16. Iterated Forcing and Martin's Axiom

In this chapter we introduce two related concepts: iterated forcing and Martin's Axiom. Iteration of forcing is one of the basic techniques used in applications of forcing. It was first used by Solovay and Tennenbaum in their proof of the independence of Suslin's Hypothesis. The idea is to repeat the generic model construction transfinitely many times. Such iterations are described in the ground model.

Martin observed that many properties of a generic extension obtained by iteration follow from a single axiom that captures the combinatorial content of the model. The general principle has become known as Martin's Axiom. Martin's Axiom has become a favorite tool in combinatorial set theory and set-theoretic topology. Its consistency is proved by iterated forcing.

Two-Step Iteration

The basic observation is that a two-step iteration can be represented by a single forcing extension. Let P be a notion of forcing, and let $\dot{Q} \in V^P$ be a name for a partial ordering in V^P.

Definition 16.1.

(i) $P * \dot{Q} = \{(p, \dot{q}) : p \in P \text{ and } \Vdash_P \dot{q} \in \dot{Q}\}$,
(ii) $(p_1, \dot{q}_1) \le (p_2, \dot{q}_2)$ if and only if $p_1 \le p_2$ and $p_1 \Vdash \dot{q}_1 \le \dot{q}_2$.

In (i), $\Vdash_P \varphi$ means that every condition in P forces φ; equivalently, $\|\varphi\|_{B(P)} = 1$.

Theorem 16.2. (i) *Let G be a V-generic filter on P, let $Q = \dot{Q}^G$, and let H be a $V[G]$-generic filter on Q. Then*

$$G * H = \{(p, \dot{q}) \in P * \dot{Q} : p \in G \text{ and } \dot{q}^G \in H\}$$

is a V-generic filter on $P * \dot{Q}$ and $V[G * H] = V[G][H]$.

(ii) *Let K be a V-generic filter on $P * \dot{Q}$. Then*

$$G = \{p \in P : \exists \dot{q}\, (p, \dot{q}) \in K\} \quad \text{and} \quad H = \{\dot{q}^G : \exists p\, (p, \dot{q}) \in K\}$$

are, respectively, a V-generic filter on P and a $V[G]$-generic filter on $Q = \dot{Q}^G$, and $K = G * H$.

Proof. (i) Let us prove that if $D \in V$ is a dense subset of $P*Q$ then $D \cap (G*H)$ is nonempty. In $V[G]$, let

$$D_1 = \{\dot{q}^G : \exists p \in G \text{ such that } (p, \dot{q}) \in D\}.$$

The set D_1 is dense in Q; this is proved by showing that for every $\dot{q}_0$, the set (in V)

$$\{p \in P : \exists \dot{q}_1\, (p \Vdash \dot{q}_1 \leq \dot{q}_0 \text{ and } (p, \dot{q}_1) \in D)\}$$

is dense in P. Hence $D_1 \cap H \neq \emptyset$ and so there exists some $q \in H$ such that for some $p \in G$ and some G-name $\dot{q}$ for q, $(p, \dot{q}) \in D$. It follows that $(p, \dot{q}) \in D \cap (G * H)$.

(ii) Let $D \in V$ be dense in P. Then $D_1 = \{(p, \dot{q}) : p \in D\}$ is dense in $P * \dot{Q}$ and so $D \cap G$ is nonempty. Hence G is a V-generic filter on P.

Let $D \in V[G]$ be dense in Q, and let $\dot{D} \in V^P$ be a G-name for D such that $\Vdash_P \dot{D}$ is dense in $\dot{Q}$. Then the set $\{(p, \dot{q}) \in P * \dot{Q} : p \Vdash \dot{q} \in \dot{D}\}$ is dense in $P * \dot{Q}$ and it follows that $D \cap H$ is nonempty. Hence H is $V[G]$-generic.

The proof of $K = G * H$ is routine. □

We shall now describe two-step iteration in terms of complete Boolean algebras. Let B be a complete Boolean algebra and let $\dot{C} \in V^B$ be such that

$$\|\dot{C} \text{ is a complete Boolean algebra}\|_B = 1.$$

Let us consider all $\dot{c} \in V^B$ such that $\|\dot{c} \in \dot{C}\| = 1$ and the equivalence relation

$$\dot{c}_1 \equiv \dot{c}_2 \quad \text{if and only if} \quad \|\dot{c}_1 = \dot{c}_2\| = 1. \tag{16.1}$$

We let D be the set of equivalence classes for (16.1). We make D a Boolean algebra as follows: If $\dot{c}_1$ and $\dot{c}_2$ are in D, there exists a unique $\dot{c} \in D$ such that $\|\dot{c} = \dot{c}_1 +_{\dot{C}} \dot{c}_2\| = 1$; we let $\dot{c} = \dot{c}_1 +_D \dot{c}_2$. The operations $\cdot_D$ and $-_D$ are defined similarly. With these operations, D is a Boolean algebra; also,

$$\dot{c}_1 \leq_D \dot{c}_2 \quad \text{if and only if} \quad \|\dot{c}_1 \leq_{\dot{C}} \dot{c}_2\| = 1.$$

Lemma 16.3. *D is a complete Boolean algebra, and B embeds in D as a complete subalgebra.*

Proof. If $X \subset D$, let $\dot{X} \in V^B$ be such that $\operatorname{dom}(\dot{X}) = X$ and $\dot{X}(\dot{c}) = 1$ for all $\dot{c} \in X$. Since $\dot{C}$ is a complete Boolean algebra in V^B and V^B is full, there exists a $\dot{c}$ such that $\|\dot{c} = \sum_{\dot{C}} \dot{X}\| = 1$. It follows that $\dot{c} = \sum_D X$.

For each $b \in B$, let $\dot{c} = \pi(b)$ be the unique $\dot{c} \in D$ such that

$$\|\dot{c} = 1_{\dot{C}}\| = b \quad \text{and} \quad \|\dot{c} = 0_{\dot{C}}\| = -b;$$

π is a complete embedding of B into D. □

We use the notation $D = B * \dot{C}$. If $B = B(P)$ and in V^B, $\dot{C} = B(\dot{Q})$, then $P * \dot{Q}$ embeds densely in $B * \dot{C}$ (Exercise 16.1).

Two-step iteration is a generalization of product: If P and Q are two notions of forcing then $P \times Q$ embeds densely in $P * \check{Q}$ (Exercise 16.2).

If B and D are complete Boolean algebras and B is a complete subalgebra of D then there exists a $\dot{C} \in V^B$ that is a complete Boolean algebra in V^B, such that $D = B * \dot{C}$: In V^B, let $\dot{F}$ be the filter on $\check{D}$ generated by the generic ultrafilter $\dot{G}$ on $\check{B}$, and let $\dot{C}$ be the quotient of $\check{D}$ by $\dot{F}$. We denote this algebra (in V^B) $\dot{C} = D : B$. $D : B$ is a complete Boolean algebra in V^B, and $B * (D : B) = D$ (Exercises 16.3 and 16.4).

It follows that if $V[G]$ and $V[H]$ are two generic extensions of V such that $V[G] \subset V[H]$, then $V[H]$ is a generic extension of $V[G]$.

