# 17. Large Cardinals

The theory of large cardinals plays central role in modern set theory. In this chapter we begin a systematic study of large cardinals. In addition to combinatorial methods, the proofs use techniques from model theory.

## Ultrapowers and Elementary Embeddings

We start with the following theorem that introduced the technique of ultrapowers to the study of large cardinals.

**Theorem 17.1 (Scott).** *If there is a measurable cardinal then $V \neq L$.*

Ultrapowers were introduced in Chapter 12. We now generalize the technique to construct ultrapowers of the universe. Let $U$ be an ultrafilter on a set $S$ and consider the class of all functions with domain $S$. Following (12.3) and (12.4) we define

$$\begin{aligned} f =^* g \quad &\text{if and only if} \quad \{x \in S : f(x) = g(x)\} \in U, \\ f \in^* g \quad &\text{if and only if} \quad \{x \in S : f(x) \in g(x)\} \in U. \end{aligned}$$

For each $f$, we denote $[f]$ the equivalence class of $f$ in $=^*$ (recall (6.4)):

$$[f] = \{g : f =^* g \text{ and } \forall h\,(h =^* f \to \operatorname{rank} g \leq \operatorname{rank} h)\}.$$

We also use the notation $[f] \in^* [g]$ when $f \in^* g$.

Let $\mathrm{Ult} = \mathrm{Ult}_U(V)$ be the class of all $[f]$, where $f$ is a function on $S$, and consider the model $\mathrm{Ult} = (\mathrm{Ult}, \in^*)$. Łoś's Theorem 12.3 holds in the present context as well: If $\varphi(x_1, \ldots, x_n)$ is a formula of set theory, then

$$\mathrm{Ult} \vDash \varphi([f_1], \ldots, [f_n]) \quad \text{if and only if} \quad \{x \in S : \varphi(f_1(x), \ldots, f_n(x))\} \in U.$$

If $\sigma$ is a sentence, then $\mathrm{Ult} \vDash \sigma$ if and only if $\sigma$ holds; the ultrapower is *elementarily equivalent* to the universe $(V, \in)$. The constant functions $c_a$ are defined, for every set $a$, by (12.12), and the function $j = j_U : V \to \mathrm{Ult}$, defined by $j_U(a) = [c_a]$ is an *elementary embedding* of $V$ in Ult:

$$\varphi(a_1, \ldots, a_n) \quad \text{if and only if} \quad \mathrm{Ult} \vDash \varphi(ja_1, \ldots, ja_n) \tag{17.1}$$

whenever $\varphi(x_1, \ldots, x_n)$ is a formula of set theory.

The most important application of ultrapowers in set theory are those in which $(\mathrm{Ult}, \in^*)$ is well-founded. As we show below, well-founded ultrapowers are closely related to measurable cardinals.

The model $\mathrm{Ult}_U(V)$ is well-founded if (i) every nonempty set $X \subset \mathrm{Ult}$ has a $\in^*$-minimal element, and (ii) $\mathrm{ext}(f)$ is a set for every $f$, where

$$\mathrm{ext}(f) = \{[g] : g \in^* f\}.$$

The second condition is clearly satisfied for any ultrafilter $U$: For every $g \in^* f$ there is some $h =^* g$ such that $\operatorname{rank} h \le \operatorname{rank} f$. As for the condition (i), this is satisfied if and only if there exists no infinite descending $\in^*$-sequence

$$f_0 \ni^* f_1 \ni^* \ldots \ni^* f_k \ni^* \ldots \qquad (k \in \omega)$$

of elements of the ultrapower.

**Lemma 17.2.** *If $U$ is a $\sigma$-complete ultrafilter, then* $(\mathrm{Ult}, \in^*)$ *is a well-founded model.*

*Proof.* We shall show that there is no infinite descending $\in^*$-sequence in Ult if $U$ is a $\sigma$-complete ultrafilter on $S$. Let us assume that $f_0, f_1, \ldots, f_n, \ldots$ is such a descending sequence. Thus for each $n$, the set

$$X_n = \{x \in S : f_{n+1}(x) \in f_n(x)\}$$

is in the ultrafilter. Since $U$ is $\sigma$-complete, the intersection $X = \bigcap_{n=0}^{\infty} X_n$ is also in $U$ and hence nonempty; let $x$ be an arbitrary element of $X$. Then we have

$$f_0(x) \ni f_1(x) \ni f_2(x) \ni \ldots$$

an infinite descending $\ni$-sequence, which is a contradiction. □

By the Mostowski Collapsing Theorem every well-founded model is isomorphic to a transitive model. Thus if $U$ is $\sigma$-complete, there exists a one-to-one mapping $\pi$ of Ult onto a transitive class such that $f \in^* g$ if and only $\pi([f]) \in \pi([g])$. In order to simplify notation, we shall identify each $[f]$ with its image $\pi([f])$. Thus if $U$ is $\sigma$-complete, the symbol Ult denotes the transitive collapse of the ultrapower, and for each function $f$ on $S$, $[f]$ is an element of the transitive class Ult; we say the function $f$ *represents* $[f] \in \mathrm{Ult}$.

Thus if $U$ is a $\sigma$-complete ultrafilter, $M = \mathrm{Ult}_U(V)$ is an inner model and $j = j_U$ is an elementary embedding $j : V \to M$.

If $\alpha$ is an ordinal, then since $j$ is elementary, $j(\alpha)$ is an ordinal; moreover, $\alpha < \beta$ implies $j(\alpha) < j(\beta)$. Thus we have $\alpha \le j(\alpha)$ for every ordinal number $\alpha$. Note that $j(\alpha + 1) = j(\alpha) + 1$, and $j(n) = n$ for all natural numbers $n$. It is also easy to see that $j(\omega) = \omega$: If $[f] < \omega$, then $f(x) < \omega$ for almost all $x \in S$, and by $\sigma$-completeness, there exists $n < \omega$ such that $f(x) = n$ for almost all $x$. By the same argument, if $U$ is $\lambda$-complete, then $j(\gamma) = \gamma$ for all $\gamma < \lambda$.

Now let $\kappa$ be a measurable cardinal, and let $U$ be a nonprincipal $\kappa$-complete ultrafilter on $\kappa$. Let $d$ (the *diagonal function*) be the function on $\kappa$ defined by

$$d(\alpha) = \alpha \qquad (\alpha < \kappa).$$

Since $U$ is $\kappa$-complete every bounded subset of $\kappa$ has measure 0 and so for every $\gamma < \kappa$, we have $d(\alpha) > \gamma$ for almost all $\alpha$. Hence $[d] > \gamma$ for all $\gamma < \kappa$ and thus $[d] \geq \kappa$. However, we clearly have $[d] < j(\kappa)$ and it follows that $j(\kappa) > \kappa$.

We have thus proved that if there is a measurable cardinal, then there is an elementary embedding $j$ of the universe in a transitive model $M$ such that $j$ is not the identity mapping; $j$ is a *nontrivial elementary embedding of the universe.*

*Proof of Theorem* 17.1. Let us assume $V = L$ and that measurable cardinals exist; let $\kappa$ be the least measurable cardinal. Let $U$ be a nonprincipal $\kappa$-complete ultrafilter on $\kappa$ and let $j : V \to M$ be the corresponding elementary embedding. As we have shown, $j(\kappa) > \kappa$.

Since $V = L$, the only transitive model containing all ordinals is the universe itself: $V = M = L$. Since $j$ is an elementary embedding and $\kappa$ is the least measurable cardinal, we have

$$M \vDash j(\kappa) \text{ is the least measurable cardinal;}$$

and hence, $j(\kappa)$ is the least measurable cardinal. This is a contradiction since $j(\kappa) > \kappa$. $\square$

If there exists a measurable cardinal, then there exists a nontrivial elementary embedding of the universe. Let us show that conversely, if $j : V \to M$ is a nontrivial elementary embedding then there exists a measurable cardinal.

**Lemma 17.3.** *If $j$ is a nontrivial elementary embedding of the universe, then there exists a measurable cardinal.*

*Proof.* Let $j : V \to M$ be a nontrivial embedding. Notice that there exists an ordinal $\alpha$ such that $j(\alpha) \neq \alpha$; otherwise, we would have $\operatorname{rank}(jx) = \operatorname{rank}(x)$ for all $x$, and then we could prove by induction on rank that $j(x) = x$ for all $x$.

Thus let $\kappa$ be the least ordinal number such that $j(\kappa) \neq \kappa$ (and hence $j(\kappa) > \kappa$). It is clear that $j(n) = n$ for all $n$ and $j(\omega) = \omega$ since 0, $n + 1$, and $\omega$ are absolute notions and $j$ is elementary. Hence $\kappa > \omega$. We shall show that $\kappa$ is a measurable cardinal.

Let $D$ be the collection of subsets of $\kappa$ defined as follows:

$$X \in D \quad \text{if and only if} \quad \kappa \in j(X) \qquad (X \subset \kappa). \tag{17.2}$$

Since $\kappa < j(\kappa)$, i.e., $\kappa \in j(\kappa)$, we have $\kappa \in D$; also $\emptyset \notin D$ because $j(\emptyset) = \emptyset$. Using the fact that $j(X \cap Y) = j(X) \cap j(Y)$ and that $j(X) \subset j(Y)$ whenever

$X \subset Y$, we see that $D$ is a filter: If $\kappa \in j(X)$ and $\kappa \in j(Y)$, then $\kappa \in j(X \cap Y)$; if $X \subset Y$ and $\kappa \in j(X)$, then $\kappa \in j(Y)$. Similarly, $j(\kappa - X) = j(\kappa) - j(X)$ and thus $D$ is an ultrafilter.

$D$ is a nonprincipal ultrafilter: For every $\alpha < \kappa$, we have $j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\}$, and so $\kappa \notin j(\{\alpha\})$ and we have $\{\alpha\} \notin D$. We shall now show that $D$ is $\kappa$-complete. Let $\gamma < \kappa$ and let $\mathcal{X} = \langle X_\alpha : \alpha < \gamma \rangle$ be a sequence of subsets of $\kappa$ such that $\kappa \in j(X_\alpha)$ for each $\alpha < \gamma$. We shall show that $\bigcap_{\alpha<\gamma} X_\alpha \in D$. In $M$ (and thus in $V$), $j(\mathcal{X})$ is a sequence of length $j(\gamma)$ of subsets of $j(\kappa)$; for each $\alpha < \gamma$, the $j(\alpha)$th term of $j(\mathcal{X})$ is $j(X_\alpha)$. Since $j(\alpha) = \alpha$ for all $\alpha < \gamma$ and $j(\gamma) = \gamma$, it follows that $j(\mathcal{X}) = \langle j(X_\alpha) : \alpha < \gamma \rangle$. Hence if $X = \bigcap_{\alpha<\gamma} X_\alpha$, we have $j(X) = \bigcap_{\alpha<\gamma} j(X_\alpha)$. Now it is clear that $\kappa \in j(X)$ and hence $X \in D$. $\square$

The construction of a $\kappa$-complete ultrafilter from an elementary embedding yields the following commutative diagram (17.3):

**Lemma 17.4.** *Let $j : V \to M$ be a nontrivial elementary embedding, let $\kappa$ be the least ordinal moved, and let $D$ be the ultrafilter on $\kappa$ defined in* (17.2). *Let $j_D : V \to \mathrm{Ult}$ be the canonical embedding of $V$ in the ultrapower $\mathrm{Ult}_D(V)$. Then there is an elementary embedding $k$ of $\mathrm{Ult}$ in $M$ such that $k(j_D(a)) = j(a)$ for all $a$:*

(17.3)
$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "j_D"'] & M \\ \mathrm{Ult} \arrow[ur, "k"'] & \end{tikzcd}$$

*Proof.* For each $[f] \in \mathrm{Ult}$, let

$$k([f]) = (j(f))(\kappa). \tag{17.4}$$

(Here $f$ is a function on $\kappa$ and $j(f)$ is a function on $j(\kappa)$.)

We shall first show that definition (17.4) does not depend on the choice of $f$ representing $[f]$. If $f =_D g$, then the set $X = \{\alpha : f(\alpha) = g(\alpha)\}$ is in $D$ and hence $\kappa$ is in the set

$$j(X) = \{\alpha < j(\kappa) : (jf)(\alpha) = (jg)(\alpha)\}.$$

Therefore $(jf)(\kappa) = (jg)(\kappa)$.

