigcup_{\beta < \alpha} (X_\beta \cup \overline{X}_\beta)\}.$$

Then $B = \bigcup_{\alpha < \theta} X_\alpha$ for some $\theta \leq |B|^+$. If we denote $G_\alpha = G \cap \overline{X}_\alpha$, $\overline{G}_\alpha = G \cap \overline{X}_\alpha$, we have

$$(15.30) \quad G_0 = A, \quad \overline{G}_\alpha = \{-a : a \in X_\alpha - G_\alpha\}, \quad \text{and} \\ G_\alpha = \{a \in X_\alpha : a = \sum Z \text{ where } Z \text{ contains at least one } b \\ \text{in some } G_\beta \text{ or } \overline{G}_\beta, \beta < \alpha\},$$

and $G = \bigcup_{\alpha < \theta} G_\alpha$. Thus given A , we define G_α and \overline{G}_α inductively using (15.30) and let $G = \bigcup_{\alpha < \theta} G_\alpha$. □

Corollary 15.41. *If B is κ -generated, then $V[G] = V[A]$ for some $A \subset \kappa$.* □

Corollary 15.42. *If G is generic on B and $A \in V[G]$ is a subset of κ , then there exists a κ -generated complete subalgebra D of B such that $V[D \cap G] = V[A]$ for some $A \subset \kappa$.*

Proof. Let \dot{A} be a name for A . We let $X = \{u_\alpha : \alpha < \kappa\}$, where $u_\alpha = \|\dot{A} \in A\|$. Now let D be the complete subalgebra completely generated by X ; by Lemma 15.40 we have $V[X \cap G] = V[D \cap G]$. It remains to show that $V[X \cap G] = V[A]$.

On the one hand, we have $A = \{\alpha : u_\alpha \in X \cap G\}$. On the other hand, $X \cap G = \{u_\alpha : \alpha \in A\}$. □

Lemma 15.43. *Let G be generic on B . If M is a model of ZFC such that $V \subset M \subset V[G]$, then there exists a complete subalgebra $D \subset B$ such that $M = V[D \cap G]$.*

Proof. We show that $M = V[A]$, where A is a set of ordinals. Then the lemma follows from Corollary 15.42. First we note that since M satisfies the Axiom of Choice, there exists for every $X \in M$ a set of ordinals $A_X \in M$ such that $X \in V[A_X]$. We let $Z = P(B) \cap M$, and let $A = A_Z$; we claim that $M = V[A]$.

If $X \in M$, consider the set of ordinals A_X ; by Corollary 15.42 there exists a subalgebra $D_X \subset B$ such that $V[A_X] = V[D_X \cap G]$. Hence $D_X \cap G \in M$, and we have $D_X \cap G \in Z$. Since $Z \in V[A]$, it follows that $D_X \cap G \in V[A]$ and hence $X \in V[A]$. Thus $M = V[A]$. \square

Let us now address the question under what conditions one generic extension embeds (as a submodel) into another generic extension. Of course, if $B(P) = B(Q)$, then $V^P = V^Q$ and if $B(P)$ is a complete subalgebra of $B(Q)$ then $V^P \subset V^Q$. But if B_1 is a complete subalgebra of B_2 , we can have $V[G \cap B_1] = V[G]$ even if $B_1 \neq B_2$. For every $a \in B_2^+$ (not necessarily in B_1), let $B_1 \upharpoonright a = \{x \cdot a : x \in B_1\}$. Now assume that the set $\{a \in B_2^+ : B_1 \upharpoonright a = B_2 \upharpoonright a\}$ is dense in B_2 . Then it is easy to see that $V[G \cap B_1] = V[G]$, for every generic G on B_2 . (One can show that this condition is also necessarily for B_1 to give the same generic extension as B_2 .)

By $V^P \subset V^Q$ we mean the following: Whenever G is a generic filter on Q then there is some $H \in V[G]$ that is a generic filter on P . In practice there are several ways how to verify $V^P \subset V^Q$. The following two lemmas are sometimes useful:

Lemma 15.44. *Let $i : P \rightarrow Q$ be such that*

- (i) *if $p_1 \leq p_2$ then $i(p_1) \leq i(p_2)$,*
- (ii) *if p_1 and p_2 are incompatible then $i(p_1)$ and $i(p_2)$ are incompatible,*
- (iii) *for every $q \in Q$ there is a $p \in P$ such that for all $p' \leq p$, $i(p')$ is comparable with q .*

Then $V^P \subset V^Q$.

Proof. If G is generic on Q then $i^{-1}(G)$ is generic on P . \square

Lemma 15.45. *Let $h : Q \rightarrow P$ be such that*

- (i) *if $q_1 \leq q_2$ then $h(q_1) \leq h(q_2)$.*
- (ii) *for every $q \in Q$ and every $p \leq h(q)$ there exists a q' compatible with q such that $h(q') \leq p$.*

Then $V^P \subset V^Q$.

Proof. If $D \subset P$ is open dense then $h^{-1}(D)$ is predense in Q . It follows that if G is generic on Q then $\{p \in P : p \geq h(q) \text{ for some } q \in G\}$ is generic on P . \square

We conclude this section with the following result that shows that for every set A of ordinals, the model $L[A]$ is a generic extension of HOD :

Theorem 15.46 (Vopěnka). *Let $V = L[A]$ where A is a set of ordinals. Then V is a generic extension of the model HOD . There is a Boolean algebra $B \in HOD$ complete in HOD , and there is an ultrafilter $G \subset B$, generic over HOD , such that $V = HOD[G]$.*

Proof. Let κ be such that $A \subset \kappa$. We let $C = OD \cap P(P(\kappa))$ be the family of all ordinal definable sets of subsets of κ . Let us consider the partial ordering (C, \subset) .

First we claim that there is a hereditarily ordinal definable partially ordered set (B, \leq) and an ordinal definable isomorphism π between (C, \subset) and (B, \leq) : There is a definable one-to-one mapping F of OD into the ordinals. The set C is an ordinal definable set of ordinal definable sets and so $F \upharpoonright C$ is an OD one-to-one mapping of C onto $F(C)$. We let $B = F(C)$, and define the partial ordering of B so that (B, \leq) is isomorphic to (C, \subset) . Since $\subset \cap C^2$ is an OD relation, we have $(B, \leq) \in HOD$.

Now (C, \subset) is clearly a Boolean algebra. Moreover, if $X \subset C$ is ordinal definable, then $\bigcup X$ is ordinal definable and so $\bigcup X = \sum^C X$. Hence the algebra C is OD -complete; and using the OD isomorphism π , we can conclude that (B, \leq) is a complete Boolean algebra in HOD .

Now we let $H = \{u \in C : A \in u\}$. Clearly, H is an ultrafilter on C , and if $X \subset H$ is OD , then $\bigcap X \in H$. Hence $G = \pi(H)$ is an HOD -generic ultrafilter on B .

It remains to show that $V = HOD[G]$. Let $f : \kappa \rightarrow B$ be the function defined by $f(\alpha) = \pi(\{Z \subset \kappa : \alpha \in Z\})$. Clearly, f is OD , and so $f \in HOD$. Now we note that for every $\alpha < \kappa$, $\alpha \in A$ if and only if $f(\alpha) \in G$ and therefore $A \in HOD[G]$. It follows that $V = L[A] = HOD[G]$. \square

Symmetric Submodels of Generic Models

In Chapter 14 we constructed a model of set theory in which the reals cannot be well-ordered, thus showing that the Axiom of Choice is independent of the axioms of ZF. What follows is a more systematic study of models in which the Axiom of Choice fails. We shall present a general method of construction of submodels of generic extensions. The construction uses symmetry arguments similar to those used in Theorem 14.36, and the models obtained are generally models of ZF and do not satisfy the Axiom of Choice. This method has been used to obtain a number of results about the relative strength of various weaker versions and consequences of the Axiom of Choice.

The main idea of the construction of symmetric models is the use of automorphisms of the Boolean-valued model V^B and the Symmetry Lemma 14.37. In fact, the idea of using automorphisms of the universe to show that the Axiom of Choice is unprovable dates back into the preforcing era of set theory. We shall describe this older construction first.