Theorem 16.4. *Let κ be a regular uncountable cardinal. If P satisfies the κ-chain condition and if in V^P, $\dot{Q}$ satisfies the κ-chain condition, then $P * \dot{Q}$ satisfies the κ-chain condition.*

Proof. Assume that $(p_\alpha, \dot{q}_\alpha)$, $\alpha < \kappa$, are mutually incompatible in $P * \dot{Q}$. Let $\dot{Z} \in V^P$ be the canonical name for the set $\{\alpha : p_\alpha \in G\}$ (where G is a generic filter on P), i.e., $\|\alpha \in \dot{Z}\| = p_\alpha$. For every α and every β, either p_α and p_β are incompatible, or every stronger condition forces that $\dot{q}_\alpha$ and $\dot{q}_\beta$ are incompatible. Thus q_α and q_β are incompatible if $\alpha \in Z$ and $\beta \in Z$, and since Q satisfies the κ-chain condition in $V[G]$, we have $|Z| < \kappa$; i.e., $\Vdash_P |\dot{Z}| < \kappa$.

Since κ is regular in $V[G]$ (by Theorem 15.3), there exists a maximal antichain $W \subset P$, and for each $p \in W$ there exists some $\gamma_p < \kappa$ such that $p \Vdash \dot{Z} \subset \gamma_p$. If we let $\gamma = \sup\{\gamma_p : p \in W\}$, we have $\gamma < \kappa$, and $\Vdash_P \dot{Z} \subset \gamma$. This is a contradiction, since $p_\gamma \Vdash \gamma \in \dot{Z}$. □

The converse of Theorem 16.4 is also true:

Lemma 16.5. *If $P * \dot{Q}$ satisfies the κ-chain condition then $\Vdash_P \dot{Q}$ satisfies the κ-chain condition.*

Of course P satisfies the κ-c.c. because $B(P)$ is a complete subalgebra of $B(P * \dot{Q})$.

Proof. Let $D = B * \dot{C}$ and assume that D satisfies the κ-chain condition. Let $\dot{W} \in V^B$ and $b_0 \in B^+$ be such that

$$b_0 \Vdash \dot{W} \text{ is a subset of } \dot{C}^+ \text{ of size } \kappa.$$

We shall find a nonzero $b \leq b_0$ such that

$$b \Vdash \dot{W} \text{ is not an antichain.} \tag{16.2}$$

Let $\dot{f} \in V^B$ be such that

$$b_0 \Vdash \dot{f} \text{ is a one-to-one function of } \kappa \text{ onto } \dot{W}.$$

For every $\alpha < \kappa$, $b_0 \Vdash (\exists x \in \dot{W})\, x = \dot{f}(\check{\alpha})$; and since V^B is full, there exists a $\dot{c}_\alpha \in D$ such that $b_0 \Vdash (\dot{c}_\alpha \in W$ and $\dot{c}_\alpha = \dot{f}(\check{\alpha}))$. Let $\dot{d}_\alpha = b_0 \cdot \dot{c}_\alpha$. Since $b_0 \Vdash \dot{c}_\alpha \neq \dot{c}_\beta$, for all $\alpha \neq \beta$, the set $\{\dot{d}_\alpha : \alpha < \kappa\}$ is a subset of D of size κ. Since D satisfies the κ-chain condition, there exist α and β such that $\dot{d}_\alpha$ and $\dot{d}_\beta$ are compatible. Hence there exists a $\dot{d} \in D^+$ such that $\dot{d} \leq \dot{d}_\alpha \cdot \dot{d}_\beta$; moreover, we can find $\dot{d}$ such that $\dot{d} = b \cdot \dot{c}$, where $0 \neq b \leq b_0$ and $b \Vdash (\dot{c} \neq 0$ and $\dot{c} \leq \dot{c}_\alpha \cdot \dot{c}_\beta)$. Now (16.2) follows. □

Corollary 16.6. *If P and Q satisfy the κ-chain condition then $P \times Q$ satisfies the κ-chain condition if and only if $\Vdash_P \check{Q}$ satisfies the κ-chain condition.* □

Lemma 16.7. *If P is κ-closed and $\Vdash_P \dot{Q}$ is κ-closed, then $P * \dot{Q}$ is κ-closed.*

Proof. Let $\lambda \leq \kappa$ and let $(p_1, \dot{q}_1) \geq (p_2, \dot{q}_2) \geq \ldots \geq (p_\alpha, \dot{q}_\alpha) \geq \ldots$ $(\alpha < \lambda)$ be a descending sequence in $P * \dot{Q}$. Then $\{p_\alpha\}_{\alpha<\lambda}$ is a descending sequence in P, and has a lower bound p. The condition p forces that $\{\dot{q}_\alpha\}_{\alpha<\lambda}$ is a descending sequence in $\dot{Q}$, and has a lower bound $\dot{q}$. Then $(p, \dot{q})$ is a lower bound of $\{(p_\alpha, \dot{q}_\alpha)\}_{\alpha<\lambda}$. □

Iteration with Finite Support

The idea of transfinite iteration of forcing is to construct sequences $\{P_\alpha\}_{\alpha<\theta}$ of forcing notions so that for every α, $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$ where $\dot{Q} \in V^{P_\alpha}$, and that at limit stages, P_α is a "limit" of $\{P_\beta\}_{\beta<\alpha}$. In this section we describe iteration with finite support, where the "limit" is the direct limit.

In Definition 16.8 below, $\dot{Q}_\alpha$ is assumed to be a forcing notion in V^{P_α}, with greatest element 1. The symbol $\leq_\alpha$ denotes the partial ordering of P_α, and $\Vdash_\alpha$ denotes the corresponding forcing relation.

Definition 16.8. Let $\alpha \geq 1$. A forcing notion P_α is an *iteration* (of *length α with finite support*) if it is a set of α-sequences with the following properties:

- (i) If $\alpha = 1$ then for some forcing notion Q_0,
  - (a) P_1 is the set of all 1-sequences $\langle p(0)\rangle$ where $p(0) \in Q_0$;
  - (b) $\langle p(0)\rangle \leq_1 \langle q(0)\rangle$ if and only if $p(0) \leq q(0)$ (in Q_0).
- (ii) If $\alpha = \beta + 1$ then $P_\beta = P_\alpha{\restriction}\beta = \{p{\restriction}\beta : p \in P_\alpha\}$ is an iteration of length β, and there is some forcing notion $\dot{Q}_\beta \in V^{P_\beta}$ such that
  - (a) $p \in P_\alpha$ if and only if $p{\restriction}\beta \in P_\beta$ and $\Vdash_\beta p(\beta) \in \dot{Q}_\beta$;
  - (b) $p \leq_\alpha q$ if and only if $p{\restriction}\beta \leq_\beta q{\restriction}\beta$ and $p{\restriction}\beta \Vdash_\beta p(\beta) \leq q(\beta)$.
- (iii) If α is a limit ordinal, then for every $\beta < \alpha$, $P_\beta = P_\alpha{\restriction}\beta = \{p{\restriction}\beta : p \in P_\alpha\}$ is an iteration of length β and
  - (a) $p \in P_\alpha$ if and only if $\forall\beta < \alpha\ p{\restriction}\beta \in P_\beta$ and for all but finitely many $\beta < \alpha$, $\Vdash_\beta p(\beta) = 1$;
  - (b) $p \leq_\alpha q$ if and only if $\forall\beta < \alpha\ p{\restriction}\beta \leq_\beta q{\restriction}\beta$.

The finite set $\{\beta < \alpha : \text{not} \Vdash_\beta p(\beta) = 1\}$ is the *support* of $p \in P_\alpha$.