Next we show that $k$ is elementary. Let $\varphi(x)$ be a formula and let $\mathrm{Ult} \vDash \varphi([f])$; we shall show that $M \vDash \varphi(k([f]))$. The set $X = \{\alpha : \varphi(f(\alpha))\}$ is in $D$ and hence $\kappa$ belongs to the set

$$j(X) = \{\alpha < j(\kappa) : M \vDash \varphi((jf)(\alpha))\}.$$

Since $(jf)(\kappa) = k([f])$, we have $M \vDash \varphi(k([f]))$.

Finally, we show that $k(j_D(a)) = j(a)$ for all $a$. Since $j_D(a) = [c_a]$, where $c_a$ is the constant function on $\kappa$ with value $a$, we have $k(j_D(a)) = (j(c_a))(\kappa)$. Now $j(c_a)$ is the constant function on $j(\kappa)$ with value $j(a)$ and hence $(j(c_a))(\kappa) = j(a)$. □

We remark that the measure $D = \{X \subset \kappa : \kappa \in j(X)\}$ defined from an elementary embedding is normal: Let $f$ be a regressive function on some $X \in D$. Then $(jf)(\kappa) < \kappa$, and if $\gamma = (jf)(\kappa)$, then $f(\alpha) = \gamma$ for almost all $\alpha$.

Normality can be expressed in terms of ultrapowers:

**Lemma 17.5.** *Let $D$ be a nonprincipal $\kappa$-complete ultrafilter on $\kappa$. Then the following are equivalent*:

(i) *$D$ is normal.*
(ii) *In the ultrapower* $\mathrm{Ult}_D(V)$,

$$\kappa = [d]$$

*where $d$ is the diagonal function.*
(iii) *For every $X \subset \kappa$, $X \in D$ if and only if $\kappa \in j_D(X)$.*

*Proof.* (i) implies (ii): Every function $f \in^* d$ is regressive, and hence represents an ordinal $\gamma < \kappa$.

(ii) implies (iii): If $X \subset \kappa$, then $X \in D$ if and only if $d(\alpha) \in X$ for almost all $\alpha$, that is, if and only if $[d] \in j_D(X)$. If $[d] = \kappa$, we get $X \in D$ if and only if $\kappa \in j_D(X)$.

(iii) implies (i): If $D = \{X \subset \kappa : \kappa \in j_D(X)\}$ then $D$ is normal, by the remark preceding the lemma. □

Let $j : V \to M$ be an elementary embedding. If $X$ is a class defined by a formula $\varphi$, then $j(X)$ is the class of the model $M$, defined in $M$ by the same formula $\varphi$. Note that $j(X) = \bigcup_{\alpha \in Ord} j(X \cap V_\alpha)$. In particular, $M = j(V)$.

**Lemma 17.6.** *Let $j$ be an elementary embedding of the universe and let $\kappa$ be the least ordinal moved (i.e., $j(\kappa) > \kappa$). If $C$ is a closed unbounded subset of $\kappa$, then $\kappa \in j(C)$.*

*Proof.* Since $j(\alpha) = \alpha$ for all $\alpha < \kappa$, we have $j(C) \cap \kappa = C$. Thus $j(C) \cap \kappa$ is unbounded in $\kappa$; and because $j(C)$ is closed (in $j(V)$ and hence in the universe), we have $\kappa \in j(C)$. □

A consequence of Lemma 17.6 is that the set of all regular cardinals below a measurable cardinal $\kappa$ is stationary (cf. Lemma 10.21): Let $X \subset \kappa$ be the set of all regular cardinals below $\kappa$. Since $\kappa$ is regular in $M$, we have $\kappa \in j(X)$, and $\kappa \in j(X \cap C)$ for every closed unbounded $C \subset \kappa$. Hence $X$ is stationary. Similarly, as $\kappa$ is Mahlo, it is Mahlo in $M$, and if $X$ is now the set of all Mahlo cardinals below $\kappa$, it follows that $X$ is stationary.

More generally, if $M(X)$ denotes the *Mahlo operation*

$$M(X) = \{\alpha : X \cap \alpha \text{ is stationary in } \alpha\} \tag{17.5}$$

where $X$ is any class of ordinals, the above argument shows that if $\kappa \in j(X)$ then $\kappa \in M(X)$ (Exercise 17.7).

The next theorem shows that there exists no nontrivial elementary embedding of $V$ into $V$. As the statement "there exists an elementary embedding of $V$" is not expressible in the language of set theory, the theorem needs to be understood as a theorem in the following modification of ZFC: The language has, in addition to $\in$, a function symbol $j$, the axioms include Separation and Replacement Axioms for formulas that contain the symbol $j$, and axioms that state that $j$ is an elementary embedding of $V$.

**Theorem 17.7 (Kunen).** *If $j : V \to M$ is a nontrivial elementary embedding, then $M \neq V$.*

First we prove the following lemma:

**Lemma 17.8.** *Let $\lambda$ be an infinite cardinal such that $2^\lambda = \lambda^{\aleph_0}$. There exists a function $F : \lambda^\omega \to \lambda$ such that whenever $A$ is a subset of $\lambda$ of size $\lambda$ and $\gamma < \lambda$, there exists some $s \in A^\omega$ such that $F(s) = \gamma$.*

*Proof.* Let $\{(A_\alpha, \gamma_\alpha) : \alpha < 2^\lambda\}$ be an enumeration of all pairs $(A, \gamma)$ where $\gamma < \lambda$ and $A$ is a subset of $\lambda$ of size $\lambda$. We define, by induction on $\alpha$, a sequence $s_\alpha$, $\alpha < 2^\lambda$, of elements of $\lambda^\omega$ as follows: If $\alpha < 2^\lambda$, then since $\lambda^{\aleph_0} = 2^\lambda > |\alpha|$, there exists an $s_\alpha \in A_\alpha^\omega$ such that $s_\alpha \neq s_\beta$ for all $\beta < \alpha$.

For each $\alpha < 2^\lambda$, we define $F(s_\alpha) = \gamma_\alpha$ (and let $F(s)$ be arbitrary if $s$ is not one of the $s_\alpha$). The function $F$ has the required property: If $A \subset \lambda$ has size $\lambda$ and $\gamma < \lambda$, then $(A, \gamma) = (A_\alpha, \gamma_\alpha)$ for some $\alpha$, and then $\gamma_\alpha = F(s_\alpha)$. □

*Proof of Theorem* 17.7. Let us assume that $j$ is a nontrivial elementary embedding of $V$ in $V$. Let $\kappa = \kappa_0$ be the least ordinal moved; $\kappa_0$ is measurable, and so are $\kappa_1 = j(\kappa_0)$, $\kappa_2 = j(\kappa_1)$, and every $\kappa_n$, where $\kappa_{n+1} = j(\kappa_n)$. Let $\lambda = \lim_{n\to\infty} \kappa_n$. Since $j(\langle \kappa_n : n < \omega\rangle) = \langle j(\kappa_n) : n < \omega\rangle = \langle \kappa_{n+1} : n < \omega\rangle$, we have $j(\lambda) = \lim_{n\to\infty} j(\kappa_n) = \lambda$. Let $G = \{j(\alpha) : \alpha < \lambda\}$; we shall use the set $G$ and Lemma 17.8 to obtain a contradiction.

The cardinal $\lambda$ is the limit of a sequence of measurable cardinals and hence is a strong limit cardinal. Since $\operatorname{cf} \lambda = \omega$, we have $2^\lambda = \lambda^{\aleph_0}$. By Lemma 17.8 there is a function $F : \lambda^\omega \to \lambda$ such that $F(A^\omega) = \lambda$ for all $A \subset \lambda$ of size $\lambda$. Since $j$ is elementary, and $j(\omega) = \omega$ and $j(\lambda) = \lambda$, the function $j(F)$ has the same property. Thus, considering the set $A = G$, there exists $s \in G^\omega$ such that $(jF)(s) = \kappa$.

Now, $s$ is a function, $s : \omega \to G$, and hence there is a $t : \omega \to \lambda$ such that $s(n) = j(t(n))$ for all $n < \omega$. It follows that $s = j(t)$. Thus we have $\kappa = (jF)(jt) = j(F(t))$; in other words, $\kappa = j(\alpha)$ where $\alpha = F(t)$. However, this is impossible since $j(\alpha) = \alpha$ for all $\alpha < \kappa$, and $j(\kappa) > \kappa$. □

Let us now consider ultrapowers and the corresponding elementary embeddings $j_U : V \to \mathrm{Ult}$. To introduce the following lemma, let us observe that if $j : V \to M$ and if $\kappa$ is the least ordinal moved, then $j(x) = x$ for every $x \in V_\kappa$, and $j(X) \cap V_\kappa = X$ for every $X \subset V_\kappa$. Hence $V^M_{\kappa+1} = V_{\kappa+1}$ (and $P^M(\kappa) = P(\kappa)$).

**Lemma 17.9.** *Let $U$ be a nonprincipal $\kappa$-complete ultrafilter on $\kappa$, let $M = \mathrm{Ult}_U(V)$ and let $j = j_U$ be the canonical elementary embedding of $V$ in $M$.*

(i) *$M^\kappa \subset M$, i.e., every $\kappa$-sequence $\langle a_\alpha : \alpha < \kappa\rangle$ of elements of $M$ is itself a member of $M$.*
(ii) *$U \notin M$.*
(iii) *$2^\kappa \le (2^\kappa)^M < j(\kappa) < (2^\kappa)^+$.*
(iv) *If $\lambda$ is a limit ordinal and if $\operatorname{cf}\lambda = \kappa$, then $j(\lambda) > \lim_{\alpha\to\lambda} j(\alpha)$; if $\operatorname{cf}\lambda \ne \kappa$, then $j(\lambda) = \lim_{\alpha\to\lambda} j(\alpha)$.*
(v) *If $\lambda > \kappa$ is a strong limit cardinal and $\operatorname{cf}\lambda \ne \kappa$, then $j(\lambda) = \lambda$.*

*Proof.* (i) Let $\langle a_\xi : \xi < \kappa\rangle$ be a $\kappa$-sequence of elements of $M$. For each $\xi < \kappa$, let $g_\xi$ be a function that represents $a_\xi$, and let $h$ be a function that represents $\kappa$:

$$[g_\xi] = a_\xi, \qquad [h] = \kappa.$$

We shall construct a function $F$ such that $[F] = \langle a_\xi : \xi < \kappa\rangle$. We let, for each $\alpha < \kappa$,

$$F(\alpha) = \langle g_\xi : \xi < h(\alpha)\rangle.$$

Since for each $\alpha$, $F(\alpha)$ is an $h(\alpha)$-sequence, $[F]$ is a $\kappa$-sequence. Let $\xi < \kappa$; we want to show that the $\xi$th term of $[F]$ is $a_\xi$. Since $[h] > \xi$, we have $\xi < h(\alpha)$ for almost all $\alpha$; and for each $\alpha$ such that $\xi < h(\alpha)$, the $\xi$th term of $F(\alpha)$ is $g_\xi(\alpha)$. But $[c_\xi] = \xi$ and $[g_\xi] = a_\xi$, and we are done.

(ii) Assume that $U \in M$, and let us consider the mapping $e$ of $\kappa^\kappa$ onto $j(\kappa)$ defined by $e(f) = [f]$. Since $\kappa^\kappa \in M$ and $U \in M$, the mapping $e$ is in $M$. It follows that $M \vDash |j(\kappa)| \le 2^\kappa$. This is a contradiction since $\kappa < j(\kappa)$ and $j(\kappa)$ is inaccessible in $M$.

(iii) $2^\kappa \le (2^\kappa)^M$ holds because $P^M(\kappa) = P(\kappa)$ and $M \subset V$; $(2^\kappa)^M$ is less than $j(\kappa)$ since $j(\kappa)$ is inaccessible in $M$; finally, we have $|j(\kappa)| = 2^\kappa$ and hence $j(\kappa) < (2^\kappa)^+$.