In order to describe this method, we introduce the theory ZFA, *set theory with atoms*. In addition to sets, ZFA has additional objects called *atoms*. These atoms do not have any elements themselves but can be collected into sets. Obviously, we have to modify the Axiom of Extensionality, for any two atoms have the same elements—none.

The language of ZFA has, in addition to the predicate \in , a constant A . The elements of A are called *atoms*; all other objects are sets. The axioms of ZFA are the axioms 1.1–1.8 of ZF plus (15.31) and (15.32):

(15.31) *If $a \in A$, then there is no x such that $x \in a$.*

The Axiom of Extensionality takes this form:

(15.32) *If two sets X and Y have the same elements, then $X = Y$.*

Other axioms of ZF remain unchanged. In particular, the Axiom of Regularity states that every nonempty set has an \in -minimal element. This minimal element may be an atom.

The effect of atoms is that the universe is no longer obtained by iterated power set operation from the empty set. In ZFA, the universe is built up from atoms.

Ordinal numbers are defined as usual except that one has to add that an ordinal does not contain any atom. For any set S , let us define the following cumulative hierarchy:

$$\begin{aligned}
 (15.33) \quad P^0(S) &= S, \\
 P^\alpha(S) &= \bigcup_{\beta < \alpha} P^\beta(S) \quad \text{if } \alpha \text{ is limit,} \\
 P^{\alpha+1}(S) &= P^\alpha(S) \cup P(P^\alpha(S)), \\
 P^\infty(S) &= \bigcup_{\alpha \in \text{Ord}} P^\alpha(S).
 \end{aligned}$$

It follows that $V = P^\infty(A)$, and that the *kernel*, the class $P^\infty(\emptyset)$ of “hereditary” sets, is a model of ZF. If A is empty, then we have just ZF.

Lemma 15.47. *The theory ZFA + AC + “ A is infinite” is consistent relative to ZFC.*

Proof. Construct a model of ZFA. Let C be an infinite set of sets of the same rank (so that $X \notin \text{TC}(Y)$ for any $X, Y \in C$). Consider one $X_0 \in C$ as the empty set, and all other $X \in C$ as atoms. Build up the model from C by iterating the operation $P^*(Z) = P(Z) - \{\emptyset\}$. \square

While in ZF, the universe does not admit nontrivial automorphisms, the important feature of ZFA is that every permutation of atoms induces an

automorphism of V : If π is a one-to-one mapping of A onto A (a *permutation* of A), then we define for every x (by \in -induction)

$$\pi(x) = \{\pi(t) : t \in x\}.$$

Clearly, π is an \in -automorphism, and we have $\pi(x) = x$ for every x in the kernel $P^\infty(\emptyset)$.

We use these automorphisms to construct transitive models of ZFA. First we point out that the analog of Theorem 13.9 is true in ZFA: If M is a transitive, almost universal class closed under Gödel operations, and if $A \in M$, then M is a model of ZFA.

Let \mathcal{G} be a group of permutations of a set S . A set \mathcal{F} of subgroups of \mathcal{G} is a *filter on \mathcal{G}* , if for all subgroups H, K of \mathcal{G} :

- (15.34) (i) $\mathcal{G} \in \mathcal{F}$;
 (ii) if $H \in \mathcal{F}$ and $H \subset K$, then $K \in \mathcal{F}$;
 (iii) if $H \in \mathcal{F}$ and $K \in \mathcal{F}$, then $H \cap K \in \mathcal{F}$;
 (iv) if $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$, then $\pi H \pi^{-1} \in \mathcal{F}$.

For a given group of permutations \mathcal{G} of the set A of atoms and a given filter \mathcal{F} on \mathcal{G} , we say that x is *symmetric* if the group

$$\text{sym}(x) = \{\pi \in \mathcal{G} : \pi(x) = x\}$$

belongs to \mathcal{F} .

Let us further assume that $\text{sym}(a) \in \mathcal{F}$ for all $a \in A$, that is, that all atoms are symmetric and let U be the class of all hereditarily symmetric objects:

(15.35) $U = \{x : \text{every } z \in \text{TC}(\{x\}) \text{ is symmetric}\}.$

The class U is called a *permutation model*. It is a transitive class and includes the kernel (because $\text{sym}(x) = \mathcal{G}$ for all $x \in P^\infty(\emptyset)$), moreover, all atoms are in U , and $A \in U$.

Lemma 15.48. *U is a transitive model of ZFA.*

Proof. We show that U is closed under Gödel operations and almost universal. It is easy to see that $G_i(\pi x, \pi y) = \pi(G_i(x, y))$ for all $i = 1, \dots, 10$, and therefore

$$\text{sym}(G_i(x, y)) \supset \text{sym}(x) \cap \text{sym}(y) \quad (i = 1, \dots, 10).$$

It follows that if x and y are hereditarily symmetric, then so is $G_i(x, y)$.

To show that U is almost universal, it suffices to verify that for each α , $U \cap P^\alpha(A)$ is symmetric. For all x and all $\pi \in \mathcal{G}$ we have $\text{rank}(\pi x) = \text{rank } x$. Also, $\text{sym}(\pi x) = \pi \cdot \text{sym}(x) \cdot \pi^{-1}$, and hence, by property (iv) in (15.34), if x is symmetric and $\pi \in \mathcal{G}$, then $\pi(x)$ is symmetric. Thus for all $\pi \in \mathcal{G}$ we have $\pi(U \cap P^\alpha(A)) = U \cap P^\alpha(A)$ and therefore, $\text{sym}(U \cap P^\alpha(A)) = \mathcal{G}$. \square

In the following examples we construct permutation models as follows: For every finite $E \subset A$, we let

$$(15.36) \quad \text{fix}(E) = \{\pi \in \mathcal{G} : \pi a = a \text{ for all } a \in E\}$$

and let \mathcal{F} be the filter on \mathcal{G} generated by $\{\text{fix}(E) : E \subset A \text{ is finite}\}$. \mathcal{F} is a filter since $\pi \cdot \text{fix}(E) \cdot \pi^{-1} = \text{fix}(\pi(E))$. Thus x is symmetric if and only if there exists a finite set of atoms E , a *support* for x , such that $\pi(x) = x$ whenever $\pi \in \mathcal{G}$ and $\pi(a) = a$ for all $a \in E$.

We shall now give two examples of permutation models.

Example 15.49. Let A be infinite, and let \mathcal{G} be the group of all permutations of A . Let \mathcal{F} be generated by $\{\text{fix}(E) : E \subset A \text{ is finite}\}$, and let U be the permutation model. In the model U the set A , although infinite, has no countable subset. Hence the Axiom of Choice fails in U .

Proof. Assume that there exists an $f \in U$ that is a one-to-one mapping of ω into A . Let E be a finite subset of A such that $\pi f = f$ for every $\pi \in \text{fix}(E)$. Since E is finite, there exists an $a \in A - E$ such that $a = f(n)$ for some n ; also, let $b \in A - E$ be arbitrary such that $b \neq a$. Now, let π be a permutation of A such that $\pi a = b$ but $\pi x = x$ for all $x \in E$. Then $\pi f = f$, and since n is in the kernel, we have $\pi n = n$. It follows that $\pi(f(n)) = (\pi f)(\pi n) = f(n)$; however, $f(n) = a$ while $\pi(f(n)) = \pi(a) \neq a$. A contradiction. \square

Example 15.50. Let A be a disjoint countable union of pairs: $A = \bigcup_{n=0}^{\infty} P_n$, $P_n = \{a_n, b_n\}$, and let \mathcal{G} be the group of all permutations of A such that $\pi(\{a_n, b_n\}) = \{a_n, b_n\}$, for all n . Let \mathcal{F} be generated by $\{\text{fix}(E) : E \subset A \text{ is finite}\}$, and let U be the permutation model. In the model U , $\{P_n : n \in \omega\}$ is a countable set of pairs and has no choice function.