An iteration with finite support is uniquely determined by the sequence $\langle \dot{Q}_\beta : \beta < \alpha \rangle$. Thus we call P_α the iteration of $\langle \dot{Q}_\beta : \beta < \alpha \rangle$. For each $\beta < \alpha$, $P_{\beta+1}$ is isomorphic to $P_\beta * \dot{Q}_\beta$. When α is a limit ordinal, $(P_\alpha, \leq_\alpha)$ is the direct limit of the P_β, $\beta < \alpha$, in the sense of Lemma 12.2. In fact $B(P_\alpha)$ is the completion of the direct limit of the $B(P_\beta)$, $\beta < \alpha$ (Exercise 16.8).

Finite support iteration preserves chain conditions:

Theorem 16.9. *Let κ be a regular uncountable cardinal. Let P_α be the iteration with finite support of $\langle \dot{Q}_\beta : \beta < \alpha \rangle$, such that for each $\beta < \alpha$, $\Vdash_\beta \dot{Q}_\beta$ satisfies the κ-chain condition. Then P_α satisfies the κ-chain condition.*

Proof. By induction on α. If $\alpha = \beta + 1$ then $P_\alpha = P_\beta * \dot{Q}_\beta$ and the assertion follows from Theorem 16.4. Thus let α be a limit ordinal. For each $p \in P_\alpha$, let $s(p)$ denote the support of p.

Let $W = \{p_\xi : \xi < \kappa\}$ be a subset of P_α of size κ. If $\operatorname{cf} \alpha \neq \kappa$ then there exist a $\beta < \alpha$ and some $Z \subset W$ of size κ such that $s(p) \subset \beta$ for each $p \in Z$. Then $\{p \restriction \beta : \beta \in Z\} \subset P_\beta$ and since P_β satisfies the κ-chain condition, there exist p and q in Z such that $p \restriction \beta$ and $q \restriction \beta$ are compatible (in P_β). Since $s(p) \subset \beta$ and $s(q) \subset \beta$, p and q are compatible.

Thus assume that $\operatorname{cf} \alpha = \kappa$, and let $\{\alpha_\xi : \xi < \kappa\}$ be a normal sequence with limit α. Let $C \subset \kappa$ be the closed unbounded set of all η such that $s(p_\xi) \subset \alpha_\eta$ for all $\xi < \eta$. For each limit $\xi \in C$ there is some $\gamma(\xi) < \xi$ such that $s(p_\xi) \cap \alpha_\xi \subset \alpha_{\gamma(\xi)}$. By Fodor's Theorem there exist a stationary set $S \subset C$ and some $\gamma < \kappa$ such that $s(p_\xi) \cap \alpha_\xi \subset \alpha_\gamma$ for all $\xi \in S$.

Now consider the set $\{p_\xi \restriction \alpha_\gamma : \xi \in S\}$. This is a subset of P_{α_γ}, of size κ, and therefore there exist ξ and η in S, $\gamma < \xi < \eta$, such that $p_\xi \restriction \alpha_\gamma$ and $p_\eta \restriction \alpha_\gamma$ are compatible. Let $q \in P_{\alpha_\gamma}$ be a condition stronger than both $p_\xi \restriction \alpha_\gamma$ and $p_\eta \restriction \alpha_\gamma$, and consider the following α-sequence r:

$$r(\beta) = \begin{cases} q(\beta) & \text{if } \beta < \alpha_\gamma, \\ p_\xi(\beta) & \text{if } \alpha_\gamma \leq \beta < \alpha_\eta, \\ p_\eta(\beta) & \text{if } \alpha_\eta \leq \beta < \alpha. \end{cases} \tag{16.3}$$

It is easily verified that r is a condition in P_α and is stronger than both p_ξ and p_η. Thus p_ξ and p_η are compatible, and W is not an antichain. □

Theorem 16.9 gives the following corollary for complete Boolean algebras:

Corollary 16.10. *Let $B_0 \subset B_1 \subset \ldots \subset B_\beta \subset \ldots$ $(\beta < \alpha)$ be a sequence of complete Boolean algebras such that for all $\beta < \gamma$, B_β is a complete subalgebra of B_γ, and that for each limit ordinal γ, $\bigcup_{\beta<\gamma} B_\beta$ is dense in B_γ. If every B_β satisfies the κ-chain condition then $\bigcup_{\beta<\alpha} B_\beta$ satisfies the κ-chain condition.* □

Martin's Axiom

Definition 16.11 (Martin's Axiom (MA)). If $(P, <)$ is partially ordered set that satisfies the countable chain condition and if $\mathcal{D}$ is a collection of fewer than $2^{\aleph_0}$ dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.

By Lemma 14.4, if $(P, <)$ is any partial ordering and if $\mathcal{D}$ is a countable collection of dense subsets of P, then a $\mathcal{D}$-generic filter on P exists. Hence Martin's Axiom is a consequence of the Continuum Hypothesis. Exercises 16.10 and 16.11 show that the restriction to fewer than continuum dense sets as well as some restriction on $(P, <)$ are necessary.

If κ is an infinite cardinal, let MA_κ be the statement

(16.4) If $(P, <)$ is a partially ordered set that satisfies the countable chain condition, and if $\mathcal{D}$ is a collection of at most κ dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.

$\mathrm{MA}_{\aleph_0}$ is true by Lemma 14.4, and Martin's Axiom states that MA_κ holds for all $\kappa < 2^{\aleph_0}$. Exercise 16.10 shows that MA_κ implies that $\kappa < 2^{\aleph_0}$.

Lemma 16.12. *Martin's Axiom is equivalent to its restriction to partial orders of cardinality* $< \mathfrak{c}$:

(16.5) *If* $(P, <)$ *is a partially ordered set that satisfies the countable chain condition and* $|P| < 2^{\aleph_0}$, *and if* $\mathcal{D}$ *is a collection of at most* κ *dense subsets of* P, *then there exists a* $\mathcal{D}$*-generic filter on* P.

Proof. Let P be a c.c.c. partially ordered set and let us assume that (16.5) holds. Let $\mathcal{D}$ be a family of fewer than $\mathfrak{c}$ dense subsets of P. For each $D \in \mathcal{D}$, we let W_D be a maximal incompatible subset of D. Since each W_D is countable, there exists a set $Q \subset P$ of size $< \mathfrak{c}$ such that $W_D \subset Q$ for all $D \in \mathcal{D}$, and if $p, q \in Q$ are compatible, then there exists some $r \in Q$ such that $r \leq p$ and $r \leq q$. Each W_D is a maximal antichain in Q; let $E_D = \{q \in Q : q \leq w \text{ for some } w \in W_D\}$. Each E_D is dense in Q.

The partially ordered set Q has size at most κ and satisfies the countable chain condition. By (16.5) there is a filter G on Q that meets every E_D. G generates a $\mathcal{D}$-generic filter on P. □

We will now show that MA is consistent with $2^{\aleph_0} > \aleph_1$:

Theorem 16.13 (Solovay and Tennenbaum). *Assume* GCH *and let* κ *be a regular cardinal greater than* $\aleph_1$. *There exists a c.c.c. notion of forcing* P *such that the generic extension* $V[G]$ *by* P *satisfies Martin's Axiom and* $2^{\aleph_0} = \kappa$.

As P satisfies the countable chain condition, the model $V[G]$ preserves cardinals and cofinalities.