(iv) If $\operatorname{cf}\lambda = \kappa$, let $\lambda = \lim_{\alpha\to\kappa} \lambda_\alpha$ and let $f(\alpha) = \lambda_\alpha$ for all $\alpha < \kappa$. Then $[f] > j(\lambda_\alpha)$ for all $\alpha < \kappa$ and $[f] < j(\lambda)$. If $\operatorname{cf}\lambda > \kappa$, then for every $f : \kappa \to \lambda$ there exists $\alpha < \lambda$ such that $[f] < j(\alpha)$. If $\operatorname{cf}\lambda = \gamma < \kappa$, let $\lambda = \lim_{\nu\to\gamma} \lambda_\nu$; for every $f : \kappa \to \lambda$ there exists (by $\kappa$-completeness) $\nu < \gamma$ such that $[f] < j(\lambda_\nu)$.

(v) For every $\alpha < \lambda$, the ordinals below $\alpha$ are represented by functions $f : \kappa \to \alpha$; hence $|j(\alpha)| \le |\alpha^\kappa| < \lambda$; by (iv) we have $j(\lambda) = \lim_{\alpha\to\lambda} j(\alpha) = \lambda$. ∎

Note that in (v) it suffices to assume that $\operatorname{cf}\lambda \neq \kappa$ and $\alpha^\kappa < \lambda$ for all cardinals $\alpha < \lambda$.

Let us recall (Lemma 10.18) that a measurable cardinal is weakly compact. We now prove a stronger result:

**Theorem 17.10.** *Every measurable cardinal $\kappa$ is weakly compact and if $D$ is a normal measure on $\kappa$ then the set $\{\alpha < \kappa : \alpha$ is weakly compact$\}$ is in $D$.*

*Proof.* The first statement was proved in Lemma 10.18. Let $D$ be a normal measure on $\kappa$, and let $j_D : V \to M$ be the canonical embedding. Since $P^M(\kappa) = P(\kappa)$, it follows that $\kappa$ is weakly compact in $M$, and since $[d]_D = \kappa$, we have $\{\alpha : \alpha$ is weakly compact$\} \in D$. $\square$

The following two results show that the existence of measurable cardinals influences cardinal arithmetic:

**Lemma 17.11.** *Let $\kappa$ be a measurable cardinal. If $2^\kappa > \kappa^+$, then the set $\{\alpha < \kappa : 2^\alpha > \alpha^+\}$ has measure one for every normal measure on $\kappa$.*

*Consequently, if $2^\alpha = \alpha^+$ for all cardinals $\alpha < \kappa$, then $2^\kappa = \kappa^+$.*

*Proof.* Let $D$ be a normal measure on $\kappa$, and let $M = \operatorname{Ult}_D(V)$. If $2^\alpha = \alpha^+$ for almost all $\alpha$, then, since $[d]_D = \kappa$, we have $M \vDash 2^\kappa = \kappa^+$. In other words, there is a one-to-one mapping in $M$ between $P^M(\kappa)$ and $(\kappa^+)^M$. However, $P^M(\kappa) = P(\kappa)$ and $(\kappa^+)^M = \kappa^+$ (because $P^M(\kappa) = P(\kappa)$), and so $2^\kappa = \kappa^+$. $\square$

**Lemma 17.12.** *Let $\kappa$ be a measurable cardinal, let $D$ be a normal measure on $\kappa$ and let $j : V \to M$ be the corresponding elementary embedding. Let $\lambda > \kappa$ be a strong limit cardinal of cofinality $\kappa$. Then $2^\lambda < j(\lambda)$.*

*Proof.* Since $\operatorname{cf}\lambda = \kappa$, we have $j(\lambda) > \lambda$. We shall show that $2^\lambda = \lambda^\kappa \leq (\lambda^\kappa)^M \leq (\lambda^{j(\kappa)})^M < j(\lambda)$. The first equality holds because $\lambda$ is strong limit. We have $\lambda^\kappa \leq (\lambda^\kappa)^M$ because every function $f : \kappa \to \lambda$ is in $M$. As for the last inequality, we have

$$M \vDash j(\lambda) \text{ is a strong limit cardinal}$$

and since $\lambda < j(\lambda)$ and $j(\kappa) < j(\lambda)$, we have $M \vDash \lambda^{j(\kappa)} < j(\lambda)$. $\square$

See Exercises 17.12–17.16.

## Weak Compactness

We shall investigate weakly compact cardinals in some detail, and give a characterization of weakly compact cardinals that explains the name "weakly compact." This aspect of weakly compact cardinals has, as many other large cardinal properties, motivation in model theory.

We shall consider infinitary languages which are generalizations of the ordinary first order language. Let $\kappa$ be an infinite cardinal number. The language $\mathcal{L}_{\kappa,\omega}$ consists of

(i) $\kappa$ variables;
(ii) various relation, function, and constant symbols;
(iii) logical connectives and infinitary connectives $\bigvee_{\xi<\alpha}\varphi_\xi$, $\bigwedge_{\xi<\alpha}\varphi_\xi$ for $\alpha<\kappa$ (infinite disjunction and conjunction);
(iv) quantifiers $\exists v$, $\forall v$.

The language $\mathcal{L}_{\kappa,\kappa}$ is like $\mathcal{L}_{\kappa,\omega}$ except that it also contains infinitary quantifiers:

(v) $\exists_{\xi<\alpha}v_\xi$, $\forall_{\xi<\alpha}v_\xi$ for $\alpha<\kappa$.

The interpretation of the infinitary symbols of $\mathcal{L}_{\kappa,\kappa}$ is the obvious generalization of the finitary case where $\bigvee_{\xi<n}\varphi_\xi$ is $\varphi_0\vee\ldots\vee\varphi_{n-1}$, $\exists_{\xi<n}v_\xi$ stands for $\exists v_0\ldots\exists v_{n-1}$, etc. The language $\mathcal{L}_{\omega,\omega}$ is just the language of the first order predicate calculus.

The finitary language $\mathcal{L}_{\omega,\omega}$ satisfies the Compactness Theorem: If $\Sigma$ is a set of sentences such that every finite $S\subset\Sigma$ has a model, then $\Sigma$ has a model. Let us say that the language $\mathcal{L}_{\kappa,\kappa}$ (or $\mathcal{L}_{\kappa,\omega}$) satisfies the Weak Compactness Theorem if whenever $\Sigma$ is a set of sentences of $\mathcal{L}_{\kappa,\kappa}$ ($\mathcal{L}_{\kappa,\omega}$) such that $|\Sigma|\le\kappa$ and that every $S\subset\Sigma$ with $|S|<\kappa$ has a model, then $\Sigma$ has a model. Clearly, if $\mathcal{L}_{\kappa,\kappa}$ satisfies the Weak Compactness Theorem, then so does $\mathcal{L}_{\kappa,\omega}$ because $\mathcal{L}_{\kappa,\omega}\subset\mathcal{L}_{\kappa,\kappa}$.

**Theorem 17.13.**

(i) *If $\kappa$ is a weakly compact cardinal, then the language $\mathcal{L}_{\kappa,\kappa}$ satisfies the Weak Compactness Theorem.*
(ii) *If $\kappa$ is an inaccessible cardinal and if $\mathcal{L}_{\kappa,\omega}$ satisfies the Weak Compactness Theorem, then $\kappa$ is weakly compact.*

*Proof.* (i) The proof of the Weak Compactness Theorem for $\mathcal{L}_{\kappa,\kappa}$ is very much like the standard proof of the Compactness Theorem for $\mathcal{L}_{\omega,\omega}$. Let $\Sigma$ be a set of sentences of $\mathcal{L}_{\kappa,\kappa}$ of size $\kappa$ such that if $S\subset\Sigma$ and $|S|<\kappa$, then $S$ has a model. Let us assume that the language $\mathcal{L}=\mathcal{L}_{\kappa,\kappa}$ has only the symbols that occur in $\Sigma$; thus $|\mathcal{L}|=\kappa$.

First we extend the language as follows: For each formula $\varphi$ with free variables $v_\xi$, $\xi<\alpha$, we introduce new constant symbols $c_\xi^\varphi$, $\xi<\alpha$ (*Skolem constants*); let $\mathcal{L}^{(1)}$ be the extended language. Then we do the same for each formula of $\mathcal{L}^{(1)}$ and obtain $\mathcal{L}^{(2)}\supset\mathcal{L}^{(1)}$. We do the same for each $n<\omega$, and then let $\mathcal{L}^*=\bigcup_{n=1}^\infty\mathcal{L}^{(n)}$. Since $\kappa$ is inaccessible, it follows that $|\mathcal{L}^*|=\kappa$. $\mathcal{L}^*$ has the property that for each formula $\varphi$ with free variables $v_\xi$, $\xi<\alpha$, there are in $\mathcal{L}^*$ constant symbols $c_\xi^\varphi$, $\xi<\alpha$ (which do not occur in $\varphi$).

For each $\varphi(v_\xi, \ldots)_{\xi<\alpha}$ let $\sigma_\varphi$ be the sentence (a *Skolem sentence*)

$$\exists_{\xi<\alpha} v_\xi\, \varphi(v_\xi, \ldots)_{\xi<\alpha} \to \varphi(c_\xi^\varphi, \ldots)_{\xi<\alpha} \tag{17.6}$$

and let $\Sigma^* = \Sigma \cup \{\sigma_\varphi : \varphi \text{ is a formula of } \mathcal{L}^*\}$.

Note that if $S \subset \Sigma^*$ and $|S| < \kappa$, then $S$ has a model: Take a model for $S \cap \Sigma$ (for $\mathcal{L}$) and then expand it to a model for $\mathcal{L}^*$ by interpreting the Skolem constants so that each sentence (17.6) is true.

Let $\{\sigma_\alpha : \alpha < \kappa\}$ be an enumeration of all the sentences in $\mathcal{L}^*$. Let $(T, \subset)$ be the binary $\kappa$-tree consisting of all $t : \gamma \to \{0, 1\}$, $\gamma < \kappa$, for which there exists a model $\mathfrak{A}$ of $\Sigma \cap \{\sigma_\alpha : \alpha \in \operatorname{dom}(t)\}$ such that for all $\alpha \in \operatorname{dom}(t)$

$$t(\alpha) = 1 \quad \text{if and only if} \quad \mathfrak{A} \vDash \sigma_\alpha.$$

Since $\kappa$ has the tree property, there exists a branch $B$ in $T$ of length $\kappa$. Let

$$\Delta = \{\sigma_\alpha : t(\alpha) = 1 \text{ for some } t \in B\}.$$

Clearly, $\Sigma^* \subset \Delta$. Let $A_0$ be the set of all constant terms of $\mathcal{L}^*$, and let $\approx$ be the equivalence relation on $A_0$ defined by

$$\tau_1 \approx \tau_2 \quad \text{if and only if} \quad (\tau_1 \approx \tau_2) \in \Delta,$$

and let $A = A_0/{\approx}$.

We make $A$ into a model $\mathfrak{A}$ for $\mathcal{L}^*$ as follows:

$$\mathfrak{A} \vDash P[[\tau_1], \ldots, [t_n]] \quad \text{if and only if} \quad P(\tau_1, \ldots, \tau_n) \in \Delta$$

and similarly for function and constant symbols. The proof is then completed by showing that $\mathfrak{A}$ is a model for $\Delta$ (and hence for $\Sigma$). The proof of

$$\mathfrak{A} \vDash \sigma \quad \text{if and only if} \quad \sigma \in \Delta \tag{17.7}$$

is done by induction on the number of quantifier blocks in $\sigma$: If $\sigma = \exists_{\xi<\alpha} v_\xi\, \varphi(v_\xi, \ldots)$, then by induction hypothesis we have

$$\mathfrak{A} \vDash \sigma(c_\xi^\varphi, \ldots)_{\xi<\alpha} \quad \text{if and only if} \quad \sigma(c_\xi^\varphi, \ldots)_{\xi<\alpha} \in \Delta$$

and (17.7) follows.