Proof. Each P_n is a symmetric set since $\pi(P_n) = P_n$ for all $\pi \in \mathcal{G}$. For the same reason, $\pi\langle P_n : n \in \omega \rangle = \pi(\{(n, P_n) : n \in \omega\}) = \langle P_n : n \in \omega \rangle$, for all $\pi \in \mathcal{G}$, and so $\langle P_n : n \in \omega \rangle \in U$. Hence $S = \{P_n : n \in \omega\}$ is a countable set in U .

We show that there is no function $f \in U$ such that $\text{dom}(f) = S$ and $f(P_n) \in P_n$ for all n . Assume that f is such a function and let E be a support of f . There exists n such that neither a_n nor b_n is in E , and we let $\pi \in \mathcal{G}$ be such that $\pi(a_n) = b_n$ but $\pi x = x$ for all $x \in E$. Then $\pi f = f$, $\pi P_n = P_n$, and so $\pi(f(P_n)) = (\pi f)(\pi P_n) = f(P_n)$ but $\pi(f(P_n)) = b_n$ while $f(P_n) = a_n$; a contradiction. \square

The method of permutation models gives numerous examples of violation of the Axiom of Choice. One usually uses the set of atoms to produce a counterexample (in the permutation model) to some consequence of the Axiom of Choice, thus showing the limitations of proofs not using the Axiom of Choice. (A typical example is a vector space that has no basis, a set that cannot be linearly ordered, etc.) However, these examples do not give any information

about the “true” sets, like real numbers, sets of real numbers, etc., since those sets are in the kernel. It is clear that a different method has to be used to investigate the role of the Axiom of Choice in ZF. We shall now describe such a method and exploit the similarities between it and permutation models.

We shall use automorphisms (symmetries) to construct submodels of generic extensions. As shown in (14.36), every automorphism π of a complete Boolean algebra B induces an automorphism of the Boolean-valued model V^B . The important property of such an automorphism is (14.36) in the Symmetry Lemma 14.37:

$$\|\varphi(\pi\dot{x}_1, \dots, \pi\dot{x}_n)\| = \pi(\|\varphi(\dot{x}_1, \dots, \dot{x}_n)\|).$$

for all names $\dot{x}_1, \dots, \dot{x}_n$.

Let \mathcal{G} be a group of automorphisms of B , and let \mathcal{F} be a *filter on \mathcal{G}* , i.e., a set of subgroups that satisfies (15.34). For each $\dot{x} \in V^B$ we define its symmetry group

$$\text{sym}(\dot{x}) = \{\pi \in \mathcal{G} : \pi(\dot{x}) = \dot{x}\}.$$

If π is an automorphism of B , then

$$(15.37) \quad \text{sym}(\pi\dot{x}) = \pi \cdot \text{sym}(\dot{x}) \cdot \pi^{-1}.$$

This is because $\sigma(\pi\dot{x}) = \pi\dot{x}$ if and only if $(\pi^{-1}\sigma\pi)(\dot{x}) = \dot{x}$. Given a filter \mathcal{F} on \mathcal{G} , we call \dot{x} *symmetric* if $\text{sym}(\dot{x}) \in \mathcal{F}$. The class *HS* of *hereditarily symmetric names* is defined by induction on $\rho(\dot{x})$:

$$\text{if } \text{dom}(\dot{x}) \subset \text{HS} \text{ and if } \dot{x} \text{ is symmetric, then } \dot{x} \in \text{HS}.$$

Note that $\pi(\check{x}) = \check{x}$ for all x and all π , and so all \check{x} are in *HS*. If a name \dot{x} is symmetric, and if $\pi \in \mathcal{G}$, then by (15.37) and (15.34)(iv), $\pi(\dot{x})$ is also symmetric. It follows that $\pi\dot{x} \in \text{HS}$ whenever $\dot{x} \in \text{HS}$ and $\pi \in \mathcal{G}$.

The class *HS* is a submodel of the Boolean-valued model V^B , and can be shown to satisfy all axioms of ZF. Instead, we prove that its interpretation is a transitive model of ZF.

Thus let M be the ground model, let B be a complete Boolean algebra in M , and let \mathcal{G} and \mathcal{F} be respectively (in M), a group of automorphisms of B and a filter on \mathcal{G} . Let G be an M -generic ultrafilter on B . We let

$$(15.38) \quad N = \{\dot{x}^G : \dot{x} \in \text{HS}\}$$

be the class of all elements of $M[G]$ that have a hereditarily symmetric name. N is called a *symmetric submodel* of $M[G]$. We will prove that N is a transitive model of ZF. Before we do so, we notice that *HS* is a Boolean-valued model (with the same $\|x \in y\|$ and $\|x = y\|$ as M^B). Thus we can define $\|\varphi\|_{\text{HS}}$ for every formula φ . Note that

$$(15.39) \quad \|\exists x \varphi(x)\|_{\text{HS}} = \sum_{\dot{x} \in \text{HS}} \|\varphi(\dot{x})\|$$

and that $\|\varphi\|_{HS} = \|\varphi\|$ whenever φ is a Δ_0 formula. We also have a forcing theorem for the model N :

$$(15.40) \quad N \models \varphi(x_1, \dots, x_n) \quad \text{if and only if} \quad \|\varphi(\dot{x}_1, \dots, \dot{x}_n)\|_{HS} \in G$$

where $\dot{x}_1, \dots, \dot{x}_n \in HS$ are names for x_1, \dots, x_n . Finally, since $\pi(HS) = HS$ for all $\pi \in \mathcal{G}$, we have the Symmetry Lemma for $\|\cdot\|_{HS}$: If $\pi \in \mathcal{G}$ and $\dot{x}_1, \dots, \dot{x}_n \in HS$, then

$$(15.41) \quad \|\varphi(\pi\dot{x}_1, \dots, \pi\dot{x}_n)\|_{HS} = \pi(\|\varphi(\dot{x}_1, \dots, \dot{x}_n)\|_{HS}).$$

Lemma 15.51. *A symmetric submodel N of $M[G]$ is a transitive model of ZF, and $M \subset N \subset M[G]$.*

Proof. Since $\check{x} \in HS$ for every $x \in M$, we have $M \subset N$. The heredity of HS implies that N is transitive. To verify that the axioms of ZF hold in N , we follow closely the proof of the Generic Model Theorem. As there, we have to show that certain sets exist in the model by exhibiting names for the sets; here we have to find such names in HS .

A. Extensionality, Regularity, Infinity. These axioms hold in N since N is transitive and $N \supset M$.

B. Separation. Let φ be a formula and let

$$Y = \{x \in X : N \models \varphi(x, p)\}$$

where $X, p \in N$. Let $\dot{X}, \dot{p} \in HS$ be names for X, p . We let $\dot{Y} \in M^B$ as follows:

$$\text{dom}(\dot{Y}) = \text{dom}(\dot{X}), \quad \dot{Y}(\dot{t}) = \dot{X}(\dot{t}) \cdot \|\varphi(\dot{t}, \dot{p})\|_{HS}.$$

A routine argument shows that \dot{Y} is a name for Y ; it remains to show that \dot{Y} is symmetric.

We shall show that $\text{sym}(\dot{Y}) \supset \text{sym}(\dot{X}) \cap \text{sym}(\dot{p})$. Thus let $\pi \in \mathcal{G}$ be such that $\pi\dot{X} = \dot{X}$ and $\pi\dot{p} = \dot{p}$. For every $\dot{t} \in \text{dom}(\dot{X})$ we have $\pi\dot{t} \in \text{dom}(\pi\dot{X}) = \text{dom}(\dot{X})$ and $\dot{X}(\pi\dot{t}) = (\pi\dot{X})(\pi\dot{t}) = \pi(\dot{X}(\dot{t}))$, and $\|\varphi(\pi\dot{t}, \dot{p})\|_{HS} = \pi\|\varphi(\dot{t}, \dot{p})\|_{HS}$, and so $\dot{Y}(\pi\dot{t}) = \pi(\dot{Y}(\dot{t}))$. Therefore, $\pi\dot{Y} = \dot{Y}$.

C. Pairing, Union, Power Set. Let $X \in N$ and let $\dot{X} \in HS$ be a name for X . For the union, we let $S = \bigcup\{\text{dom}(\dot{y}) : \dot{y} \in \text{dom}(\dot{X})\}$. If $\pi \in \text{sym}(\dot{X})$ then $\pi(S) = S$ and so the set $Y = \{t^G : t \in S\}$ has a hereditarily symmetric name $\dot{Y} : \dot{Y}(\dot{t}) = 1$ for all $\dot{t} \in S$. Moreover, $Y \supset \bigcup X$.