Proof. We construct P as a finite support iteration of length κ, of a certain (yet to be determined) sequence $\langle \dot{Q}_\alpha : \alpha < \kappa \rangle$. At each stage, we'll have $\Vdash_\alpha \dot{Q}_\alpha$ satisfies the countable chain condition, and so P will satisfy c.c.c. as well. We shall also have, for each $\alpha < \kappa$, $\Vdash_\alpha |\dot{Q}_\alpha| < \kappa$. It follows, by induction on α, that $|P_\alpha| \le \kappa$ for every $\alpha \le \kappa$: If α is a limit ordinal and if $|P_\beta| \le \kappa$ for all $\beta < \alpha$, then $|P_\alpha| \le \kappa$ since the elements of P_α are α-sequences with finite support. Thus assume that $|P_\alpha| \le \kappa$ and let us prove $|P_{\alpha+1}| \le \kappa$. Because P_α satisfies c.c.c. and κ is regular, there exists a $\lambda < \kappa$ such that $\Vdash_\alpha |\dot{Q}_\alpha| \le \lambda$. Every name $\dot{q}$ for an element of $\dot{Q}_\alpha$ can be represented by a function from an antichain in P_α into λ. As every antichain in P_α is countable, the number of such functions is at most $\kappa^{\aleph_0}$ which is κ (by GCH). It follows that $|P_{\alpha+1}| \le \kappa$; in fact $|B(P_{\alpha+1})| \le \kappa$.

Note that because GCH holds in V, and because P_α is a c.c.c. forcing of size $\le \kappa$, we have $\Vdash_\alpha 2^\lambda \le \kappa$, for every $\lambda < \kappa$. In particular, $\Vdash_P 2^{\aleph_0} \le \kappa$.

We shall now define the $\dot{Q}_\alpha$, by induction on $\alpha < \kappa$. Let us fix a function π that maps κ onto $\kappa \times \kappa$ such that if $\pi(\alpha) = (\beta, \gamma)$ then $\beta \le \alpha$. For every $\alpha < \kappa$, the model V^{P_α} has at most κ nonisomorphic partial orderings of size $< \kappa$ (because $\Vdash_\alpha \kappa^{<\kappa} = \kappa$). Since P_α satisfies c.c.c., there are at most κ distinct names in V^{P_α} for such partial orderings.

Thus let us assume that $\alpha < \kappa$ and that $\langle \dot{Q}_\beta : \beta < \alpha \rangle$ has been defined. Let $\pi(\alpha) = (\beta, \gamma)$. Let $\dot{Q}$ be the γth name in V^{P_β} for a partial order with a greatest element 1, of size $< \kappa$. Let $b = \|\dot{Q}$ satisfies the countable chain condition$\|_{P_\alpha}$ and let $\dot{Q}_\alpha \in V^{P_\alpha}$ be such that $\|\dot{Q}_\alpha = \dot{Q}\|_{P_\alpha} = b$ and $\|\dot{Q}_\alpha = \{1\}\|_{P_\alpha} = -b$.

Now let P be the finite support iteration of $\langle \dot{Q}_\alpha : \alpha < \kappa \rangle$. We shall prove that V^P satisfies Martin's Axiom as well as $2^{\aleph_0} = \kappa$. Let G be a generic filter on P, and let $G_\alpha = G \restriction P_\alpha$ for all $\alpha < \kappa$.

Lemma 16.14. *If $\lambda < \kappa$ and $X \subset \lambda$ is in $V[G]$ then $X \in V[G_\alpha]$ for some $\alpha < \kappa$.*

Proof. Let $\dot{X}$ be a name for X. Every Boolean value $\|\xi \in \dot{X}\|$ (where $\xi < \lambda$) is determined by a countable antichain in P and hence $\dot{X}$ is determined by at most λ conditions in P. Every condition has finite support which in turn is included in some $\alpha < \kappa$. Therefore there exists some $\alpha < \kappa$ such that all these λ conditions have support included in α. It follows that X has a name in V^{P_α}; hence $X \in V[G_\alpha]$. □

Lemma 16.15. *Let $(Q, <) \in V[G]$ and $\mathcal{D} \in G$ be such that $(Q, <)$ is a c.c.c. partial order, $|Q| < \kappa$ and $|\mathcal{D}| < \kappa$. There exists in $V[G]$ a $\mathcal{D}$-generic filter on Q.*

Once we prove Lemma 16.15, we finish the proof of Theorem 16.13 as follows: Let Q be the forcing for adding one Cohen generic real; Q is countable. For any set $X \subset \{0,1\}^\omega$ of size $< \kappa$, let $\mathcal{D}_X = \{D_g : g \in X\}$ where $D_g = \{q \in Q : q \not\subset g\}$ (see Exercise 16.10). Lemma 16.15 applied to $\mathcal{D}_X$ shows that $X \neq \{0,1\}^\omega$ and therefore $V[G]$ satisfies $2^{\aleph_0} \geq \kappa$. However, we already proved that $2^{\aleph_0} \leq \kappa$, and so $V[G] \models 2^{\aleph_0} = \kappa$. Thus $V[G]$ satisfies (16.5) and therefore MA, completing the proof. □

Proof of Lemma 16.15. By Lemma 16.14, both $(Q, <)$ and $\mathcal{D}$ are in $V[G_\beta]$, for some $\beta < \kappa$. Let $\dot{Q}$ be a name for Q in V^{P_β}. We may assume that Q has a greatest element, and let γ be such that $\dot{Q}$ is the γth name for such partial order. Let α be such that $\pi(\alpha) = (\beta, \gamma)$. As Q satisfies the countable chain condition in $V[G]$, it also satisfies the countable chain condition in $V[G_\alpha]$. Thus $Q = \dot{Q}_\alpha^{G_\alpha}$.

In $V[G_{\alpha+1}]$ there is a generic filter H on Q over $V[G_\alpha]$, because $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$. The filter H meets every dense subset of Q that is in $V[G_\alpha]$, and therefore it meets every $D \in \mathcal{D}$. Hence H is $\mathcal{D}$-generic. □

Independence of Suslin's Hypothesis

Suslin's Hypothesis (SH) is the statement there are no Suslin lines. In Chapter 15 we showed that the negation of SH is consistent; by the following theorem, SH is independent.

Theorem 16.16. *If* $\text{MA}_{\aleph_1}$ *holds, then there is no Suslin tree.*

Proof. Let us assume that T is a normal Suslin tree and let P_T be the partially ordered set obtained from T by reversing the order. P_T satisfies the countable chain condition. For each $\alpha < \omega_1$, let D_α be the union of all levels above α: $D_\alpha = \{x \in T : o(x) > \alpha\}$. Each D_α is dense in P_T; if we let $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\}$ and if G is a $\mathcal{D}$-generic filter on P, then G is a branch in T of length ω_1. A contradiction. □

The proof of independence of SH was the first application of iterated forcing (and led to the formulation of Martin's Axiom). The model for SH, due to Solovay and Tennenbaum [1971], was constructed by iteration of the forcing notions P_T, for all prospective Suslin trees in the final model. The forcing P_T "kills" the Suslin tree T by forcing an ω_1-branch in T.

In the proof of the following theorem, Suslin trees are killed by a different method: by specializing the tree. Recalling the definition in Chapter 9 (and Exercise 9.9), an Aronszajn tree T is *special* if there exists a function $f : T \to \omega$ such that each $f_{-1}(\{n\})$ is an antichain.

Theorem 16.17 (Baumgartner, Malitz, and Reinhardt [1970]). *If* $\text{MA}_{\aleph_1}$ *holds, then every Aronszajn tree is special.*

Lemma 16.18. *If T is an Aronszajn tree and W is an uncountable collection of finite pairwise disjoint subsets of T, then there exist $S, S' \in W$ such that any $x \in S$ is incomparable with any $y \in S'$.*

Proof. Since uncountably many elements of W have the same size, we may as well assume that there exists a natural number n such that $|S| = n$ for all $S \in W$; furthermore let us consider a fixed enumeration $\{z_1, \ldots, z_n\}$ of each set $S \in W$. Let D be an ultrafilter on W such that every $X \in D$ is uncountable.