(ii) Let $\kappa$ be inaccessible and assume that the language $\mathcal{L}_{\kappa,\omega}$ satisfies the Weak Compactness Theorem. We shall show that $\kappa$ has the tree property. Let $(T, <)$ be a tree of height $\kappa$ such that each level of $T$ has size $< \kappa$. Let us consider the $\mathcal{L}_{\kappa,\omega}$ language with one unary predicate $B$ and constant symbols $c_x$ for all $x \in T$. Let $\Sigma$ be the following set of sentences:

$$\begin{array}{rl} \neg(B(c_x) \wedge B(c_y)) & \text{for all } x, y \in T \text{ that are incomparable,} \\ \bigvee_{x \in U_\alpha} B(c_x) & \text{for all } \alpha < \kappa, \text{ where } U_\alpha \text{ is the } \alpha\text{th level of } T \end{array}$$

($\Sigma$ says that $B$ is branch in $T$ of length $\kappa$). If $S \subset \Sigma$ and $|S| < \kappa$, then we get a model for $S$ by taking a sufficiently large initial segment of $T$ and some branch in this segment. By the Weak Compactness Theorem for $\mathcal{L}_{\kappa,\omega}$, $\Sigma$ has a model, which obviously yields a branch of length $\kappa$. $\square$

## Indescribability

Let $n > 0$ be a natural number and let us consider the $n$th order predicate calculus. There are variables of orders 1, 2, $\ldots$, $n$, and the quantifiers are applied to variables of all orders. An $n$th *order formula* contains, in addition to first order symbols and higher order quantifiers, predicates $X(z)$ where $X$ and $z$ are variables of order $k+1$ and $k$ respectively (for any $k < n$). Satisfaction for an $n$th order formula in a model $\mathfrak{A} = (A, P, \ldots, f, \ldots, c, \ldots)$ is defined as follows: Variables of first order are interpreted as elements of the set $A$, variables of second order as elements of $P(A)$ (as subsets of $A$), etc.; variables of order $n$ are interpreted as elements of $P^{n-1}(A)$. The predicate $X(z)$ is interpreted as $z \in X$. A $\Pi^n_m$ *formula* is a formula of order $n+1$ of the form

$$\underbrace{\forall X\, \exists Y \ldots}_{m \text{ quantifiers}}\ \psi \tag{17.8}$$

where $X, Y, \ldots$ are $(n+1)$th order variables and $\psi$ is such that all quantified variables are of order at most $n$. Similarly, a $\Sigma^n_m$ formula is as in (17.8), but with $\exists$ and $\forall$ interchanged.

We shall often exhibit a sentence $\sigma$ and claim that it is $\Pi^n_m$ (or $\Sigma^n_m$) although it is only *equivalent* to a $\Pi^n_m$ (or $\Sigma^n_m$) sentence, in the following sense: We are considering a specific type of models in which $\sigma$ is interpreted (e.g., the models $(V_\alpha, \in)$) and there is a $\Pi^n_m$ (or $\Sigma^n_m$) sentence $\bar{\sigma}$ such that the equivalence $\sigma \leftrightarrow \bar{\sigma}$ holds in all these models.

Note that every first order formula is equivalent to some $\Pi^0_n$ formula (and also to some $\Sigma^0_k$ formula).

**Definition 17.14.** A cardinal $\kappa$ is $\Pi^n_m$-*indescribable* if whenever $U \subset V_\kappa$ and $\sigma$ is a $\Pi^n_m$ sentence such that $(V_\kappa, \in, U) \vDash \sigma$, then for some $\alpha < \kappa$, $(V_\alpha, \in, U \cap V_\alpha) \vDash \sigma$.

**Lemma 17.15.** *Every measurable cardinal is* $\Pi^2_1$*-indescribable.*

*Proof.* Let $\kappa$ be a measurable cardinal, let $U \subset V_\kappa$ and let $\sigma$ be a $\Pi^2_1$ sentence of the (third order) language $\{\in, U\}$. Let us assume that $(V_\kappa, \in, U) \vDash \sigma$.

We have $\sigma = \forall X\, \varphi(X)$ where $X$ is a third order variable and $\varphi(X)$ contains only second and first order quantifiers. Thus

$$\forall X \subset V_{\kappa+1}\ (V_{\kappa+1}, \in, X, V_\kappa, U) \vDash \tilde{\varphi} \tag{17.9}$$

where $\tilde{\varphi}$ is the (first order) sentence obtained from $\varphi$ by replacing the first order quantifiers by the restricted quantifiers $\forall x \in V_\kappa$ and $\exists x \in V_\kappa$.

Now let $D$ be a normal measure on $\kappa$ and let $M = \mathrm{Ult}_D(V)$. Since $V^M_{\kappa+1} = V_{\kappa+1}$, we know that (17.9) holds also in $M$. Using the fact that $V_\kappa$ is represented in the ultrapower by the function $\alpha \mapsto V_\alpha$, $V_{\kappa+1}$ by $\alpha \mapsto V_{\alpha+1}$, and $U$ by $\alpha \mapsto U \cap V_\alpha$, we conclude that for almost all $\alpha$,

$$\forall X \subset V_{\alpha+1}\ (V_{\alpha+1}, \in, X, V_\alpha, U \cap V_\alpha) \models \tilde{\varphi}. \tag{17.10}$$

Then, translating (17.10) back into the third order language, we obtain

$$(V_\alpha, \in, U \cap V_\alpha) \models \sigma$$

for almost all, and hence for some, $\alpha < \kappa$. $\square$

**Lemma 17.16.** *If $\kappa$ is not inaccessible, then it is describable by a first order sentence, i.e., $\Pi^0_m$-describable for some $m$.*

*Proof.* Let $\kappa$ be a singular cardinal, and let $f$ be a function with $\operatorname{dom}(f) = \lambda < \kappa$ and $\operatorname{ran}(f)$ cofinal in $\kappa$. Let $U_1 = f$ and $U_2 = \{\lambda\}$, and let $\sigma$ be the first order sentence saying that $U_2$ is nonempty and that the unique element of $U_2$ is the domain of $U_1$. Clearly, $\kappa$ is describable in the sense that $(V_\kappa, \in, U_1, U_2) \models \sigma$ and there is no $\alpha < \kappa$ such that $(V_\alpha, \in, U_1 \cap V_\alpha, U_2 \cap V_\alpha) \models \sigma$. It is routine to find a single $U \subset V_\kappa$ and an $(\in, U)$-sentence $\tilde{\sigma}$ attesting to the describability of $\kappa$.

If $\kappa \le 2^\lambda$ for some $\lambda < \kappa$, there is a function $f$ that maps $P(\lambda)$ onto $\kappa$. We let $U_1 = f$ and $U_2 = \{P(\lambda)\}$; then $\kappa$ is described by the same sentence as above.

Finally, $\kappa = \omega$ is describable as follows: $(V_\kappa, \in) \models \forall x\, \exists y\, x \in y$. $\square$

The converse is also true; cf. Exercise 17.23.

We shall now present a result of Hanf and Scott that shows that $\Pi^1_1$-indescribable cardinals are exactly the weakly compact cardinals. First we need a lemma.

**Lemma 17.17.** *If $\kappa$ is a weakly compact cardinal, then for every $U \subset V_\kappa$, the model $(V_\kappa, \in, U)$ has a transitive elementary extension $(M, \in, U')$ such that $\kappa \in M$.*

*Proof.* Let $\Sigma$ be the set of all $\mathcal{L}_{\kappa,\kappa}$ sentences true in the model $(V_\kappa, \in, U, x)_{x \in V_\kappa}$ plus the sentences

$$\begin{aligned} &c \text{ is an ordinal,} \\ &c > \alpha, \qquad (\text{all } \alpha < \kappa). \end{aligned}$$

Clearly $|\Sigma| = \kappa$, and if $S \subset \Sigma$ is such that $|S| < \kappa$, then $S$ has a model (namely $V_\kappa$, where the constant $c$ can be interpreted as some ordinal greater than all the $\alpha$'s mentioned in $S$).

Hence $\Sigma$ has a model $\mathfrak{A} = (A, E, U^\mathfrak{A}, x^\mathfrak{A})_{x \in V_\kappa}$; we may assume that $A \supset V_\kappa$, $E \cap (V_\kappa \times V_\kappa) = \in$, $U^\mathfrak{A} \cap V_\kappa = U$, and $x^\mathfrak{A} = x$ for all $x \in V_\kappa$. Moreover, $V_\kappa \prec (A, E, U^\mathfrak{A})$ because $\mathfrak{A}$ satisfies all formulas true in $V_\kappa$ of all $x \in V_\kappa$. If we show that the model $(A, E)$ is well-founded, then the lemma follows.

Here we make use of the expressive power of the infinitary language $\mathcal{L}_{\kappa,\kappa}$: We consider the sentence

$$\neg\exists v_0\,\exists v_1\,\ldots\,\exists v_n\,\ldots\,\bigwedge_{n\in\omega}(v_{n+1}\in v_n). \tag{17.11}$$

The sentence (17.11) holds in a model $\mathfrak{A}=(A,E)$ if and only if $\mathfrak{A}$ is well-founded. Since $\Sigma$ contains the sentence (17.11), every model of $\Sigma$ is well-founded. □

The converse is also true; this will follow from the proof of Theorem 17.18.

**Theorem 17.18 (Hanf-Scott).** *A cardinal $\kappa$ is $\Pi^1_1$-indescribable if and only if it is weakly compact.*

*Proof.* First we show that every $\Pi^1_1$-indescribable cardinal is weakly compact. If $\kappa$ is $\Pi^1_1$-indescribable, then by Lemma 17.16, $\kappa$ is inaccessible, and it suffices to show that $\kappa$ has the tree property. In fact, by the proof of Theorem 17.13(i) it suffices to consider trees $(T,<)$ consisting of sequences $t:\gamma\to\{0,1\}$, $\gamma<\kappa$. Let $T$ be such a tree. For every $\alpha<\kappa$, the model $(V_\alpha,\in,T\cap V_\alpha)$ satisfies the $\Sigma^1_1$ sentence

$$\exists B\,(B\subset T\text{ and }B\text{ is a branch of unbounded length}). \tag{17.12}$$

Namely, let $B=\{t{\restriction}\xi:\xi<\alpha\}$ where $t$ is any $t\in T$ with domain $\alpha$. Since $\kappa$ is $\Pi^1_1$-indescribable, the sentence (17.12) holds in $(V_\kappa,\in,T)$ and hence $T$ has a branch of length $\kappa$.

To show that a weakly compact cardinal is $\Pi^1_1$-indescribable, we use Lemma 17.17. Let $\kappa$ be weakly compact, let $U\subset V_\kappa$ and let $\sigma$ be a $\Pi^1_1$ sentence true in $(V_\kappa,\in,U)$. We have $\sigma=\forall X\,\varphi(X)$ where $X$ is a second order variable and $\varphi$ has only first order quantifiers.

Let $(M,\in,U')$ be a transitive elementary extension of $(V_\kappa,\in,U)$ such that $\kappa\in M$. Since

$$(\forall X\subset V_\kappa)\ (V_\kappa,\in,U)\vDash\varphi(X)$$

and $V_\kappa^M=V_\kappa$, we have

$$(M,\in,U')\vDash(\forall X\subset V_\kappa)\ (V_\kappa,\in,U'\cap V_\kappa)\vDash\varphi(X).$$

Therefore,

$$(M,\in,U')\vDash\exists\alpha\,(\forall X\subset V_\alpha)\ (V_\alpha,\in,U'\cap V_\alpha)\vDash\varphi(X),$$

and so

$$(V_\kappa,\in,U)\vDash\exists\alpha\,(\forall X\subset V_\alpha)\ (V_\alpha,\in,U'\cap V_\alpha)\vDash\varphi(X).$$

Hence for some $\alpha<\kappa$, $(V_\alpha,\in,U\cap V_\alpha)\vDash\sigma$. □

**Corollary 17.19.** *Every weakly compact cardinal $\kappa$ is a Mahlo cardinal, and the set of Mahlo cardinals below $\kappa$ is stationary.*

*Proof.* Let $C \subset \kappa$ be a closed unbounded set. Since $\kappa$ is inaccessible, $(V_\kappa, \in, C)$ satisfies the following $\Pi^1_1$ sentence:

$$\neg\exists F\,(F \text{ is a function from some } \lambda < \kappa \text{ cofinally into } \kappa)$$
$$\text{and } C \text{ is unbounded in } \kappa.$$

By $\Pi^1_1$-indescribability, there exists a regular $\alpha < \kappa$ such that $C \cap \alpha$ is unbounded in $\alpha$; hence $\alpha \in C$. Thus $\kappa$ is Mahlo.