Pairing and Power Set are handled similarly.

D. Replacement. We show that if $X \in N$, then there exists a $Y \in N$ such that for all $u \in X$, N satisfies

$$\exists v \varphi(u, v) \rightarrow (\exists v \in Y) \varphi(u, v).$$

We proceed as in (14.15) except that (we deal with $\|\cdot\|_{HS}$ instead of $\|\cdot\|$ and that) we look for $S \subset HS$ such that $\pi(S) = S$ for all $\pi \in \mathcal{G}$ (for then

$Y = \{t^G : t \in S\}$ has a name in HS). This is accomplished by taking for S the set $HS \cap M_\alpha^B$ for large enough α . Since every π preserves the rank and since each $\pi \in \mathcal{G}$ preserves HS , we have $\pi(S) = S$ for all $\pi \in \mathcal{G}$. \square

In general, the set G is not a member of N , and N does not satisfy the Axiom of Choice.

The model in Example 15.52 is due to Cohen. It is an analog of the permutation model in Example 15.49, and in fact, it is the same model that was used in Theorem 14.36.

Example 15.52. Let $V[G]$ be the generic extension adjoining countably many Cohen reals: P is the set of all finite 0–1 functions p with domain $\text{dom}(p) \subset \omega \times \omega$. We define $a_n, n \in \omega$, and $A = \{a_n : n \in \omega\}$, as well as their canonical names as in (14.40) and (14.41).

Every permutation π of ω induces an automorphism of P (and in turn an automorphism of B) by (14.44). We can view such permutations as permutations of the set $\{\dot{a}_n : n \in \omega\}$. Let \mathcal{G} be the group of all automorphisms of B that are induced by such permutations. For every finite $E \subset \omega$, let

$$\text{fix}(E) = \{\pi \in \mathcal{G} : \pi n = n \text{ for each } n \in E\},$$

and let \mathcal{F} be the filter on \mathcal{G} generated by the $\{\text{fix}(E) : E \subset \omega \text{ is finite}\}$.

Now let HS be the class of all hereditarily symmetric names, and let N be the corresponding symmetric submodel of $V[G]$. It is easy to see that all \dot{a}_n are in HS and so is \dot{A} . Moreover, the a_n are distinct subsets of ω and so A is an infinite set of reals in N .

We claim that in N , A has no countable subset. Thus assume that some $f \in N$ is a one-to-one function from ω into A . Let $\dot{f} \in HS$ and let $p_0 \in G$ be such that

$$p_0 \Vdash \dot{f} \text{ maps } \check{\omega} \text{ one-to-one into } \dot{A}.$$

The contradiction is obtained as in Lemma 14.39. We let E be a support of \dot{f} , i.e., a finite subset of ω such that $\text{sym}(\dot{f}) \supset \text{fix}(E)$. We pick $i \in \omega$ such that $i \notin E$, and find $p \leq p_0$ and $n \in \omega$ such that

$$p \Vdash \dot{f}(\check{n}) = \dot{a}_i.$$

Then we find a permutation $\pi \in \mathcal{G}$ such that:

- (i) πp and p are compatible;
- (ii) $\pi \in \text{fix}(E)$;
- (iii) $\pi i = j \neq i$.

Then $\pi \dot{f} = \dot{f}$, $\pi(\check{n}) = \check{n}$, and we have $p \cup \pi p \Vdash \dot{f}(\check{n}) = \dot{a}_i$ and $p \cup \pi p \Vdash \dot{f}(\check{n}) = \dot{a}_j$, a contradiction. \square

The set A in Example 15.52 is a set of reals and is therefore linearly ordered. Lévy proved that in the model N in Example 15.52, every set can

be linearly ordered. In fact, Halpern and Lévy proved that the model even satisfies the Prime Ideal Theorem, thus establishing the independence of the Axiom of Choice from the Prime Ideal Theorem. We note that numerous consequences of the Axiom of Choice in mathematics can be proved using the Prime Ideal Theorem instead—among others the Hahn-Banach Theorem, Compactification Theorems, the Completeness Theorem, the Tikhonov Theorem for Hausdorff spaces, etc.

Another construction of Cohen yields a model that has similar properties as the permutation model in Example 15.50. The atoms are replaced not by reals, but by sets of reals.

The similarity between permutation models and symmetric submodels is made precise by the following result that shows that every permutation model can be embedded in a symmetric model of ZF, “with a prescribed degree of accuracy.”

Theorem 15.53 (Jech-Sochor). *Let U be a permutation model, let A be its set of atoms, and let α be an ordinal. There exist a symmetric model N of ZF and an embedding $x \mapsto \tilde{x}$ of U into N such that*

$$(P_\alpha(A))^U \text{ is } \in\text{-isomorphic to } (P_\alpha(\tilde{A}))^N.$$

Proof. We work in the theory ZFA, plus the Axiom of Choice. We denote A the set of all atoms, and let M be the kernel, $M = P^\infty(\emptyset)$. We consider a group \mathcal{G} of permutations of A , and a filter \mathcal{F} on \mathcal{G} , and let U be the permutation model given by \mathcal{G} and \mathcal{F} . Let α be an ordinal number.

We shall construct a generic extension $M[G]$ of the kernel, and then the model N as a symmetric submodel of $M[G]$. We construct $M[G]$ by adjoining to M a number of subsets of a regular cardinal κ , κ of them for each $a \in A$. We use these to embed U in $M[G]$.

Let κ be a regular cardinal such that $\kappa > |P^\alpha(A)|$. The set P of forcing conditions consists of 0–1 functions p such that $|\text{dom}(p)| < \kappa$ and $\text{dom}(p) \subset (A \times \kappa) \times \kappa$; as usual, $p < q$ if and only if $p \supset q$.

Let G be an M -generic filter on P . For each $a \in A$ and each $\xi < \kappa$, we let

$$x_{a,\xi} = \{\eta \in \kappa : p(a, \xi, \eta) = 1 \text{ for some } p \in G\}.$$

Each $x_{a,\xi}$ has a canonical name $\dot{x}_{a,\xi}$:

$$\dot{x}_{a,\xi}(\check{\eta}) = \sum \{p \in P : p(a, \xi, \eta) = 1\} \quad (\eta \in \kappa).$$

Then we define, for every $a \in A$,

$$\tilde{a} = \{x_{a,\xi} : \xi < \kappa\}$$

and let $\tilde{A} = \{\tilde{a} : a \in A\}$. The sets \tilde{a} and \tilde{A} have obvious canonical names.

Having defined \tilde{a} for each $a \in A$, we can define \tilde{x} (and its canonical name $\dot{\tilde{x}}$) for each x by \in -induction:

$$(15.42) \quad \tilde{x} = \{\tilde{y} : y \in x\}.$$

We shall show that the function $x \mapsto \tilde{x}$ is an \in -isomorphism.

Lemma 15.54. *For all x and y , $x \in y$ if and only if $\tilde{x} \in \tilde{y}$, and $x = y$ if and only if $\tilde{x} = \tilde{y}$.*

Proof. First we note that $\|\dot{x}_{a,\xi} - \dot{x}_{a',\xi'}\| = 0$ whenever $(a, \xi) \neq (a', \xi')$, and that $\|\dot{x}_{a,\xi} - \dot{z}\| = 0$ for all $z \in M$. Consequently, we have $\tilde{a} \neq \tilde{b}$ whenever $a \neq b$ are atoms. We claim that for all x , $\tilde{x} \neq x_{a,\xi}$ for any a, ξ . If $x \in M$, then $\tilde{x} = x$ and so $\tilde{x} \neq x_{a,\xi}$. If $x \notin M$, then \tilde{x} is of higher rank than any $x_{a,\xi}$: $x_{a,\xi}$ is a subset of κ , while the transitive closure of \tilde{x} contains some of the $x_{a,\xi}$.