Let us assume that the lemma is false. For each $x \in T$ and each $k = 1, \ldots, n$, let $Y_{x,k}$ be the set of all $S \in W$ such that x is comparable with the kth element of S. Since any S and S' contain comparable elements, we have

$$\bigcup_{x \in S} \bigcup_{k=1}^{n} Y_{x,k} = W$$

for every $S \in W$. Thus pick, for each $S \in W$, an element $x = x_S$ of S and $k = k_S$ such that $Y_{x,k} \in D$. Now, there is $k \leq n$ such that the set $Z = \{S \in W : k_S = k\}$ is uncountable. We shall show that the elements x_S, $S \in Z$, are pairwise comparable; and that will be a contradiction since T has no uncountable branch.

If $S_1, S_2 \in Z$ and $x = x_{S_1}, y = x_{S_2}$, then $Y = Y_{x,k} \cap Y_{y,k}$ is in the ultrafilter and thus uncountable. If $S \in Y$, then the kth element of S is comparable with both x and y. Since Y is uncountable, there must exist $S \in Y$ such that the kth element of S is greater than both x and y. But then it follows that x and y are comparable. □

Let T be an Aronszajn tree and let us consider the following notion of forcing $(P, <)$: Forcing conditions are functions p such that

(16.6)
(i) $\mathrm{dom}(p)$ is a finite subset of T;
(ii) $\mathrm{ran}(p) \subset \omega$;
(iii) if $x, y \in \mathrm{dom}(p)$ and x and y are comparable, then $p(x) \neq p(y)$;
(iv) p is stronger than q if and only if p extends q.

Lemma 16.19. *$(P, <)$ satisfies the countable chain condition.*

Proof. Let W be an uncountable subset of P. Note that the set $\{\mathrm{dom}(p) : p \in W\}$ is uncountable (there are only countably many functions from a finite set into ω). By Δ-Lemma, there is an uncountable $W_1 \subset W$, and a finite set $S \subset T$ such that $\mathrm{dom}(p) \cap \mathrm{dom}(q) = S$ for any distinct elements $p, q \in W_1$. Then there is an uncountable $W_2 \subset W_1$ such that $p \restriction S = q \restriction S$ for any $p, q \in W_2$. By Lemma 16.18 there exist p and $q \in W_2$ such that any $x \in \mathrm{dom}(p) - S$ is incomparable with any $y \in \mathrm{dom}(q) - S$. Then $p \cup q$ is a function that satisfies (16.6) and extends both p and q. Thus p and q are compatible elements of W and so $(P, <)$ satisfies the countable chain condition. □

Proof of Theorem 16.17. For each $x \in T$, let D_x be the set of all $p \in P$ such that $x \in \mathrm{dom}(p)$; clearly, each D_x is dense in P. Let $\mathcal{D} = \{D_x : x \in T\}$.

It follows from $\mathrm{MA}_{\aleph_1}$, that $(P,<)$ has a $\mathcal{D}$-generic filter G. The elements of G are pairwise compatible and since G is $\mathcal{D}$-generic, every $x \in T$ is in the domain of the function $f = \bigcup G$. The function f maps T into ω and witnesses that T is a special Aronszajn tree. $\square$

More Applications of Martin's Axiom

Theorem 16.20 (Martin-Solovay). *Martin's Axiom implies that $\mathfrak{c}$ is regular, and $2^\kappa = \mathfrak{c}$ for all infinite cardinals $\kappa < \mathfrak{c}$.*

Proof. Assuming MA, we prove that $2^\kappa = 2^{\aleph_0}$ for every $\kappa < 2^{\aleph_0}$. Regularity of $\mathfrak{c}$ follows, as $\operatorname{cf} 2^{\aleph_0} = \operatorname{cf} 2^\kappa > \kappa$ for all $\kappa < 2^{\aleph_0}$. Let $\kappa < 2^{\aleph_0}$ and let $\{A_\alpha : \alpha < \kappa\}$ be an almost disjoint family of subsets of ω.

Let X be a subset of κ. We shall find a set $A \subset \omega$ such that for all $\alpha < \kappa$

$$\alpha \in X \quad \text{if and only if} \quad A \cap A_\alpha \text{ is infinite.} \tag{16.7}$$

In other words, $X = \{\alpha \in \kappa : A \cap A_\alpha \text{ is infinite}\}$ is "coded" by the set A. Therefore there exists a mapping of $P(\omega)$ onto $P(\kappa)$, and so $2^\kappa \le 2^{\aleph_0}$.

Let $(P,<)$ be the following notion of forcing: A condition is a function p from a subset of ω into $\{0,1\}$ such that:

(16.8) (i) $\operatorname{dom}(p) \cap A_\alpha$ is finite for every $\alpha \in X$;
(ii) $\{n : p(n) = 1\}$ is finite.

The set P is partially ordered by reverse inclusion: $p \le q$ if and only if p extends q.

We first show that P satisfies the countable chain condition. If p and q are incompatible, then $\{n : p(n) = 1\} \ne \{n : q(n) = 1\}$ and since there are only countably many finite subsets of ω, it follows that P satisfies c.c.c.

For each $\beta \in \kappa - X$, let $D_\beta = \{p \in P : A_\beta \subset \operatorname{dom}(p)\}$. Any $q \in P$ can be extended to some $p \in D_\beta$: Simply let $p(n) = 0$ for all $n \in A_\beta - \operatorname{dom}(p)$. Since A_β is almost disjoint from all A_α, $\alpha \in X$, p has property (16.8)(i) and hence is a condition. Thus each D_β is dense.

For each $\alpha \in X$ and each $k \in \omega$, let

$$E_{\alpha,k} = \{p \in P : \{n \in A_\alpha : p(n) = 1\} \text{ has size at least } k\}.$$

It is easy to see that each $E_{\alpha,k}$ is dense in P.

Let $\mathcal{D}$ be the collection of all D_β for $\beta \in \kappa - X$ and all $E_{\alpha,k}$ for $\alpha \in X$ and $k \in \omega$. By MA, there exists a $\mathcal{D}$-generic filter G on P. Note that $f = \bigcup G$ is a function on a subset of ω. We let

$$A = \{n : f(n) = 1\} = \{n : p(n) = 1 \text{ for some } p \in G\}. \tag{16.9}$$

If $\alpha \in X$, then $A \cap A_\alpha$ is infinite because for each k there is some $p \in G \cap E_{\alpha,k}$. If $\beta \in \kappa - X$, then $A \cap A_\beta$ is finite because for some $p \in G$, $A_\beta \subset \operatorname{dom}(p)$ and $\{n : p(n) = 1\}$ is finite. $\square$

The *almost disjoint forcing* defined in the proof of Theorem 16.20 is often used to code generically uncountable sets. A typical application is the following:

Let $V[X]$ be a generic extension where $X \subset \omega_1$; furthermore, assume that $\omega_1^{V[X]} = \omega_1$. Let $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ be an almost disjoint family in V, and let us consider the almost disjoint forcing P in $V[X]$. If $G \subset P$ is generic over $V[X]$, then $V[X][G] = V[X][A]$, where A is defined by (16.9). Note that $\omega_1^{V[X][A]} = \omega_1$.

Now in $V[X][A]$, the set X satisfies (16.7), and since $\mathcal{A} \in V$, it follows that $X \in V[A]$, and we have $V[X][A] = V[A]$. Thus we have found a generic extension $V[A]$ such that $A \subset \omega$ and $X \in V[A]$. See Exercise 16.15.