Being Mahlo is also expressible by a $\Pi^1_1$ sentence:

$$\forall X\,(\text{if } X \text{ is closed unbounded, then } \exists \text{ a regular } \alpha \text{ in } X)$$

and so the same argument as above shows that there is a stationary set of Mahlo cardinals below $\kappa$. $\square$

**Corollary 17.20.** *If $\kappa$ is weakly compact and if $S \subset \kappa$ is stationary, then there is a regular uncountable $\lambda < \kappa$ such that $S \cap \lambda$ is stationary in $\lambda$.*

*Proof.* "$\kappa$ is regular" is expressible by a $\Pi^1_1$ sentence in $(V_\kappa, \in)$ and so is "$\kappa$ is uncountable." "$S$ is stationary" is $\Pi^1_1$ in $(V_\kappa, \in, S)$: For every $C$, if $C$ is closed unbounded, then $S \cap C \neq \emptyset$. $\square$

**Lemma 17.21.** *If $\kappa$ is weakly compact and if $A \subset \kappa$ is such that $A \cap \alpha \in L$ for every $\alpha < \kappa$, then $A$ is constructible.*

*Proof.* Let $A \subset \kappa$ be such that $A \cap \alpha \in L$ for all $\alpha < \kappa$. By Lemma 17.17 there is a transitive model $(M, \in, A') \succ (V_\kappa, \in, A)$ such that $\kappa \in M$. Consider the sentence $\forall\alpha\,\exists x\,(x \text{ is constructible and } x = A \cap \alpha)$ and let $\alpha = \kappa$. $\square$

Unlike measurability, weak compactness is consistent with $V = L$:

**Theorem 17.22.** *If $\kappa$ is weakly compact then $\kappa$ is weakly compact in $L$.*

*Proof.* In $L$, let $T = (\kappa, <_T)$ be a tree of height $\kappa$ such that each level of $T$ has size less than $\kappa$. If $\kappa$ is weakly compact then $T$ has a branch $B$ (in the universe), and by Lemma 17.21, $B \in L$. Hence $\kappa$ has the tree property in $L$, and since $\kappa$ is inaccessible, it is weakly compact in $L$. $\square$

## Partitions and Models

Let us consider a model $\mathfrak{A} = (A, P^{\mathfrak{A}}, \ldots, F^{\mathfrak{A}}, \ldots, c^{\mathfrak{A}}, \ldots)$ of a (not necessarily countable) language $\mathcal{L} = \{P, \ldots, F, \ldots, c, \ldots\}$. Let $\kappa$ be an infinite cardinal and let us assume that the universe $A$ of the model $\mathfrak{A}$ contains all ordinals $\alpha < \kappa$, i.e., $\kappa \subset A$.

**Definition 17.23.** A set $I \subset \kappa$ is a *set of indiscernibles* for the model $\mathfrak{A}$ if for every $n \in \omega$, and every formula $\varphi(v_1, \ldots, v_n)$,

$$\mathfrak{A} \vDash \varphi[\alpha_1, \ldots, \alpha_n] \quad \text{if and only if} \quad \mathfrak{A} \vDash \varphi[\beta_1, \ldots, \beta_n]$$

whenever $\alpha_1 < \ldots < \alpha_n$ and $\beta_1 < \ldots < \beta_n$ are two increasing sequences of elements of $I$.

**Lemma 17.24.** *Let $\kappa$ be an infinite cardinal and assume that*

$$\kappa \to (\alpha)^{<\omega}_{2^\lambda}$$

*where $\alpha$ is a limit ordinal and $\lambda$ is an infinite cardinal. Let $\mathcal{L}$ be a language of size $\leq \lambda$ and let $\mathfrak{A}$ be a model of $\mathcal{L}$ such that $\kappa \subset A$. Then $\mathfrak{A}$ has a set of indiscernibles of order-type $\alpha$.*

*Proof.* Let $\Phi$ be the set of all formulas of the language $\mathcal{L}$. We consider the function $F : [\kappa]^{<\omega} \to P(\Phi)$ defined as follows: If $x \in [\kappa]^n$ and $x = \{\alpha_1, \ldots, \alpha_n\}$ where $\alpha_1 < \ldots < \alpha_n$, then

$$F(x) = \{\varphi(v_1, \ldots, v_n) \in \Phi : \mathfrak{A} \vDash \varphi[\alpha_1, \ldots, \alpha_n]\}.$$

The function $F$ is a partition into at most $2^\lambda$ pieces and thus has a homogeneous set $I \subset \kappa$ of order-type $\alpha$. It is now easy to verify that $I$ is a set of indiscernibles for $\mathfrak{A}$. $\square$

We shall see later that for a given limit ordinal $\alpha$, the least $\kappa$ that satisfies $\kappa \to (\alpha)^{<\omega}$ is inaccessible and satisfies $\kappa \to (\alpha)^{<\omega}_\lambda$ for all $\lambda < \kappa$. Now we shall prove this for Ramsey cardinals.

**Lemma 17.25.** *If $\kappa \to (\kappa)^{<\omega}$ and if $\lambda < \kappa$ is a cardinal, then $\kappa \to (\kappa)^{<\omega}_\lambda$.*

*Proof.* Let $F : [\kappa]^{<\omega} \to \lambda$ be a partition into $\lambda < \kappa$ pieces. We consider the following partition $G$ of $[\kappa]^{<\omega}$ into two pieces: If $\alpha_1 < \ldots < \alpha_k < \alpha_{k+1} < \ldots < \alpha_{2k}$ are elements of $\kappa$ and if $F(\{\alpha_1, \ldots, \alpha_k\}) = F(\{\alpha_{k+1}, \ldots, \alpha_{2k}\})$, then we let $G(\{\alpha_1, \ldots, \alpha_{2k}\}) = 1$; for all other $x \in [\kappa]^{<\omega}$, we let $G(x) = 0$.

Now, let $H \subset \kappa$ be a homogeneous set for $G$, $|H| = \kappa$. We claim that for each $k$ and each $x \in [H]^{2k}$, $G(x) = 1$: This is because $|H| = \kappa > \lambda$, and therefore we can find $\alpha_1 < \ldots < \alpha_k < \alpha_{k+1} < \ldots < \alpha_{2k}$ in $H$ such that $F(\{\alpha_1, \ldots, \alpha_k\}) = F(\{\alpha_{k+1}, \ldots, \alpha_{2k}\})$.

It follows that $H$ is homogeneous for $F$: If $\alpha_1 < \ldots < \alpha_n$ and $\beta_1 < \ldots < \beta_n$ are two sequences in $H$, we choose a sequence $\gamma_1 < \ldots < \gamma_n$ in $H$ such that both $\alpha_n < \gamma_1$ and $\beta_n < \gamma_1$. Then

$$G(\{\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_n\}) = G(\{\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n\}) = 1,$$

and hence

$$F(\{\alpha_1, \ldots, \alpha_n\}) = F(\{\gamma_1, \ldots, \gamma_n\}) = F(\{\beta_1, \ldots, \beta_n\}). \qquad \square$$

**Corollary 17.26.** *If $\kappa$ is a Ramsey cardinal and if $\mathfrak{A} \supset \kappa$ is a model of a language of size $< \kappa$, then $\mathfrak{A}$ has a set of indiscernibles of size $\kappa$.* □

The combinatorial methods introduced in this section will now be employed to obtain a result on measurable cardinals considerably stronger than Scott's Theorem. It will be shown that if a Ramsey cardinal exists then $V = L$ fails in a strong way. A more extensive theory will be developed in Chapter 18.

Let us make a few observations about models with definable Skolem functions. Let $\mathfrak{A}$ be a model of a language $\mathcal{L}$ such that $\mathfrak{A} \supset \kappa$ and let $I \subset \kappa$ be a set of indiscernibles for $\mathfrak{A}$. Let us assume that the model $\mathfrak{A}$ has *definable Skolem functions*; i.e., for every formula $\varphi(u, v_1, \ldots, v_n)$, where $n \geq 0$, there exists an $n$-ary function $h_\varphi$ in $\mathfrak{A}$ such that:

(i) $h_\varphi$ is definable in $\mathfrak{A}$, i.e., there is a formula $\psi$ such that

$$y = h_\varphi(x_1, \ldots, x_n) \quad \text{if and only if} \quad \mathfrak{A} \vDash \psi[y, x_1, \ldots, x_n]$$

for all $y, x_1, \ldots, x_n \in A$; and

(ii) $h_\varphi$ is a Skolem function for $\varphi$.

Let $\mathfrak{B} \subset \mathfrak{A}$ be the closure of $I$ under all functions in $\mathcal{L}$ and the functions $h_\varphi$ for all formulas $\varphi$. $\mathfrak{B}$ is an elementary submodel of $\mathfrak{A}$, and in fact is the smallest elementary submodel of $\mathfrak{A}$ that includes the set $I$; we call $\mathfrak{B}$ the *Skolem hull* of $I$ and say that $I$ *generates* $\mathfrak{B}$.

We augment the language of $\mathfrak{A}$ by adding function symbols for all the Skolem functions $h_\varphi$ and call *Skolem terms* the terms built from variables and constant symbols (0-ary functions) by applications of functions in $\mathcal{L}$ and the Skolem functions. Since $\mathfrak{B}$ is an elementary submodel of $\mathfrak{A}$, the interpretation of each Skolem term $t$ is the same in $\mathfrak{B}$ as in $\mathfrak{A}$. For every element $x \in \mathfrak{B}$ there is a Skolem term $t$ and indiscernibles $\gamma_1 < \ldots < \gamma_n$, elements of $I$, such that $x = t^{\mathfrak{A}}[\gamma_1, \ldots, \gamma_n] = t^{\mathfrak{B}}[\gamma_1, \ldots, \gamma_n]$. Now if $\psi$ is a formula of the augmented language, i.e., if $\psi$ also contains the Skolem terms, it still does not distinguish between the indiscernibles: If $\alpha_1 < \ldots < \alpha_n$ and $\beta_1 < \ldots < \beta_n$ are two sequences in $I$, then $\psi(\alpha_1, \ldots, \alpha_n)$ holds (either in $\mathfrak{A}$ or in $\mathfrak{B}$) if and only if $\psi(\beta_1, \ldots, \beta_n)$ holds.

**Theorem 17.27 (Rowbottom).** *If $\kappa$ is a Ramsey cardinal, then the set of all constructible reals is countable. More generally, if $\lambda$ is an infinite cardinal less than $\kappa$, then $|P^L(\lambda)| = \lambda$.*

*Proof.* Let $\kappa$ be a Ramsey cardinal and let $\lambda < \kappa$. Since $\kappa$ is inaccessible, we have $P^L(\lambda) \subset L_\kappa$. Consider the model

$$\mathfrak{A} = (L_\kappa, \in, P^L(\lambda), \alpha)_{\alpha \leq \lambda}.$$

$\mathfrak{A}$ is a model of the language $\mathcal{L} = \{\in, Q, c_\alpha\}_{\alpha \leq \lambda}$ where $Q$ is a one-place predicate (interpreted in $\mathfrak{A}$ as $P(\lambda) \cap L$) and $c_\alpha$, $\alpha \leq \lambda$, are constant symbols (interpreted as ordinals less than or equal to $\lambda$). Since $\kappa$ is Ramsey, there exists a set $I$ of size $\kappa$ of indiscernibles for $\mathfrak{A}$.

The model $\mathfrak{A}$ has definable Skolem functions: Since $\kappa$ is inaccessible, $L_\kappa$ is a model of ZFC + $V = L$ and therefore has a definable well-ordering. Thus let $\mathfrak{B} \subset L_\kappa$ be the elementary submodel of $\mathfrak{A}$ generated by the set $I$. Every element $x \in \mathfrak{B}$ is expressible as $x = t(\gamma_1, \ldots, \gamma_n)$ where $t$ is a Skolem term and $\gamma_1 < \ldots < \gamma_n$ are elements of $I$.

We shall now show that the set $S = P^L(\lambda) \cap \mathfrak{B}$ has at most $\lambda$ elements. Since $S$ is the interpretation in $\mathfrak{B}$ of the one-place predicate $Q$, it suffices to show that there are at most $\lambda$ elements $x \in \mathfrak{B}$ such that $\mathfrak{B} \vDash Q(x)$.