Now we can prove the lemma, simultaneously for \in and $=$, by induction on rank:

(a) If $x \in y$, then $\tilde{x} \in \tilde{y}$ follows from the definition (15.42). If $\tilde{x} \in \tilde{y}$, then y cannot be an atom because then we would have $\tilde{x} = x_{a,\xi}$ for some a, ξ , which is impossible. Hence $\tilde{x} = \tilde{z}$ for some $z \in y$ and we have $x = z$ by the induction hypothesis; thus $x \in y$.

(b) If $x = y$, then $\tilde{x} = \tilde{y}$. Conversely, if $x \neq y$, then either both x and y are atoms and then $\tilde{x} \neq \tilde{y}$; or, e.g., x contains some z that is not in y , and then, by the induction hypothesis, $\tilde{z} \in \tilde{x}$ and $\tilde{z} \notin \tilde{y}$; thus $\tilde{x} \neq \tilde{y}$. \square

Note that the proof of Lemma 15.54 does not depend on the particular G and so in fact we have proved

$$(15.43) \quad x = y \quad \text{if and only if} \quad \|\dot{x} = \dot{y}\| \neq 0 \quad \text{if and only if} \quad \|\dot{x} = \dot{y}\| = 1$$

and similarly for \in .

Now we shall construct a symmetric submodel N of $M[G]$. We construct N so that for every $x \in U$, \tilde{x} is in N and that $(P^\alpha(A))^U$ is isomorphic to $(P^\alpha(\tilde{A}))^N$. For every permutation σ of A , let $\bar{\sigma}$ be the group of all permutations π of $A \times \kappa$ such that for all a, ξ ,

$$\pi(a, \xi) = (\sigma a, \xi') \quad \text{for some } \xi'.$$

We let $\bar{H} = \bigcup\{\bar{\sigma} : \sigma \in H\}$ for every subgroup H of \mathcal{G} . Since every permutation π of $A \times \kappa$ induces an automorphism of P by

$$(\pi p)(\pi(a, \xi), \eta) = p(a, \xi, \eta) \quad (\text{all } a, \xi, \eta)$$

we consider $\bar{\mathcal{G}}$ as a group of automorphisms of $B = B(P)$. For every finite $A \subset A \times \kappa$ we let

$$\text{fix}(E) = \{\pi \in \bar{\mathcal{G}} : \pi(a, \xi) = (a, \xi) \text{ for all } (a, \xi) \in E\},$$

and we let $\bar{\mathcal{F}}$ be the filter on $\bar{\mathcal{G}}$ generated by the set

$$(15.44) \quad \{\bar{H} : H \in \mathcal{F}\} \cup \{\text{fix}(E) : E \subset A \times \kappa \text{ finite}\}.$$

Let HS be the class of all hereditarily symmetric names and let N be the corresponding symmetric submodel of $M[G]$. It is an immediate consequence of (15.44) that all $\dot{x}_{a,\xi}$, all \dot{a} ($a \in A$), and \dot{A} are symmetric, and so \dot{A} is in N . The following two lemmas show that for any x , \dot{x} is in N if and only if $x \in U$.

Lemma 15.55. *For all x , $x \in U$ if and only if $\dot{x} \in HS$.*

Proof. It suffices to show that x is symmetric if and only if \dot{x} is symmetric. If $\sigma \in \bar{\mathcal{G}}$ and $\pi \in \bar{\sigma}$, then $\pi\dot{x}$ is the canonical name for (σx) , and so $\text{sym}_{\bar{\mathcal{G}}}(\dot{x}) = \text{sym}_{\bar{\mathcal{G}}}(x)$; thus if $\text{sym}(x) \in \mathcal{F}$, then $\text{sym}_{\bar{\mathcal{G}}}(\dot{x}) \in \bar{\mathcal{F}}$. On the other hand, if $\text{sym}_{\bar{\mathcal{G}}}(\dot{x}) \in \bar{\mathcal{F}}$, then $\overline{\text{sym}(x)} \supset \bar{H} \cap \text{fix}(E)$ for some $H \in \mathcal{F}$ and a finite $E \subset A \times \kappa$. If $e = \{a \in A : (a, \xi) \in E \text{ for some } \xi\}$, then $\text{sym}(x) \supset H \cap \text{fix}(e)$, and since $\text{fix}(e) \in \mathcal{F}$, we have $\text{sym}(x) \in \mathcal{F}$. \square

Lemma 15.56. *For all x , $x \in U$ if and only if $\dot{x} \in N$.*

Proof. By Lemma 15.55, it suffices to show that if $\dot{x} \in N$, then $x \in U$. Assume otherwise, and let x be of least rank such that $\dot{x} \in N$ and $x \notin U$. Thus $x \subset U$, and since $\dot{x} \in N$, there exist a name $\dot{z} \in HS$ and some $p \in G$ such that $p \Vdash \dot{z} = \dot{x}$. Since $\text{sym}_{\bar{\mathcal{G}}}(\dot{z}) \in \bar{\mathcal{F}}$, we have $\text{sym}_{\bar{\mathcal{G}}}(\dot{z}) \supset \bar{H} \cap \text{fix}(E)$ for some $H \in \mathcal{F}$ and a finite $E \subset A \times \kappa$. We shall find $\sigma \in \bar{\mathcal{G}}$ and $\pi \in \bar{\sigma}$ such that:

- (i) πp and p are compatible;
- (ii) $\pi \in \bar{H} \cap \text{fix}(E)$;
- (iii) $\sigma x \neq x$.

Then we have $\pi\dot{z} = \dot{z}$ by (ii), $\|\pi\dot{x} = \dot{x}\| = 0$ by (iii) and (15.43); and since $\pi p \Vdash \pi\dot{z} = \pi\dot{x}$, we have

$$\pi p \cup p \Vdash \dot{z} = \dot{x}, \quad \pi p \cup p \Vdash \dot{z} = \pi\dot{x},$$

a contradiction.

To find π , note that x is not symmetric, so that there is a $\sigma \in \bar{\mathcal{G}}$ such that $\sigma x \neq x$ and $\sigma \in H \cap \text{fix}(e)$, where $e = \{a \in A : (a, \xi) \in E \text{ for some } \xi\}$. Since $|p| < \kappa$, there exists a $\gamma < \kappa$ such that $(a, \xi) \notin \text{dom}(p)$ for all $a \in A$ and all $\xi > \gamma$. Thus we define $\pi \in \bar{\sigma}$ as follows:

- if $a \in e$, then $\pi(a, \xi) = (a, \xi)$ for all ξ ;
- if $a \notin e$, then $\begin{cases} \pi(a, \xi) = \pi(\sigma a, \gamma + \xi) \text{ and } \pi(a, \gamma + \xi) = \pi(\sigma a, \xi) & \text{if } \xi < \gamma; \\ \pi(a, \xi) = (\sigma a, \xi) & \text{if } \xi > \gamma \cdot 2. \end{cases}$

It follows that $\pi \in \bar{H} \cap \text{fix}(E)$ and that p and πp are compatible. \square

We complete the proof of Theorem 15.53 by showing that

$$((P^\alpha(A))^U)^\sim = (P^\alpha(\tilde{A}))^N.$$

The left-hand side is clearly included in the right-hand side; we prove the converse by induction. Thus let $x \in P^\alpha(A) \cap U$ and let $y \in N$ be a subset of x ; we shall show that $y = \tilde{z}$ for some $z \in U$. Let \dot{y} be a name for y . The notion of forcing that we are using here is $<\kappa$ -closed; and since we have chosen κ large, it follows that there is a $p \in G$ that decides $\dot{t} \in \dot{y}$ for all $t \in x$. Hence $y = \tilde{z}$, where $z = \{t \in x : p \Vdash \dot{t} \in \dot{y}\}$, and by Lemma 15.56 we have $z \in U$. □

As for applications of Theorem 15.53, consider a formula $\varphi(X, \gamma)$ such that the only quantifiers in φ are $\exists u \in P^\gamma(X)$ and $\forall u \in P^\gamma(X)$. Let U be a permutation model such that

$$U \models \exists X \varphi(X, \gamma).$$

Let $X \in U$ be such that $U \models \varphi(X, \gamma)$; let α be such that $P^\gamma(X) \subset P^\alpha(A)$. By the theorem, U can be embedded in a model N of ZF such that $(P^\alpha(A))^U$ is isomorphic to $(P^\alpha(\tilde{A}))^N$. Since the quantifiers in φ are restricted to $P^\gamma(X)$, it follows that $N \models \varphi(\tilde{X}, \gamma)$, and so

$$N \models \exists X \varphi(X, \gamma).$$

Therefore, if we wish to prove consistency (with ZF) of an existential statement of the kind just described, it suffices to construct a permutation model (of ZFA).