The next theorem shows that under $\mathrm{MA}_{\aleph_1}$, countable chain condition is preserved by products. Compare with Exercise 15.28.

Theorem 16.21. $\mathrm{MA}_{\aleph_1}$ *implies that every partially ordered set that satisfies the countable chain condition has property* (K).

Proof. Let P be a partially ordered set that satisfies the countable chain condition and let $W = \{w_\alpha : \alpha < \omega_1\}$ be an uncountable subset of P. We will use $\mathrm{MA}_{\aleph_1}$ to find a filter G such that $Z = G \cap W$ is uncountable.

First we claim that there is some $p_0 \in W$ such that every $p \le p_0$ is compatible with uncountably many w_α. Otherwise, for each $\alpha < \omega_1$ there is $\beta > \alpha$ and some $v_\alpha \le w_\alpha$ which is incompatible with all w_γ, $\gamma \ge \beta$; then we can construct an ω_1-sequence $\{v_{\alpha_i} : i < \omega_1\}$ of pairwise incompatible elements.

For each $\alpha < \omega_1$, let

$$D_\alpha = \{p \le p_0 : p \le w_\gamma \text{ for some } \gamma \ge \alpha\}.$$

By the above claim, each D_α is dense below p_0. By $\mathrm{MA}_{\aleph_1}$, there exists a filter G on P such that $p_0 \in G$ and $G \cap D_\alpha \ne \emptyset$ for all $\alpha < \omega_1$. It follows that $G \cap W$ is uncountable. Hence P has property (K). □

Corollary 16.22. $\mathrm{MA}_{\aleph_1}$ *implies that if every* P_i, $i \in I$, *satisfies the countable chain condition then so does the product* $\prod_{i \in I} P_i$ (*with finite support*).

Proof. Theorem 15.15. □

The next result generalizes the Baire Category Theorem:

Theorem 16.23. *Martin's Axiom implies that the intersection of fewer than* $\mathfrak{c}$ *dense open sets of reals is dense.*

Proof. Let $\kappa < \mathfrak{c}$ and let U_α, $\alpha < \kappa$, be dense open sets of reals. Let I be a bounded open interval. We'll show that $\bigcap_{\alpha<\kappa} U_\alpha \cap I \ne \emptyset$. Let P be the following notion of forcing: Conditions are nonempty open sets p such that $\bar{p} \subset I$, with $p \le q$ if and only if $p \subset q$. Since every collection of disjoint open sets is at most countable, P satisfies the countable chain condition. For each $\alpha < \kappa$, let $D_\alpha = \{p \in P : \bar{p} \subset U_\alpha\}$; each D_α is dense in P. Let G be a $\mathcal{D}$-generic filter on P where $\mathcal{D} = \{D_\alpha : \alpha < \kappa\}$. Since G is a filter, the intersection $\bigcap\{\bar{p} : p \in G\}$ is nonempty, and is contained in each U_α since $G \cap D_\alpha \neq \emptyset$. □

If f and g are functions from ω to ω, we say that f *eventually dominates* g if $f(n) > g(n)$ for all but finitely many $n \in \omega$ (i.e., $f > g$ in the notation of Lemma 10.16). A set of functions $\mathcal{G}$ is eventually dominated by f if $f > g$ for all $g \in \mathcal{G}$.

Theorem 16.24. *Martin's Axiom implies that every family $\mathcal{G}$ of fewer than $\mathfrak{c}$ functions from ω to ω is eventually dominated by some $f \in \omega^\omega$.*

Corollary 16.25. MA *implies that there exists a $\mathfrak{c}$-scale.*

Proof. A scale is constructed by transfinite induction, using an enumeration of ω^ω of order-type $\mathfrak{c}$. □

Corollary 16.26. MA *implies that $\mathfrak{c}$ is not real-valued measurable.*

Proof. Lemma 10.16. □

The proof of Theorem 16.24 uses the *Hechler forcing*: Let $\mathcal{G}$ be a given family of functions $h : \omega \to \omega$. A forcing condition is a pair $p = (s, E)$, where $s = \langle s(0), \ldots, s(n-1)\rangle$ is a finite sequence of natural numbers and E is a finite subset of $\mathcal{G}$. A condition (s', E') is stronger than (s, E) if:

(16.10) (i) $s \subset s'$, and $E \subset E'$;
(ii) if $k \in \operatorname{dom}(s') - \operatorname{dom}(s)$, then $s(k) > h(k)$ for all $h \in E$.

If (s_1, E_1) and (s_2, E_2) are conditions and $s_1 = s_2$, then (s_1, E_1) and (s_2, E_2) are compatible. Hence $(P, <)$ satisfies the countable chain condition. Let $G \subset P$ be generic; we let $f = \bigcup\{s : (s, E) \in G \text{ for some } E\}$. We claim that $\mathcal{G}$ is eventually dominated by f. Let $h \in \mathcal{G}$ be arbitrary. First there is a condition $(s, E) \in G$ such that $h \in E$ (by genericity). Secondly, every condition $(s', E') < (s, E)$ satisfies (16.10)(ii), and so $f(k) > g(k)$ for all $k \notin \operatorname{dom}(s)$. Thus in $V[G]$, there is $f : \omega \to \omega$ such that $h < f$ for all $h \in \mathcal{G}$.

Proof of Theorem 16.24. If $\mathcal{G} \subset \omega^\omega$ and $|\mathcal{G}| < \mathfrak{c}$, let P be the Hechler forcing for the family $\mathcal{G}$. Let $\mathcal{D} = \{D_h : h \in \mathcal{G}\} \cup \{E_n : n \in \omega\}$ where $D_h = \{(s, E) : h \in E\}$ and $E_n = \{(s, E) : n \in \operatorname{dom}(s)\}$. Then if G is a $\mathcal{D}$-generic filter, the function $f = \bigcup\{s : (s, E) \in G \text{ for some } E\}$ eventually dominates all $h \in \mathcal{G}$. □

Theorem 16.27 (Booth). *Martin's Axiom implies that there exists a p-point.*

Proof. Let $\mathcal{A}_\alpha$, $\alpha < 2^{\aleph_0}$, be an enumeration of all decreasing sequences $\{A_n\}_{n=0}^\infty$ of subsets of ω. We construct, by induction on $\alpha < 2^{\aleph_0}$, a chain of families $\mathcal{G}_0 \subset \ldots \subset \mathcal{G}_\alpha \subset \ldots$ of nonempty subsets of ω, such that each $\mathcal{G}_\alpha$ is closed under finite intersections and $|\mathcal{G}_\alpha| < 2^{\aleph_0}$ for all α.

We let $\mathcal{G}_0 = \{X \subset \omega : \omega - X \text{ is finite}\}$. If α is a limit ordinal, we let $\mathcal{G}_\alpha = \bigcup_{\beta<\alpha} \mathcal{G}_\alpha$. Having constructed $\mathcal{G}_\alpha$, we construct $\mathcal{G}_{\alpha+1}$ as follows: Let $\mathcal{A}_\alpha = \{A_n\}_{n=0}^\infty$ be a decreasing sequence of subsets of ω. If some A_n is disjoint from some $X \in \mathcal{G}_\alpha$, then we let $\mathcal{G}_{\alpha+1} = \mathcal{G}_\alpha$. Otherwise, the family $\mathcal{G} = \mathcal{G}_\alpha \cup \{A_n : n \in \omega\}$ has the finite intersection property and we claim (see Lemma 16.28 below), that there exists a $Z \subset \omega$ such that $Z - A_n$ is finite for all n, and $\mathcal{G}' = \mathcal{G} \cup \{Z\}$ has the finite intersection property. Then we let $\mathcal{G}_{\alpha+1}$ consist of all finite intersections $X_1 \cap \ldots \cap X_k$ of elements of $\mathcal{G}'$.