Let $t$ be a Skolem term. Let us consider the truth value of the formula

$$t(\alpha_1, \ldots, \alpha_n) = t(\beta_1, \ldots, \beta_n) \tag{17.13}$$

for a sequence of indiscernibles $\alpha_1 < \ldots < \alpha_n < \beta_1 < \ldots < \beta_n$. The formula (17.13) is either true for all increasing sequences in $I$ or false for all increasing sequences in $I$. If (17.13) is true, then it is true for any two sequences $\alpha_1 < \ldots < \alpha_n$, $\beta_1 < \ldots < \beta_n$, in $I$: Pick $\gamma_1, \ldots, \gamma_n$ bigger than both $\alpha_n$ and $\beta_n$ and then $t(\alpha_1, \ldots, \alpha_n) = t(\gamma_1, \ldots, \gamma_n) = t(\beta_1, \ldots, \beta_n)$. If (17.13) is false, then we choose $\kappa$ increasing sequences

$$\alpha_1^0 < \ldots < \alpha_n^0 < \alpha_1^1 < \ldots < \alpha_n^1 < \ldots < \alpha_1^\xi < \ldots < \alpha_n^\xi < \ldots \qquad (\xi < \kappa)$$

in $I$ and then $t(\alpha_1^\xi, \ldots, \alpha_n^\xi) \neq t(\alpha_1^\eta, \ldots, \alpha_n^\eta)$ whenever $\xi \neq \eta$. In conclusion, the set

$$\{t(\alpha_1, \ldots, \alpha_n) : \alpha_1 < \ldots < \alpha_n \text{ are in } I\} \tag{17.14}$$

has either one or $\kappa$ elements.

Now we apply this to evaluate the size of the set $S$. We know that $|S| < \kappa$ because $S \subset P^L(\lambda) \subset P(\lambda)$ and $\kappa$ is inaccessible. If $t$ is a Skolem term for which the set (17.14) has size $\kappa$, then $t(\alpha_1, \ldots, \alpha_n)$ is not in $S$, for any $\alpha_1 < \ldots < \alpha_n$ in $I$; by indiscernibility, $Q(t(\alpha_1, \ldots, \alpha_n))$ is true or false simultaneously for all increasing sequences in $I$. Thus if $t(\alpha_1, \ldots, \alpha_n) \in S$, the set (17.14) has only one element.

However, since $|\mathcal{L}| \leq \lambda$, there are at most $\lambda$ Skolem terms. And since every $x \in \mathfrak{B}$ has the form $t(\alpha_1, \ldots, \alpha_n)$ for some Skolem term and $\alpha_1 < \ldots < \alpha_n$ in $I$, it follows that $|S| \leq \lambda$.

Thus we have proved that $S = Q^{\mathfrak{B}} = P^L(\lambda) \cap \mathfrak{B}$ has at most $\lambda$ elements. Now $\mathfrak{B} \prec L_\kappa$ and $|\mathfrak{B}| = \kappa$; hence the transitive collapse of $\mathfrak{B}$ is $L_\kappa$ and we have an isomorphism

$$\pi : B \simeq L_\kappa.$$

Since each $\alpha \leq \lambda$ has a name in $\mathfrak{A}$, we have $\lambda \cup \{\lambda\} \subset \mathfrak{B}$ and so $\pi(X) = X$ for each $X \subset \lambda$ in $\mathfrak{B}$. In particular $\pi(X) = X$ for all $X \in S$; and since $Q^{\pi(\mathfrak{B})} = \pi(S) = S$, we have

$$S = P^L(\lambda) \cap \pi(\mathfrak{B}) = P^L(\lambda) \cap L_\kappa = P^L(\lambda).$$

This completes the proof: On the one hand, we proved that $|S| \le \lambda$; and on the other hand, $|P^L(\lambda)| \ge \lambda$; thus $|P^L(\lambda)| = \lambda$. $\square$

Every Ramsey cardinal is weakly compact. Not only is the least Ramsey cardinal greater than the least weakly compact but, as we show below, there is a hierarchy of large cardinals below each Ramsey cardinal, exceeding the least weakly compact.

**Definition 17.28.** For every limit ordinal $\alpha$, the *Erdős cardinal* $\eta_\alpha$ is the least $\kappa$ such that $\kappa \to (\alpha)^{<\omega}$.

We shall prove that each $\eta_\alpha$, if it exists, is inaccessible, and if $\alpha < \beta$ then $\eta_\alpha < \eta_\beta$. Note that $\kappa$ is a Ramsey cardinal if and only if $\kappa = \eta_\kappa$.

**Lemma 17.29.** *If $\kappa \to (\alpha)^{<\omega}$, then $\kappa \to (\alpha)^{<\omega}_{2^{\aleph_0}}$.*

*Proof.* Let $f$ be a partition, $f : [\kappa]^{<\omega} \to \{0,1\}^\omega$. For each $n < \omega$, let $f_n = f{\restriction}[\kappa]^n$, and for each $\kappa < \omega$, let $f_{n,k} : [\kappa]^n \to \{0,1\}$ be as follows:

$$f_{n,k}(\{\alpha_1,\dots,\alpha_n\}) = h(k), \qquad \text{where } h = f_n(\{\alpha_1,\dots,\alpha_n\}).$$

Let $\pi$ be a one-to-one correspondence between $\omega$ and $\omega \times \omega$ such that if $\pi(m) = (n,k)$, then $m \ge n$; for each $m$, let $g_m : [\kappa]^m \to \{0,1\}$ be the partition defined by

$$g_m(\{\alpha_1,\dots,\alpha_m\}) = f_{n,k}(\{\alpha_1,\dots,\alpha_n\})$$

where $(n,k) = \pi(m)$.

By the assumption, there exists $H \subset \kappa$ of order-type $\alpha$ which is homogeneous for all $g_m$. We claim that $H$ is homogeneous for $f$. If not, then $f_n(\{\alpha_1,\dots,\alpha_n\}) \ne f_n(\{\beta_1,\dots,\beta_n\})$ for some $\alpha$'s and $\beta$'s in $H$. Then for some $k$, $f_{n,k}(\{\alpha_1,\dots,\alpha_n\}) \ne f_{n,k}(\{\beta_1,\dots,\beta_n\})$, contrary to the assumption that $H$ is homogeneous for $g_m$, where $\pi(m) = (n,k)$. $\square$

**Lemma 17.30.** *For every $\kappa < \eta_\alpha$, $\eta_\alpha \to (\alpha)^{<\omega}_\kappa$.*

*Proof.* Let $\kappa < \eta_\alpha$, and let $f : [\eta_\alpha]^{<\omega} \to \kappa$. We wish to find a homogeneous set for $f$ of order-type $\alpha$. Since $\kappa < \eta_\alpha$, there exists $g : [\kappa]^{<\omega} \to \{0,1\}$ that has no homogeneous set of order-type $\alpha$. For each $n$, let $f_n = f{\restriction}[\eta_\alpha]^n$ and $g_n = g{\restriction}[\kappa]^n$, and let $\mathfrak{A}$ be the model $(V_{\eta_\alpha}, \in, f_n, g_n)_{n=0,1,\dots}$.

By Lemmas 17.29 and 17.24, the model $\mathfrak{A}$ has a set of indiscernibles $H$ of order-type $\alpha$. We shall show that $H$ is homogeneous for $f$. It suffices to show that for each $n$, the formula

$$f_n(\{\alpha_1,\dots,\alpha_n\}) = f_n(\{\beta_1,\dots,\beta_n\}) \tag{17.15}$$

holds in $\mathfrak{A}$ for any increasing sequence $\alpha_1 < \ldots < \alpha_n < \beta_1 < \ldots < \beta_n$ of indiscernibles: Then if $\alpha_1 < \ldots < \alpha_n$ and $\alpha'_1 < \ldots < \alpha'_n$ are arbitrary in $H$, we choose $\beta_1 < \ldots < \beta_n$ in $H$ such that $\alpha_n < \beta_1$ and $\alpha'_n < \beta_1$, and $f_n(\{\alpha_1, \ldots, \alpha_n\}) = f_n(\{\alpha'_1, \ldots, \alpha'_n\})$ follows from (17.15).

Thus let us assume that the negation of (17.15) holds for any $\alpha_1 < \ldots < \alpha_n < \beta_1 < \ldots < \beta_n$ in $H$. Let $u_\xi$, $\xi < \alpha$, be increasing $n$-sequences in $H$ such that the last element of $u_\xi$ is less than the first element of $u_\eta$ whenever $\xi < \eta$. Let $\gamma_\xi = f(u_\xi)$ for all $\xi < \alpha$, and let $G = \{\gamma_\xi : \xi < \alpha\}$. By indiscernibility, and since $\gamma_0 > \gamma_1 > \ldots > \gamma_\xi > \ldots$ is impossible, we have $\gamma_0 < \gamma_1 < \ldots < \gamma_\xi < \ldots$.

We shall reach a contradiction by showing that $G$ is homogeneous for $g$. For each $k$, consider the formula

$$g(\{f(u_{\xi_1}), \ldots, f(u_{\xi_k})\}) = g(\{f(u_{\nu_1}), \ldots, f(u_{\nu_k})\}). \tag{17.16}$$

By indiscernibility, either (17.16) or its negation holds for all increasing sequences $\xi_1 < \ldots < \xi_k < \nu_1 < \ldots < \nu_k$. The inequality cannot hold because $g$ takes only two values, 0 and 1, and three sequences $\langle \xi_1, \ldots, \xi_k \rangle$ would give three different values. Thus (17.16) holds, and the same argument as earlier in this proof shows that $g$ is constant on $[G]^k$. □

**Theorem 17.31.** *Every Erdős cardinal $\eta_\alpha$ is inaccessible, and if $\alpha < \beta$ then $\eta_\alpha < \eta_\beta$.*

*Proof.* First we claim that $\eta_\alpha$ is a strong limit cardinal. If $\kappa < \eta_\alpha$ then because $2^\kappa \not\to (\alpha)^2_\kappa$ (by Lemma 9.3) and $\eta_\alpha \to (\alpha)^2_\kappa$, we have $2^\kappa < \eta_\alpha$. We shall show that $\eta_\alpha$ is regular.

Let us assume that $\eta_\alpha$ is singular and that $\kappa = \operatorname{cf} \eta_\alpha$; let $\eta_\alpha = \lim_{\nu \to \kappa} \lambda_\nu$. For each $\nu < \kappa$, let $f^\nu : [\lambda_\nu]^{<\omega} \to \{0, 1\}$ be such that $f^\nu$ has no homogeneous set of order-type $\alpha$. For each $n$, let $f^\nu_n = f^\nu \restriction [\lambda_\nu]^n$; let $\mathfrak{A}$ be the model $(V_{\eta_\alpha}, \in, \lambda_\nu, f^\nu_n)_{\nu<\kappa, n=0,1,\ldots}$ Since $\eta_\alpha$ is a strong limit and $\kappa < \eta_\alpha$, the model $\mathfrak{A}$ has a set of indiscernibles $H$ of order-type $\alpha$.

Let $\nu$ be such that $\lambda_\nu$ is greater than the least element of $H$. Then by indiscernibility, all elements of $H$ are smaller than $\lambda_\nu$. Since the function $f^\nu$ takes only two values, it follows that for each $n$, it is the equality

$$f^\nu_n(\{\alpha_1, \ldots, \alpha_n\}) = f^\nu_n(\{\beta_1, \ldots, \beta_n\})$$

that holds for all increasing sequences $\alpha_1 < \ldots < \alpha_n < \beta_1 < \ldots < \beta_n$ in $H$, and not its negation. Hence $H$ is homogeneous for $f^\nu$, contrary to the assumption on $f^\nu$.

Finally, let $\alpha < \beta$ be limit ordinals, and let us assume that $\eta_\alpha = \eta_\beta$. For each $\xi < \eta_\alpha$, there exists a function $f_\xi : [\xi]^{<\omega} \to \{0, 1\}$ that has no homogeneous subset of $\xi$ of order-type $\alpha$. Let us define $g : [\eta_\beta]^{<\omega} \to \{0, 1\}$ by

$$g(\{\xi_1, \ldots, \xi_n\}) = f_{\xi_n}(\{\xi_1, \ldots, \xi_{n-1}\}).$$

Now if $H$ is homogeneous for $g$, then for each $\xi \in H$, $H \cap \xi$ is homogeneous for $f_\xi$. Hence the order-type of each $H \cap \xi$ is less than $\alpha$, and therefore the order-type of $H$ is at most $\alpha$, which is less than $\beta$. A contradiction. □

We shall now show that the least Erdős cardinal $\eta_\omega$ is greater than the least weakly compact cardinal. We use the following lemma, of independent interest:

**Lemma 17.32.** *Let $M$ and $N$ be transitive models of* ZFC *and let $j : M \to N$ be a nontrivial elementary embedding; let $\kappa$ be the least ordinal moved. If $P^M(\kappa) = P^N(\kappa)$, then $\kappa$ is a weakly compact cardinal in $M$.*

*Proof.* We prove a somewhat stronger statement: $\kappa$ is ineffable in $M$ (see Exercise 17.26).