Note that “ X cannot be well ordered,” “ X cannot be linearly ordered” are formulas of the above type and so is “ X is a countable set of pairs without a choice function.”

Theorem 15.53, in conjunction with the construction of permutation models, has interesting applications in algebra. One can construct various abstract counterexamples to theorems whose proofs use the Axiom of Choice. For example, one can construct a vector space that has no basis, etc.

We conclude this section by sketching two examples of models of ZF in which the Axiom of Choice fails. The first model was constructed by Feferman and Lévy, the other by Feferman.

Example 15.57. Let M be a transitive model of ZFC. There is a model $N \supset M$ such that $(\aleph_1)^N = (\aleph_\omega)^M$; hence \aleph_1 is singular in N .

Proof. First we construct a generic extension $M[G]$ by adjoining collapsing maps $f_n : \omega \rightarrow \omega_n$, for all $n \in \omega$: We let (P, \supset) consist of finite functions with domain $\subset \omega \times \omega$, such that $p(n, i) < \omega_n$ for all $(n, i) \in \text{dom}(p)$. If G is a generic filter on P , then $f = \bigcup G$ is a function on $\omega \times \omega$, and for every n ,

the function f_n defined on ω by $f_n(i) = f(n, i)$ maps ω onto ω_n . We shall construct a symmetric model $N \subset M[G]$ such that each f_n is in N but \aleph_ω is a cardinal in N .

Let \mathcal{G} be the group of all permutations π of $\omega \times \omega$ such that for every n , $\pi(n, i) = (n, j)$, for some j . Every π induces an automorphism of P by

$$\text{dom}(\pi p) = \{ \pi(n, i) : (n, i) \in \text{dom}(p) \}, \quad (\pi p)(\pi(n, i)) = p(n, i).$$

Let \mathcal{F} be the filter on \mathcal{G} generated by $\{H_n : n \in \omega\}$, where H_n consists of all π such that $\pi(k, i) = (k, i)$ for all $k \leq n$, all $i \in \omega$. Let HS be the class of all hereditarily symmetric names and let N be the symmetric model.

It is easy to verify that for each n , the canonical name \dot{f}_n of f_n is symmetric and so $f_n \in N$. To show that \aleph_ω remains a cardinal in N , we use the following lemma:

Lemma 15.58. *If $\text{sym}(\dot{x}) \supset H_n$ and $p \Vdash \varphi(\dot{x})$, then $p \upharpoonright n \Vdash \varphi(\dot{x})$, where $p \upharpoonright n$ is the restriction of p to $\{(k, i) : k \leq n\}$.*

Proof. Let us assume that $p \upharpoonright n$ does not force $\varphi(\dot{x})$ and let $q \supset p \upharpoonright n$ be such that $q \Vdash \neg \varphi(\dot{x})$. It is easy to find some $\pi \in H_n$ such that πp and q are compatible; since $\pi p \Vdash \varphi(\pi \dot{x})$ and $\pi \dot{x} = \dot{x}$, we get a contradiction. \square

Now let us assume that $g \in N$ is a function of ω onto \aleph_ω , and let \dot{g} be a symmetric name for g . Let $p_0 \in G$ be such that p_0 forces “ \dot{g} is a function from $\check{\omega}$ onto $\check{\aleph}_\omega$.” Let n be such that $p_0 \upharpoonright n = p_0$ and that $\text{sym}(\dot{g}) \supset H_n$. Since g takes \aleph_ω values, it follows that for some $k \in \omega$, there exists an incompatible set W of conditions $p \supset p_0$ such that $|W| \geq \aleph_{n+1}$, and distinct ordinals α_p , $p \in W$, such that for each $p \in W$, $p \Vdash \dot{g}(k) = \alpha_p$. By Lemma 15.58, we have $p \upharpoonright n \Vdash \dot{g}(k) = \alpha_p$, for each $p \in W$, which is a contradiction: On the one hand, the conditions $p \upharpoonright n$, $p \in W$, must be mutually incompatible, and on the other hand, the set $\{p \upharpoonright n : p \in P\}$ has size only \aleph_n . \square

If the ground model M in the above example satisfies GCH, then one can show that in N , the set of all reals is the countable union of countable sets.

Example 15.59. Let M be a transitive model of ZFC. There is a model $N \supset M$ such that in N , there is no nonprincipal ultrafilter on ω .

Proof. The model N is obtained by adjoining to M infinitely many generic reals a_n , $n < \omega$, without putting in N the set $\{a_n : n \in \omega\}$ (unlike in Example 15.52 where $\{a_n : n \in \omega\}$ is in N). First we construct $M[G]$ as in Example 15.52: (P, \supset) is the set of all finite 0-1 functions with domain $\subset \omega \times \omega$. Let G be generic and let $a_n = \{m : p(n, m) = 1 \text{ for some } p \in G\}$, for each $n \in \omega$.

Now let N be as follows. Every $X \subset \omega \times \omega$ induces a *symmetry* σ_X , an automorphism of P defined by

$$(\sigma_X p)(n, m) = \begin{cases} p(n, m) & \text{if } (n, m) \notin X, \\ 1 - p(n, m) & \text{if } (n, m) \in X. \end{cases}$$

Let \mathcal{G} be the group of all $\sigma_X, X \subset \omega \times \omega$, and let \mathcal{F} be the filter on \mathcal{G} generated by $\{\text{fix}(E) : E \subset \omega \text{ finite}\}$, where $\text{fix}(E) = \{\sigma_X : X \cap (E \times \omega) = \emptyset\}$. Let N be the symmetric model.

Let $D \in N$ be an ultrafilter on ω ; we shall show that D is principal. Let $\dot{D} \in HS$ be a name for D and let $p \in G$ be such that p forces “ \dot{D} is an ultrafilter on $\check{\omega}$.” Let $E \subset \omega$ be finite, such that $\text{sym}(\dot{D}) \supset \text{fix}(E)$, and let $n \notin E$. Then there is some $q \leq p, q \in G$, that decides $\dot{a}_n \in \dot{D}$ (where \dot{a}_n is the canonical name for a_n). For example, assume that $q \Vdash \dot{a}_n \in \dot{D}$ (the proof is similar if $q \Vdash \dot{a}_n \notin \dot{D}$).

Let m_0 be such that for all $m \geq m_0, (n, m) \notin \text{dom}(q)$, and let $X = \{(n, m) : m \geq m_0\}$. Let $\dot{b}_n = \sigma_X(\dot{a}_n)$. Since for each $m \geq m_0, \|\check{m} \in \dot{b}_n\| = -\|\check{m} \in \dot{a}_n\|$, it follows that $a_n \cap b_n$ is a finite set. However, $\sigma_X q \Vdash \sigma_X \dot{a}_n \in \sigma_X \dot{D}$; it is fairly obvious that $\sigma_X q = q$ and since $\sigma_X \in \text{fix}(E)$, we have $\sigma_X \dot{D} = \dot{D}$. Thus $q \Vdash \dot{b}_n \in \dot{D}$ and hence $a_n \cap b_n \in D$. Consequently, D is principal. \square

Exercises

15.1. If P satisfies the κ -chain condition then $|B(P)| \leq |P|^{<\kappa}$.
 [Every $u \in B^+$ is $\sum W$ for some antichain in P .]

15.2. Let P be as in (15.2) and let $Q = \{p \in P : \text{dom}(p) \text{ is an initial segment of } \kappa\}$. Then Q is dense in P and hence $B(Q) = B(P)$.

15.3. Let κ be a singular cardinal and let $(P, <)$ be defined as in (15.2). Then P collapses κ to $\text{cf}(\kappa)$: In the generic extension, there is a one-to-one function g from κ into $\text{cf}(\kappa)$.