Finally, we let $\mathcal{G} = \bigcup\{\mathcal{G}_\alpha : \alpha < 2^{\aleph_0}\}$, and let D be any ultrafilter such that $D \supset \mathcal{G}$. We claim that D is a p-point: If $A_0 \supset A_1 \supset \ldots A_n \supset \ldots$ is any decreasing sequence of elements of D, then $\{A_n\}_{n=0}^\infty = \mathcal{A}_\alpha$ for some $\alpha < 2^{\aleph_0}$ and we have $Z \in \mathcal{G}_{\alpha+1}$ such that $Z - A_n$ is finite for all n. By Exercise 7.7, D is a p-point. □

It remains to prove the claim:

Lemma 16.28. *Assume* MA, *and let $\mathcal{G}$ be a family of subsets of ω with the finite intersection property such that $|\mathcal{G}| < 2^{\aleph_0}$. Let $A_0 \supset A_1 \supset \ldots A_n \supset \ldots$ be a decreasing sequence of elements of $\mathcal{G}$. Then there exists a $Z \subset \omega$ such that*:

(i) *$\mathcal{G} \cup \{Z\}$ has the finite intersection property*;
(ii) *$Z - A_n$ is finite for all $n \in \omega$.*

Proof. We may assume that that if $X, Y \in \mathcal{G}$, then $X \cap Y \in \mathcal{G}$. For each $X \in \mathcal{G}$, let $h_X : \omega \to \omega$ be some function such that $h_X(n) \in X \cap A_n$. By Theorem 16.24 the family $\{h_X : X \in \mathcal{G}\}$ is eventually dominated by a function f; in particular for every $X \in \mathcal{G}$ there exists some n such that $f(n) \geq h_X(n)$. Now we let $Z = \bigcup_{n=0}^\infty \{k \in A_n : k \leq f(n)\}$. It is readily verified that $Z - A_n$ is finite for each n, and that $Z \cap X \neq \emptyset$ for every $X \in \mathcal{G}$. □

Iterated Forcing

We conclude this chapter with the general definition of iterated forcing. We shall return to the general method in later chapters. Below we follow closely Definition 16.8 of finite support iteration. As before, for each ordinal $\alpha \geq 1$, P_α denotes an iteration of length α, $\leq_\alpha$ is the partial ordering of P_α and $\Vdash_\alpha$ is the corresponding forcing relation, and $\dot{Q}_\alpha$ is a name in V^{P_α} for a forcing notion with a greatest element 1. The general definition differs from Definition 16.8 by its handling of limit stages.

Definition 16.29. Let $\alpha \geq 1$. A forcing notion P_α is an iteration (of length α) if it is a set of α-sequences with the following properties:

(i) If $\alpha = 1$ then for some forcing notion Q_0,
 (a) P_1 is the set of all 1-sequences $\langle p(0) \rangle$ where $p(0) \in Q_0$;
 (b) $\langle p(0) \rangle \leq_1 \langle q(0) \rangle$ if and only if $p(0) \leq q(0)$.

(ii) If $\alpha = \beta + 1$ then $P_\beta = P_\alpha{\restriction}\beta = \{p{\restriction}\beta : p \in P_\alpha\}$ is an iteration of length β, and there is some forcing notion $\dot{Q}_\beta \in V^{P_\beta}$ such that
 (a) $p \in P_\alpha$ if and only if $p{\restriction}\beta \in P_\beta$ and $\Vdash_\beta p(\beta) \in \dot{Q}_\beta$;
 (b) $p \leq_\alpha q$ if and only if $p{\restriction}\beta \leq_\beta q{\restriction}\beta$ and $p{\restriction}\beta \Vdash_\beta p(\beta) \leq q(\beta)$.

(iii) If α is a limit ordinal, then for every $\beta < \alpha$, $P_\beta = P_\alpha{\restriction}\beta = \{p{\restriction}\beta : p \in P_\alpha\}$ is an iteration of length β and
 (a) the α-sequence $\langle 1, 1, \ldots, 1, \ldots \rangle$ is in P_α;
 (b) if $p \in P_\alpha$, $\beta < \alpha$ and if $q \in P_\beta$ is such that $q \leq_\beta p{\restriction}\beta$, then $r \in P_\alpha$ where for all $\xi < \alpha$, $r(\xi) = q(\xi)$ if $\xi < \beta$ and $r(\xi) = p(\xi)$ if $\beta \leq \xi < \alpha$;
 (c) $p \leq_\alpha q$ if and only if $\forall \beta < \alpha \; p{\restriction}\beta \leq_\beta q{\restriction}\beta$.

Clearly, an iteration with finite support is an iteration. In general, property (iii)(b) guarantees that if $P_\beta = P_\alpha{\restriction}\beta$ then $V^{P_\beta} \subset V^{P_\alpha}$; see Exercise 16.17.

A general iteration depends not only on the $\dot{Q}_\beta$ but also on the limit stages of the iteration. Let P_α be an iteration of length α where α is a limit ordinal. P_α is a *direct limit* if for every α-sequence p,

$$p \in P_\alpha \quad \text{if and only if} \quad \exists \beta < \alpha \; p{\restriction}\beta \in P_\beta \text{ and } \forall \xi \geq \beta \; p(\xi) = 1. \tag{16.11}$$

P_α is an *inverse limit* if for every α-sequence p,

$$p \in P_\alpha \quad \text{if and only if} \quad \forall \beta < \alpha \; p{\restriction}\alpha \in P_\beta. \tag{16.12}$$

In practice, forcing iterations combine direct and inverse limits. Finite support iterations are exactly those that use direct limits at all limit stages. In general, let $s(p)$, the *support* of $p \in P_\alpha$, be the set of all $\beta < \alpha$ such that it is not the case that $\Vdash_\beta p(\beta) = 1$. If I is an ideal on α containing all finite sets then an *iteration with I-support* is an iteration that satisfies for every limit ordinal $\gamma \leq \alpha$,

$$p \in P_\gamma \quad \text{if and only if} \quad \forall \beta < \gamma \; p{\restriction}\beta \in P_\beta \text{ and } s(p) \in I. \tag{16.13}$$

One of the most useful tools in forcing are iterations with *countable support*, where in (16.13) I is the ideal of at most countable sets. A countable support iteration is an iteration such that for every limit ordinal γ if $\operatorname{cf} \gamma = \omega$ then P_γ is an inverse limit, and if $\operatorname{cf} \gamma > \omega$ then P_γ is a direct limit. We shall return to countable support iterations later in the book.

The following generalizes Theorem 16.9:

Theorem 16.30. *Let κ be a regular uncountable cardinal and let α be a limit ordinal. Let P_α be an iteration such that for each $\beta < \alpha$, $P_\beta = P_\alpha \restriction \beta$ satisfies the κ-chain condition. If P_α is a direct limit, and either $\operatorname{cf} \alpha \neq \kappa$ or (if $\operatorname{cf} \alpha = \kappa$) for a stationary set of $\beta < \alpha$, P_β is a direct limit, then P_α satisfies the κ-chain condition.*

Proof. Exactly as the proof of Theorem 16.9. The only difference is that we apply Fodor's Theorem not to C, but to the stationary subset of C consisting of all ξ such that P_{α_ξ} is a direct limit. ☐

Exercises

16.1. $B(P * \dot{Q}) = B(P) * B(\dot{Q})$.