Let $\langle A_\alpha : \alpha < \kappa\rangle \in M$ be such that $A_\alpha \subset \alpha$ for all $\alpha$. We have $j(A_\alpha) = A_\alpha$ for all $\alpha < \kappa$, and hence $j(\langle A_\alpha : \alpha < \kappa\rangle) = (\langle A_\alpha : \alpha < j(\kappa)\rangle$ (for some $A_\alpha$, $\kappa \le \alpha < j(\kappa)$). The set $A_\kappa$ is in $M$ and witnesses ineffability of $\kappa$ in $M$. □

**Theorem 17.33.** *If $\eta_\omega$ exists then there exists a weakly compact cardinal below $\eta_\omega$.*

*Proof.* Let $h_\varphi$, $\varphi \in \mathit{Form}$, be Skolem functions for the language $\{\in\}$ of set theory, and let us consider the model $\mathfrak{A} = (V_{\eta_\omega}, \in, h_\varphi^{\mathfrak{A}})_{\varphi\in \mathit{Form}}$ where for each $\varphi$, $h_\varphi^{\mathfrak{A}}$ is a Skolem function for $\varphi$ in $(V_{\eta_\omega}, \in)$. The model $\mathfrak{A}$ has a set of indiscernibles $I$ of order-type $\omega$. Let $B$ be the closure of $I$ under the Skolem functions $h_\varphi^{\mathfrak{A}}$.

Let us consider some nontrivial order-preserving mapping of $H$ into $H$. Using the Skolem functions, we extend this mapping (in the unique way) to a nontrivial elementary embedding of $B$ into $B$. Let $M$ be the transitive set isomorphic to $B$ and let $j : M \to M$ be the corresponding nontrivial elementary embedding.

Since $\eta_\omega$ is inaccessible, $V_{\eta_\omega}$ is a model of ZFC and thus $M$ is a transitive model of ZFC. By Lemma 17.32 there exists a weakly compact cardinal in $M$, and therefore in $V_{\eta_\omega}$. □

The next result shows that the Erdős cardinal $\eta_\omega$ is consistent with $V = L$. In Chapter 18 we show that the existence of $\eta_{\omega_1}$ implies $V \ne L$.

**Theorem 17.34.** *If $\kappa \to (\omega)^{<\omega}$ then $L \vDash \kappa \to (\omega)^{<\omega}$.*

*Proof.* Let $f$ be a constructible partition $f : [\kappa]^{<\omega} \to \{0, 1\}$. We claim that if there is an infinite homogeneous set for $f$, then there is one in $L$. Let $T$ be the set of all finite increasing sequences $t = \langle\alpha_0, \dots, \alpha_{n-1}\rangle$ in $\kappa$ such that for every $k \le n$, $f$ is constant on $[\{\alpha_0, \dots, \alpha_{n-1}\}]^k$, and let us consider the tree $(T, \supset)$; clearly, $T$ is constructible. We note that an infinite homogeneous set for $f$ exists if and only if $(T, \supset)$ is not well-founded. However, being well-founded is an absolute property for models of ZFC; and so if the tree is not well-founded, then it is not well-founded in $L$, and the claim follows. □

Let us consider models of a countable language $\mathcal{L}$, with a distinguished one-place predicate $Q$. A model $\mathfrak{A} = (A, Q^{\mathfrak{A}}, \dots)$ of $\mathcal{L}$ has *type* $(\kappa, \lambda)$ if $|A| = \kappa$ and $|Q^{\mathfrak{A}}| = \lambda$.

**Definition 17.35.** A cardinal $\kappa > \aleph_1$ is a *Rowbottom cardinal* if for every uncountable $\lambda < \kappa$, every model of type $(\kappa, \lambda)$ has an elementary submodel of type $(\kappa, \aleph_0)$.

An infinite cardinal is a *Jónsson cardinal* if every model of size $\kappa$ has a proper elementary submodel of size $\kappa$.

Every Rowbottom cardinal is a Jónsson cardinal and the following lemma, a variation on Rowbottom's Theorem, shows that every Ramsey cardinal is a Rowbottom cardinal.

**Lemma 17.36.** *Let $\kappa$ be a Ramsey cardinal, and let $\lambda$ be an infinite cardinal less than $\kappa$. Let $\mathfrak{A} = (A, \dots)$ be a model of a language $\mathcal{L}$ such that $|\mathcal{L}| \le \lambda$, and let $A \supset \kappa$. If $P \subset A$ is such that $|P| < \kappa$ then $\mathfrak{A}$ has an elementary submodel $\mathfrak{B} = (B, \dots)$ such that $|B| = \kappa$ and $|P \cap B| \le \lambda$.*

*Moreover, if $X \subset A$ is of size at most $\lambda$, then we can find $\mathfrak{B}$ such that $X \subset B$.*

*Moreover, if $\kappa$ is a measurable and $D$ is a normal measure on $\kappa$, then we can find $\mathfrak{B}$ such that $B \cap \kappa \in D$.*

*Proof.* First we add to the language $\mathcal{L}$ one unary predicate whose interpretation is the set $P$; we also add constant symbols for all $x \in X$. Next we find some Skolem functions $h_\varphi$ (in $(A, \dots, P, x)_{x \in X}$) for every formula $\varphi$, and extend the language further by adding function symbols for the functions $h_\varphi$.

Next we find a set of indiscernibles $I \subset \kappa$, of size $\kappa$, for the expanded model $\mathfrak{A}'$; if $\kappa$ is measurable and $D$ is a normal measure, we find $I \in D$. We let $B$ be the elementary submodel of $\mathfrak{A}'$ generated by $I$. As in the proof of Theorem 17.27, one proves that if $|P \cap B| < \kappa$ then $|P \cap B| \le \lambda$. □

In Chapter 18 we show that if there exists a Jónsson cardinal then $V \ne L$.

## Exercises

**17.1.** Let $U$ be a nonprincipal ultrafilter on $\omega$. Then $\mathrm{Ult}_U(V)$ is not well-founded.

[For each $k \in \omega$, let $f_k$ be a function on $\omega$ such that $f_k(n) = n - k$ for all $n \ge k$. Then $f_0 \ni^* f_1 \ni^* f_2 \ni^* \dots$ is a descending $\in^*$-sequence in Ult.]

**17.2.** If $U$ is not $\sigma$-complete, then $\mathrm{Ult}_U(V)$ is not well-founded.

[There exists a countable partition $\{X_n : n = 0, 1, 2, \dots\}$ of $S$ such that $X_n \notin U$ for all $n$. For each $k$, let $f_k$ be a function on $S$ such that $f_k(x) = n - k$ for all $x \in X_n$.]

**17.3.** If Ult is well-founded, then every ordinal number $\alpha$ is represented by a function $f : S \to Ord$.

**17.4.** If $U$ is a principal ultrafilter $\{X \in S : x_0 \in S\}$ then $[f] = f(x_0)$ for each $f$, and $j_U$ is the identity mapping.

**17.5.** Let $U$ be a nonprincipal $\sigma$-complete ultrafilter on $S$ and let $\lambda$ be the largest cardinal such that $U$ is $\lambda$-complete. Then $j_U(\lambda) > \lambda$.

[Let $\{X_\alpha : \alpha < \lambda\}$ be a partition of $S$ into sets of measure 0; let $f$ be a function on $S$ such that $f(x) = \alpha$ if $x \in X_\alpha$. Then $[f] \geq \lambda$.]

**17.6.** If $j$ is an elementary embedding of the universe into a transitive model $M$, then $M = \bigcup_{\alpha \in Ord} j(V_\alpha)$.

**17.7.** Let $j$ be an elementary embedding of the universe and let $\kappa$ be the least ordinal moved. If $X$ is a class of ordinals such that $\kappa \in j(X)$, then $\kappa \in M(X)$.

**17.8.** If $j : V \to M$ is a nontrivial elementary embedding, if $\kappa$ is the least ordinal moved, and if $\lambda = \lim\{\kappa, j(\kappa), j(j(\kappa)), \ldots\}$, then there exists $A \subset \lambda$ such that $A \notin M$.

[Assuming that $M$ contains all bounded subsets of $\lambda$, the proof of Theorem 17.7 shows that $G \notin M$.]

**17.9.** If $\kappa$ is measurable, then there exists a normal measure $D$ on $\kappa$ such that $\mathrm{Ult}_D(V) \models \kappa$ is not measurable.

[Let $D$ be a normal measure such that $j_D(\kappa)$ is the least possible ordinal; let $M = \mathrm{Ult}_D(V)$. If $\kappa$ is measurable in $M$, then there is a normal measure $U$ on $\kappa$ such that $U \in M$. Since $P(\kappa) \subset M$, we have $\mathrm{Ult}_U(\kappa, <) \in M$. By Lemma 17.9(iii) we have $j_U(\kappa) < ((2^\kappa)^+)^M$. Since $((2^\kappa)^+)^M < j_D(\kappa)$, we get a contradiction.]

A function $f$ on $\kappa$ is *monotone* if $f(\alpha) \leq f(\beta)$ whenever $\alpha < \beta$.

**17.10.** Let $U$ be a nonprincipal $\kappa$-complete ultrafilter on $\kappa$. Then $U$ extends the closed unbounded filter if and only if the diagonal function is the least nonconstant monotone function in $\mathrm{Ult}_U$.

[If $U$ extends the closed unbounded filter and if $f$ is monotone and regressive on some $X \in U$, then since $X$ is stationary, $f$ is constant on an unbounded set and hence constant almost everywhere. If $U$ does not extend the closed unbounded filter, then $f(\alpha) = \sup(C \cap \alpha)$ (where $C \notin U$ is closed unbounded) is a nonconstant monotone function regressive on $X \in U$.]

**17.11.** Let $U$ be a $\kappa$-complete ultrafilter on $\kappa$, and let $h : \kappa \to \kappa$. If $D = h_*(U)$, then the mapping $k : \mathrm{Ult}_D(V) \to \mathrm{Ult}_U(V)$ defined by $k([f]_D) = [f \circ h]_U$ is an elementary embedding.

**17.12.** If $D$ is a normal measure on $\kappa$ and $\{\alpha : 2^\alpha \leq \alpha^{++}\} \in D$, then $2^\kappa \leq \kappa^{++}$. More generally, if $\beta < \kappa$ and $\{\aleph_\alpha : 2^{\aleph_\alpha} \leq \aleph_{\alpha+\beta}\} \in D$, then $2^{\aleph_\kappa} \leq \aleph_{\kappa+\beta}$.

[If $f$ is such that $f(\aleph_\alpha) = \aleph_{\alpha+\beta}$ for all $\alpha < \kappa$, then $[f]_D = (\aleph_{\kappa+j(\beta)})^M \leq \aleph_{\kappa+\beta}$.]

**17.13.** If $D$ is a normal measure on $\kappa$ and $\{\alpha : 2^{\aleph_\alpha} < \aleph_{\alpha+\alpha}\} \in D$, then $2^{\aleph_\kappa} < \aleph_{\kappa+\kappa}$.

[If $f(\alpha) = \aleph_{\alpha+\alpha}$, then $[f] = (\aleph_{\kappa+\kappa})^M$.]

**17.14.** Let $\kappa$ be measurable and let $\lambda = \aleph_{\kappa+\kappa}$ be strong limit. Then $2^\lambda < \aleph_{(2^\kappa)^+}$.

[$j(\lambda) = (\aleph_{j(\kappa+\kappa)})^M \leq \aleph_{j(\kappa)+j(\kappa)}$; $j(\kappa) + j(\kappa) < (2^\kappa)^+$.]

**17.15.** Let $\kappa$ be measurable, let $\lambda$ be strong limit, $\mathrm{cf}\,\lambda = \kappa$, such that $\lambda < \aleph_\lambda$. Then $2^\lambda < \aleph_\lambda$.

[If $\lambda = \aleph_\alpha$, then $j(\lambda) = (\aleph_{j(\alpha)})^M \leq \aleph_{j(\alpha)}$, and $j(\alpha) < (\alpha^\kappa)^+ < \lambda$.]