[Let $\kappa = \aleph_\omega$, and let X be the added subset of \aleph_ω . For each $\alpha < \aleph_\omega$, let $g(\alpha) =$ the least n such that the order-type of $X \cap (\omega_{n+1} - \omega_n)$ is $\omega_n + \alpha$. Show that for every α and every $p \in P$ there is $q \supset p$ and some n such that $\text{dom}(q) \supset \omega_{n+1} - \omega_n$ and that the set $\{\xi \in \omega_{n+1} - \omega_n : q(\xi) = 1\}$ has the order-type $\omega_n + \alpha$. By the genericity of G , the function g is defined for every $\alpha < \aleph_\omega$; it is clearly one-to-one.]

15.4. Again let κ be singular, and let P be the set of all 0–1 functions whose domains are bounded subsets of κ ; P is ordered by \supset . Show that P collapses κ to $\text{cf}(\kappa)$.

15.5. If every $f : \kappa \rightarrow V$ in V^B is in the ground model, then B is κ -distributive.
 [Let $W_\alpha, \alpha < \kappa$, be partitions of B . Consider $\dot{f} \in V^B$ such that $\|\dot{f}(\alpha) = u\| = u$ for $u \in W_\alpha$, and find a common refinement of the W_α .]

15.6. If $B(P_1) = B(P_2)$ and $B(Q_1) = B(Q_2)$ then $B(P_1 \times Q_1) = B(P_2 \times Q_2)$.

15.7. $B(P \times Q)$ is the completion of the direct sum of the algebras $B(P)$ and $B(Q)$.

15.8. Let P be such that for every p there exist incompatible $q \leq p$ and $r \leq p$. Show that if $G \subset P$ then $G \times G$ is not generic on $P \times P$.

15.9. If $B(P_i) = B(Q_i)$ for each $i \in I$, then $B(P) = B(Q)$ where $P = \prod_i P_i$ and $Q = \prod_i Q_i$.

15.10. Let P be the notion of forcing (15.1) that adjoins κ Cohen reals. Then P is (isomorphic to) the product of κ copies of the forcing for adding a single Cohen real (Example 14.2).

15.11. If P satisfies c.c.c. and Q has property (K) then $P \times Q$ satisfies c.c.c.

15.12. The Singular Cardinal Hypothesis holds in Easton's model.

[If κ is singular then every $f : \text{cf}(\kappa) \rightarrow \kappa$ is in $N = V[G^{\leq \text{cf} \kappa}]$, and so if $F(\text{cf} \kappa) < \kappa$ then $(\kappa^{\text{cf} \kappa})^{V[G]} = (\kappa^{\text{cf} \kappa})^N \leq (2^\kappa)^N \leq |B(P^{\leq \text{cf} \kappa})|^\kappa = (F(\text{cf} \kappa))^\kappa = \kappa^+.$]

15.13. In (15.18), let $\kappa = \aleph_1$ and $\lambda = \aleph_\omega$. Then in $V[G]$ there is a one-to-one function $g : \aleph_\omega^{\aleph_0} \rightarrow \aleph_1$.

[If X is a countable subset of \aleph_ω , let $g(X) =$ the least α such that $f(\alpha + \omega - \alpha) = X$ (where $f = \bigcup G$ is the collapsing function). Use the fact that $X \in V$.]

15.14. In (15.18), let $\kappa = \aleph_\omega$. Then in $V[G]$ there is a one-to-one function g from λ into ω .

[Let $f = \bigcup G$, and let $g(\alpha) =$ the least n such the function $f \upharpoonright (\omega_{n+1} - \omega_n)$ is eventually equal to α .]

15.15. There is a generic extension $V[G]$ such that $V[G]$ satisfies the GCH.

[For each α , let P_α be the notion of forcing which collapses $\lambda = \beth_{\alpha+1}$ onto $\kappa = (\beth_\alpha)^+$ (see (15.18)). P_α is \beth_α -closed and satisfies the λ^+ -chain condition. Let P be an Easton product of P_α , $\alpha \in \text{Ord}$; namely, we require that $|s(p) \cap \gamma| < \gamma$ for every inaccessible $\gamma = \beth_\alpha$. Show that for each α , $\kappa = (\beth_\alpha)^+$ is a cardinal in $V[G]$, $\kappa = \aleph_{\alpha+1}^{V[G]}$, and $V[G] \models 2^{\aleph_\alpha} = \aleph_{\alpha+1}$. Apply Lemma 15.19 in two ways: (a) For each α , consider $P^{\leq \alpha} \times P^{> \alpha}$: $P^{\leq \alpha}$ satisfies the $\beth_{\alpha+1}^+$ -chain condition and $P^{> \alpha}$ is $\beth_{\alpha+1}$ -closed; (b) if α is inaccessible and $\alpha = \beth_\alpha$, consider $P^{< \alpha} \times P^{\geq \alpha}$: $P^{< \alpha}$ satisfies the \beth_α^+ -chain condition and $P^{\geq \alpha}$ is \beth_α -closed.]

15.16. Let $(P, <)$ be the notion of forcing that adds a subset of ω_1 (15.2), and let $(Q, <)$ be the notion of forcing that collapses 2^{\aleph_0} onto \aleph_1 (15.18). Then $B(P) = B(Q)$.

[Let $Q' = \{q \in Q : \text{dom}(q) \text{ is an initial segment of } \omega_1\}$; Q' is dense in Q . Show that P has a dense set P' isomorphic to Q' : Use the fact that every $p \in P$ has 2^{\aleph_0} mutually incompatible extensions.]

[Another way to show that $(P, <)$ from (15.2) adjoins a one-to-one mapping of 2^{\aleph_0} into \aleph_1 : Let $f = \bigcup G$, and for every $g \in \{0, 1\}^\omega$, let $F(g) =$ least α such that $f(\alpha + n) = g(n)$ for all n .]

15.17. Let P be the forcing that adds a subset of ω_1 , and let Q be the forcing that adds a Suslin tree as in (15.9). Then $B(P) = B(Q)$.

If T_1 and T_2 are trees, then an *isomorphism* $\pi : T_1 \rightarrow T_2$ between T_1 and T_2 is a one-to-one mapping of T_1 onto T_2 such that $x < y$ if and only if $\pi(x) < \pi(y)$. An isomorphism maps level α of T_1 onto level α of T_2 (for all α); and if b is a branch in T_1 , then $\pi(b)$ is a branch in T_2 . An *automorphism* of T is an isomorphism of T_1 onto T_2 . A tree T is *rigid* if it has no nontrivial automorphism, i.e., the only automorphism of T is the identity mapping. T is *homogeneous* if for any x, y at the same level of T , there exists an automorphism π of T such that $\pi(x) = y$.

15.18. If T is a normal α -tree where $\alpha < \omega_1$ is a limit ordinal and if π is a nontrivial automorphism of T , then T has an extension $T' \in P$ of height $\alpha + 1$ such that π cannot be extended to an automorphism of T' .

[Construct T' so that for some branch b in T , b is extended while $\pi(b)$ is not.]

15.19. The generic Suslin tree constructed in Theorem 15.23 is rigid.

[If $T \Vdash \dot{\rho}$ is a nontrivial automorphism of \mathcal{T} , then the set $\{T' \leq T : \exists \text{automorphism } \pi \text{ of an initial segment of } T' \text{ that cannot be extended to an automorphism of } T' \text{ and } T' \Vdash \pi \subset \dot{\rho}\}$ is dense below T ; a contradiction.]

If $s : \alpha \rightarrow \omega$ and $t : \alpha \rightarrow \omega$, let $s \sim t$ if and only if $s(\xi) = t(\xi)$ for all but finitely many $\xi < \alpha$.

15.20. There is a generic model $V[G]$ in which there exists a homogeneous Suslin tree.

[Let the forcing conditions be normal countable trees with the additional properties: (vi) if $t \in T$ and $s \sim t$, then $s \in T$; and (vii) if $s \in T$ and $t \in T$ are at the same level, then $s \sim t$.]