16.2. $P \times Q$ embeds densely in $P * \check{Q}$.

16.3. In V^B, $D : B = D/\dot{I}$ where for each $d \in D$, $\|d \in \dot{I}\|_B = \sum\{b \in B : b \cdot d = 0\}$.

16.4. $\|D : B$ is a complete Boolean algebra$\|_B = 1$, and D is isomorphic to $B * (D : B)$.

[Every name for an element of $D : B$ has the form d/I where $d \in D$. To see that $D : B$ is complete in V^B, let A be a name for a subset of $D : B$, and let $e = \sum\{d : \|d/I \in A\|\} = 1$. Then $\|e/I = \sum A\| = 1$.]

16.5. Let $h : P * \dot{Q} \to P$ be defined by $h(p, \dot{q}) = p$. Then h satisfies the conditions in Lemma 15.45.

16.6. If P has property (K) and $\Vdash_P \dot{Q}$ has property (K), then $P * \dot{Q}$ has property (K).

16.7. If P is κ-distributive and $\Vdash_P \dot{Q}$ is κ-distributive then $P * \dot{Q}$ is κ-distributive.

16.8. Let P_α, α a limit ordinal, be a finite support iteration, and $B_\beta = B(P_\alpha \restriction \beta)$ for all $\beta \leq \alpha$. Then B_α is the completion of the direct limit of the algebras B_β, $\beta < \alpha$.

16.9. If P_α is a finite support iteration and $P_\beta = P_\alpha \restriction \beta$ then $V^{P_\beta} \subset V^{P_\alpha}$. The projection $h(p) = p \restriction \beta$ satisfies Lemma 15.45; $G_\beta = \{p \restriction \beta : p \in G\}$ is a generic filter on P_β.

16.10. Let $(P, <)$ be the notion of forcing producing a Cohen generic real. There is a collection $\mathcal{D}$ of size $2^{\aleph_0}$ of dense subsets of P such that there is no $\mathcal{D}$-generic filter on P.

[For each $g : \omega \to \{0, 1\}$, let $D_g = \{p \in P : p \not\subset g\}$.]

16.11. Let $(P, <)$ be the notion of forcing that collapses ω_1. There is a collection $\mathcal{D}$ of size $\aleph_1$ of dense subsets of P such that there is no $\mathcal{D}$-generic filter on P.

[For each $\alpha < \omega_1$, let $D_\alpha = \{p \in P : \alpha \in \operatorname{ran}(p)\}$.]

16.12. MA_κ is equivalent to the statement of MA_κ restricted to complete Boolean algebras.

16.13. MA_κ is equivalent to of MA_κ restricted to partial orders of cardinality $\le \kappa$.

16.14. Let T be a Suslin tree and let P be the notion of forcing that adjoins κ Cohen generic reals. Let G be a generic filter on P. Then T is a Suslin tree in $V[G]$.

[Let P_T be the notion of forcing associated with the Suslin tree T. P satisfies the c.c.c. in any $V[H]$ where H is a generic filter on P_T. Thus $P_T \times P$ is c.c.c., and so P_T is c.c.c. in V[G].]

It follows that the existence of a Suslin tree is consistent with $2^{\aleph_0} > \aleph_1$.

16.15. There is a generic extension $V[A]$ where $A \subset \omega$, such that $\omega_1^{V[A]} = \omega_1$, and ω_2 is collapsed.

[Let f be a generic mapping of ω_1 onto ω_2 and let $X \subset \omega_1$ be such that $V[f] = V[X]$. Use almost disjoint forcing to find $A \subset \omega$ such that $V[A] = V[X][A]$.]

16.16. Assume MA_κ and let $\{X_\alpha : \alpha < \kappa\}$ be a sequence of infinite subsets of ω such that $X_\beta - X_\alpha$ is finite if $\alpha < \beta$. Show that there exists an infinite X such that $X - X_\alpha$ is finite for all $\alpha < \kappa$.

[A forcing condition is a pair (s, F) where s is a finite subset of ω and F is a finite subset of κ; $(s', F') \le (s, F)$ just in case $s' \supset s$, $F' \supset F$, and $s' - s \subset X_\alpha$ for all $\alpha \in F$. Consider the dense sets $D_n = \{(s, F) : |s| \ge n\}$, $n < \omega$, and $E_\alpha = \{(s, F) : \alpha \in F\}$, $\alpha < \kappa$.]

16.17. If P_α is an iteration and $P_\beta = P_\alpha{\restriction}\beta$ then $V^{P_\beta} \subset V^{P_\alpha}$.

[Use (iii)(b) in Definition 16.29 and Lemma 15.45.]

16.18. Let P_α and P'_α be countable support iterations of $\{\dot{Q}_\beta\}_\beta$ and $\{\dot{Q}'_\beta\}_\beta$, respectively. Assume that for every $\beta < \alpha$, if $B(P_\beta) = B(P'_\beta)$ then $\Vdash_\beta B(\dot{Q}_\beta) = B(\dot{Q}'_\beta)$. Then $B(P_\alpha) = B(P'_\alpha)$.

16.19. Let I be a κ-closed ideal on α, and let P_α be an iteration of $\{\dot{Q}_\beta\}_\beta$ with I-support. If for each $\beta < \alpha$, $\Vdash_\beta \dot{Q}_\beta$ is $<\kappa$-closed, then P_α is $<\kappa$-closed.

16.20. Let $\kappa \ge \aleph_2$ be a regular cardinal. Let P be a countable support iteration of length κ such that for all $\beta < \kappa$, $P{\restriction}\beta$ has a dense subset of size $< \kappa$. Then P satisfies the κ-chain condition.

[Use Theorem 16.30.]

Historical Notes

Iterated forcing was introduced by Solovay and Tennenbaum [1971]. The formulation in terms of Boolean algebras is based on their paper. Our presentation of general iteration (Definitions 16.8 and 16.29) follows Baumgartner [1983].

Following Solovay and Tennenbaum's construction of a model in which there are no Suslin trees (Theorem 16.13), Martin formulated an axiom ($\mathrm{MA}_{\aleph_1}$) which implies that there are no Suslin trees, and whose consistency was obtained by Solovay-Tennenbaum's method. The consistency proof of $\mathrm{MA} + 2^{\aleph_0} > \aleph_1$ appears in Solovay and Tennenbaum [1971].

Martin's Axiom is investigated in detail in the paper [1970] of Martin and Solovay. The paper contains various equivalent formulations of Martin's Axiom and numerous applications (including Theorem 16.20). Theorem 16.21 was discovered by Kunen, Rowbottom, Solovay and possibly others.

Baumgartner, Malitz, and Reinhardt [1970] proved that $\mathrm{MA}_{\aleph_1}$ implies that every Aronszajn tree is special (Theorem 16.17). Special Aronszajn trees have applications in model theory (this fact is due to Rowbottom and Silver) and are investigated in Mitchell's paper [1972/73].

Scales were investigated extensively by Hechler [1974]. Hechler introduced the notion of forcing used in the proof of Theorem 16.24. Hechler, among others, showed that if $\operatorname{cf}\kappa > \omega$, then there is a generic extension in which $2^{\aleph_0} > \kappa$ and a κ-scale exists.

The construction of p-points (and Ramsey ultrafilters) under the assumption of Martin's Axiom is due to Booth [1970]. Our proof of Theorem 16.27 follows Ketonen [1976].