Let $(P, <)$ be the notion of forcing consisting of finite trees $(T, <_T)$ such that $T \subset \omega_1$, and such that $\alpha < \beta$ if $\alpha <_T \beta$; $(T_1, <_{T_1})$ is stronger than $(T_2, <_{T_2})$ if and only if $T_1 \supset T_2$ and $<_{T_1} = <_{T_2} \cap (T_2 \times T_2)$. If G is a generic set of conditions, then $\mathcal{T} = \bigcup\{T : T \in G\}$ is a Suslin tree. The crucial properties to verify are: (a) $(P, <)$ satisfies the countable chain condition, and (b) \mathcal{T} has no uncountable antichain:

15.21. $(P, <)$ satisfies c.c.c.

[Given an uncountable set W of conditions, use Δ -Lemma to find an uncountable $Z \subset W$ such that any $X, Y \in Z$ are compatible.]

15.22. \mathcal{T} has no uncountable antichain.

[If $T_0 \Vdash \dot{A}$ is uncountable, we first find an uncountable set W of pairs (T, α_T) such that $T \leq T_0$ and $T \Vdash \alpha_T \in \dot{A}$. By Δ -Lemma, find an uncountable $Z \subset W$ with the property that if $T_1, T_2 \in Z$, then there is T stronger than both T_1 and T_2 such that $T \Vdash \alpha_{T_1}$ is compatible with α_{T_2} . Then some $T' \leq T_0$ forces that \dot{A} is not an antichain.]

Let Q consist of all countable sequences $p = \langle S_\xi : \xi < \alpha \rangle$ ($\alpha < \omega_1$) where $S_\xi \subset \xi$ for all $\xi < \alpha$; let $p \leq q$ if and only if p extends q . Q is \aleph_0 -closed.

15.23. Let G be Q -generic. Then $V[G] \models \diamond$.

[If $p \Vdash \dot{C}$ is closed unbounded set and $\dot{X} \subset \omega_1$, find $q \leq p$ such that $q = \langle S_\xi : \xi \leq \alpha \rangle$ and $q \Vdash (\alpha \in \dot{C} \text{ and } \dot{X} \cap \alpha = S_\alpha)$.]

15.24. Let P be the forcing that adds a subset of ω_1 (15.2) and let Q be the forcing that adds a \diamond -sequence (Exercise 15.23). Then $B(P) = B(Q)$.

A purely combinatorial argument can be used to show that \diamond is equivalent to the following statement:

(\diamond') There exists a sequence of functions h_α , $\alpha < \omega_1$, such that for every $f : \omega_1 \rightarrow \omega_1$, the set $\{\alpha < \omega_1 : f \upharpoonright \alpha = h_\alpha\}$ is stationary.

15.25. $V = L$ implies \diamond' .

15.26. If $V = L$ then there exists a rigid Suslin tree.

15.27. If $V = L$ then there exists a homogeneous Suslin tree.

15.28. If T is a normal Suslin tree then $P_T \times P_T$ does not satisfy the countable chain condition.

[For each $x \in T$, pick two immediate successors p_x and q_x of x . The set $\{(p_x, q_x) : x \in T\} \subset P_T \times P_T$ is an antichain in $P_T \times P_T$.]

15.29. A Cohen-generic real is not minimal over the ground model.

[Show that P is isomorphic to $P \times P$, and therefore $V[x] = V[x_1][x_2]$, where x_1 is Cohen-generic over V and x_2 is Cohen-generic over $V[x_1]$. Consequently, $x_1 \notin V$ and $x \notin V[x_1]$.]

15.30. If a is a Sacks real, then in $V[a]$, every $f : \omega \rightarrow \omega$ is dominated by some $g : \omega \rightarrow \omega$ in the ground model.

15.31. If B is $(\kappa, 2)$ -distributive then it is $(\kappa, 2^\kappa)$ -distributive.

[Given $f : \kappa \rightarrow P(\kappa)$, consider $\{(\alpha, \beta) : \beta \in f(\alpha)\} \in P(\kappa \times \kappa)$.]

15.32. If κ is singular and B is $<\kappa$ -distributive then it is κ -distributive.

[Given a function f on κ , consider $\{f \upharpoonright \kappa_\alpha : \alpha < \text{cf } \kappa\}$.]

15.33. Let P be the forcing that adds a Cohen real. The algebra $B(P)$ is not weakly (ω, ω) -distributive.

[See Lemma 15.30(ii).]

15.34. B is weakly (ω, ω_1) -distributive if and only if ω_1 is a cardinal in $V[G]$.

15.35. If a complete Boolean algebra is κ -generated and λ -saturated, then $|B| \leq \kappa^{<\lambda}$.

15.36. Every infinite countably generated c.c.c. complete Boolean algebra has size 2^{\aleph_0} .

15.37. Show that in either Example 15.49 or 15.50, the set A cannot be linearly ordered.

Historical Notes

The forcing that adds Cohen reals is due to Cohen. Shortly after Cohen's discoveries, Solovay (in [1963]) noticed that Cohen's construction of a model for $2^{\aleph_0} = \aleph_2$ can be generalized so that for a regular cardinal κ one obtains a model of with $2^\kappa = \lambda$ (assuming $2^{<\kappa} = \kappa$ and $\lambda^\kappa = \lambda$ in the ground model).

The relation between the chain condition and preservation of cardinals is basically due to Cohen; the observation that a λ -closed notion of forcing does not produce new subsets of λ is due to Solovay. The Product Lemma 15.9 is due to Solovay [1970]. Theorem 15.15 comes from general topology and is due to Engelking and Karłowicz [1965].

Easton's Theorem (Theorem 15.18) was published in [1970]. The generalization of Cohen's method allowing a class of forcing conditions is due to Easton. The Lévy collapse (Theorem 15.22) was constructed by Lévy; cf. [1970].

Suslin's Problem was formulated by Suslin in [1920]. Tennenbaum [1968] and Jech [1967] discovered models of set theory in which a Suslin line exists; Solovay and Tennenbaum [1971] proved that existence of a Suslin line is not provable in ZFC. Subsequently, Jensen proved that a Suslin line exists in the constructible universe (cf. [1968, 1972]).

The present proof of Theorem 15.23 is as in Jech [1967] (countable conditions); Tennenbaum's proof (finite conditions) is presented in Exercises 15.21 and 15.22.

Random reals were introduced by Solovay [1970]. Forcing with perfect trees to obtain a minimal degree (Theorem 15.34) is due to Sacks [1971].

Theorem 15.46 is due to Vopěnka and appears in the book [1972] of Vopěnka and Hájek.

The idea of using symmetry arguments to construct models in which the Axiom of Choice fails goes back to Fraenkel [1922b]; the two examples of models of ZFA (an infinite set of atoms without a countable subset, and a countable set of pairs that has no choice function) are basically due to him. Further examples of permutation models were given by Mostowski who (in [1939]) developed a theory of such models. The present definition using filters was given by Specker [1957].

Cohen incorporated the symmetry arguments into his method and constructed the model in Example 15.52. The formulation of Cohen's method in terms of symmetric submodels of Boolean-valued models is due to Scott (unpublished) and Jech [1971a]; the latter's version was a reformulation of a topological version of Vopěnka and Hájek [1965].

Theorem 15.53 is due to Jech and Sochor [1966a, 1966b]. Numerous applications of the theorem are given in the second paper [1966b]. The method has been generalized by Pincus in [1971] and in [1972], extending further the analogy between permutations models of ZFA and symmetric models of ZF.

Lévy showed that in Cohen's model in Example 15.52 every set can be linearly ordered; consequently, Halpern and Lévy [1971] proved that the Prime Ideal Theorem holds in the model. Example 15.57 (singularity of \aleph_1) is due to Feferman and Lévy [1963]. Example 15.59 (independence of the Prime Ideal Theorem) is due to Feferman [1964/65]. A. Blass constructed in [1977] a model, similar to Feferman's model, in which every ultrafilter is principal.

Exercise 15.15: Jensen [1965].

Exercise 15.20: Fukson [1971].

Exercises 15.25–15.28: Jensen [1969].

The results in Exercises 15.31 and 15.32 had been known before forcing; see Sikorski [1964].