**17.16.** Let $\Phi(\alpha)$ denote the $\alpha$th fixed point of $\aleph$, i.e., the $\alpha$th ordinal $\xi$ such that $\aleph_\xi = \xi$. Let $\kappa$ be measurable and let $\lambda = \Phi(\kappa + \kappa)$ be strong limit. Then $2^\lambda < \Phi((2^\kappa)^+)$.

[Use the fact that $(\Phi(\alpha))^M \leq \Phi(\alpha)$ for all $\alpha$.]

**17.17.** If $\kappa = \lambda^+$ is a successor cardinal, then the Weak Compactness Theorem for $\mathcal{L}_{\kappa,\omega}$ is false.

[Consider constants $c_\alpha$, $\alpha \le \kappa$, a binary relation $<$ and a ternary relation $R$. Consider the sentences saying that (a) $<$ is a linear ordering; (b) $c_\alpha < c_\beta$ for $\alpha < \beta$; (c) each $f_x$ is a function, where $f_x(y) = z$ stands for $R(x,y,z)$. Let $\Sigma$ consist of these sentences, the sentence $z < x \to \exists y\, R(x,y,z)$ (saying that $\operatorname{ran}(f_x) \supset \{z : z < x\}$), and the infinitary sentence $R(x,y,z) \to \bigvee_{\xi<\lambda}(y = c_\xi)$ (saying that $\operatorname{dom}(f_x) \subset \{c_\xi : \xi < \lambda\}$). Show that each $S \subset \Sigma$, $|S| \le \lambda$, has a model, but $\Sigma$ does not.]

**17.18.** If $\kappa$ is a singular cardinal, then the Weak Compactness Theorem for $\mathcal{L}_{\kappa,\omega}$ is false.

[Let $A \subset \kappa$ be a cofinal subset of size $< \kappa$. Consider constants $c_\alpha$, $\alpha \le \kappa$, and a linear ordering $<$. There is $\Sigma$ that says on the one hand that $\{c_\alpha : \alpha \in A\}$ is cofinal in the universe, and on the other hand that for each $\alpha < \kappa$, if $(\forall \beta < \alpha)\, c_\kappa > c_\beta$ then $c_\kappa > c_\alpha$; and each $S \subset \Sigma$, $|S| < \kappa$, has a model.]

**17.19.** If $\kappa$ is weakly compact and if $(\mathcal{B}, \subset)$ is a $\kappa$-complete algebra of subsets of $\kappa$ such that $|\mathcal{B}| = \kappa$, then every $\kappa$-complete filter $F$ on $\mathcal{B}$ can be extended to a $\kappa$-complete ultrafilter on $\mathcal{B}$.

[Consider constants $c_X$ for all $X \in \mathcal{B}$, and a unary predicate $U$. Let $\Sigma$ be the following set of $\mathcal{L}_{\kappa,\omega}$-sentences: $\neg U(c_\emptyset)$, $U(c_X) \vee U(c_{\kappa - X})$ for all $X \in \mathcal{B}$, $U(c_X) \to U(c_Y)$ for all $X \subset Y$ in $\mathcal{B}$, $U(c_X)$ for all $X \in F$, and $\bigwedge_{X\in\mathcal{A}} U(c_X) \to U(c_{\bigcap \mathcal{A}})$ for all $\mathcal{A} \subset \mathcal{B}$ such that $|\mathcal{A}| < \kappa$. Show that $\Sigma$ has a model.]

**17.20.** If $\kappa$ is inaccessible and if every $\kappa$-complete filter on any $\kappa$-complete algebra $\mathcal{B}$ of subsets of $\kappa$ such that $|\mathcal{B}| = \kappa$ can be extended to a $\kappa$-complete ultrafilter, then $\kappa$ is weakly compact.

[As in Lemma 10.18.]

**17.21.** If $(P, <)$ is a linearly ordered set of size $\kappa$, and $\kappa$ is weakly compact, then there is a subset $W \subset P$ of size $\kappa$ that is either well-ordered or conversely well-ordered by $<$.

**17.22.** The least measurable cardinal is $\Sigma^2_1$-describable.

[$\exists U\, (U$ is $\kappa$-complete nonprincipal ultrafilter on $\kappa)$.]

**17.23.** Every inaccessible cardinal is $\Pi^0_m$-indescribable for all $m$.

[Let $U \subset V_\kappa$. The model $(V_\kappa, \in, U)$ has a countable elementary submodel $M_0$. Let $\alpha_0 < \kappa$ be such that $M_0 \subset V_{\alpha_0}$. For each $n$, let $M_{n+1}$ be an elementary submodel of $(V_\kappa, \in, U)$ such that $V_{\alpha_n} \subset M_{n+1}$, and let $M_{n+1} \subset V_{\alpha_{n+1}}$. Let $\alpha = \lim_{n\to\omega} \alpha_n$; then $V_\alpha$ is an elementary submodel of $(V_\kappa, \in, U)$.]

**17.24.** If $\kappa$ is weakly compact, then there is no countably generated complete Boolean algebra $B$ such that $|B| = \kappa$.

[Assume that $B$ is such. Note that $\operatorname{sat}(B) = \kappa$. We may assume that $B = (\kappa, +, \cdot, -)$; let $A \subset \kappa$ be a countable set of generators. Let $U_1$ be the set of all pairs $(u, x)$ such that $u \in \kappa$, $x \subset \kappa$, $|x| < \kappa$, and $u = \sum x$, let $U_2 = \{A\}$. Let $\sigma$ be the conjunction of these sentences: (a) $B$ is a Boolean algebra and $B \supset A$ (first order), (b) $\forall x\, \exists u\,$(if $x \subset \kappa$, then $u = \sum x$) (first order), and (c) $\forall X\,$(if $X \subset \kappa$ and $X$ is a partition of $B$, then $\exists x\, (x = X)$) (here $x$ is a first order variable; the sentence (c) is $\Pi^1_1$). Since $(V_\kappa, \in, U_1, U_2)$ satisfies $\sigma$, there is some $\alpha < \kappa$ such that $(V_\alpha, \in, U_1 \cap V_\alpha, U_2 \cap V_\alpha) \models \sigma$. Then $(\alpha, +, \cdot, -)$ is a complete subalgebra of $B$ containing $A$.]

**17.25.** Let $\kappa$ be a measurable cardinal. If $\langle A_\alpha : \alpha < \kappa \rangle$ is a sequence of sets such that $A_\alpha \subset \alpha$ for all $\alpha < \kappa$, then there exists an $A \subset \kappa$ such that $\{\alpha \in \kappa : A \cap \alpha = A_\alpha\}$ is stationary.

A cardinal $\kappa$ with the property from Exercise 17.25 is called *ineffable.*

**17.26.** Let $\kappa$ be ineffable and let $f : [\kappa]^2 \to \{0, 1\}$ be a partition. Then there exists a stationary homogeneous set. (Hence $\kappa$ is weakly compact.)

[For each $\alpha < \kappa$ let $A_\alpha = \{\xi < \alpha : f(\{\xi, \alpha\}) = 1\}$, and let $A \subset \kappa$ be such that $S = \{\alpha : A \cap \alpha = A_\alpha\}$ is stationary. Either $S \cap A$ or $S - A$ is stationary, and is homogeneous.]

**17.27.** If $\kappa$ is ineffable then $\kappa$ is ineffable in $L$.

[Use Lemma 17.21.]

**17.28.** If $\kappa$ is Ramsey then $\aleph_1$ is inaccessible in $L$.

[Show that $P^{L[x]}(\omega)$ is countable for every $x \subset \omega$.]

**17.29.** If $M$ is a transitive model of ZFC and if $j : M \to M$ is a nontrivial elementary embedding, then the least ordinal $\kappa$ moved by $j$ is $\Pi^n_m$-indescribable in $M$, for all $n$ and $m$.

[If $U \subset V^M_\kappa$, then $U = j(U) \cap V^M_\kappa$. If $\sigma$ is a $\Pi^n_m$ sentence and $M \vDash ((V_\kappa, \in, U) \vDash \sigma)$, then $M \vDash ((\exists \alpha < j(\kappa))\, (V_\alpha, \in, j(U) \cap V_\alpha) \vDash \sigma)$.]

**17.30.** The cardinal $\eta_\omega$ is not weakly compact.

[$\eta_\omega$ is $\Pi^1_1$-describable.]

**17.31.** An infinite cardinal $\kappa$ is a Jónsson cardinal if and only if for every $F : [\kappa]^{<\omega} \to \kappa$ there exists a set $H \subset \kappa$ of size $\kappa$ such that the image of $[H]^{<\omega}$ under $F$ is not the whole set $\kappa$.

[To show that the condition is necessary, consider the model $(\kappa, <, F_1, F_2, \ldots)$ where $F_n = F{\restriction}[\kappa]^n$. To show that the condition is sufficient, let $\mathfrak{A} = (\kappa, \ldots)$ be a model. Let $\{h_n : n < \omega\}$ be a set of Skolem functions for $\mathfrak{A}$, closed under composition and arranged so that each $h_\alpha$ is $n$-ary. For each $x \in [\kappa]^n$, let $F(x) = h_n(x)$. If $H \subset \kappa$, then the image of $[H]^{<\omega}$ under $F$ is an elementary submodel of $\mathfrak{A}$.]

**17.32.** $\aleph_0$ is not a Jónsson cardinal.

[Let $\mathfrak{A} = (\omega, f)$ where $f(n) = n - 1$ for all $n > 0$.]

**17.33.** If $\kappa$ is a Rowbottom cardinal, then either $\kappa$ is weakly inaccessible or $\operatorname{cf} \kappa = \omega$.

[To show that $\kappa = \lambda^+$ is not Rowbottom, let $f_\alpha$ be a one-to-one mapping of $\alpha$ onto $\lambda$, for each $\alpha$, such that $\lambda \le \alpha < \kappa$. Let $\mathfrak{A} = (\kappa, \lambda, <, R)$ where $R(\alpha, \beta, \gamma)$ if and only if $f_\alpha(\beta) = \gamma$. If $(B, B \cap \lambda, <, R \cap B^3) \prec \mathfrak{A}$ and $|B| = \kappa$, let $\alpha$ be the $\lambda$th element of $B$; then $f_\alpha(B \cap \alpha) \subset B \cap \lambda$ and hence $|B \cap \lambda| = \lambda > \aleph_0$.

To show that $\kappa$ is not Rowbottom if $\kappa > \operatorname{cf} \kappa = \lambda > \aleph_0$, let $f$ be a nondecreasing function of $\kappa$ onto $\lambda$ and use $f$ to produce a counterexample.]

## Historical Notes

In [1963/64] Keisler and Tarski introduced the method of ultraproducts in the study of measurable cardinals, and it was established that the least measurable cardinal is greater than the least inaccessible cardinal. Scott used the method of ultrapowers to prove that existence of measurable cardinals contradicts the Axiom of Constructibility. Rowbottom and Silver initiated applications of infinitary combinatorics developed by Erdős and his collaborators. Scott's Theorem appeared in [1961] and Kunen's Theorem in [1971a]. (Lemma 17.8 is due to Erdős and Hajnal [1966].)

In [1963/64a] Hanf studied compactness of infinitary languages; his work let to the systematic study of Keisler and Tarski. Hanf proved that the least inaccessible cardinal is not measurable (in fact not weakly compact); Erdős and Hajnal then pointed out (cf. [1962]) that the same result can be proved using infinitary combinatorics. Keisler and Tarski introduced the Mahlo operation and showed that the least measurable cardinal is much greater than, e.g., the least Mahlo cardinal, etc.

The equivalence of various formulations of weak compactness is a result of several papers. In [1963/64a] Hanf initiated investigations of compactness of infinitary languages. Erdős and Tarski listed in [1961] several properties that were subsequently shown mutually equivalent (for inaccessible cardinals) and proved several implications. These properties included the partition property $\kappa \to (\kappa)^2_2$, the tree property, and several other properties. Hanf and Scott [1961] introduced $\Pi^n_m$-indescribability and announced Theorem 17.18. Further contributions were made in the papers Hanf [1963/64b], Hajnal [1964], Keisler [1962], Monk and Scott [1963/64], Tarski [1962], and Keisler and Tarski [1963/64]. A complete list of equivalent formulations with the proofs appeared in Silver [1971b]. Theorem 17.22 is due to Silver [1971b]. Rowbottom's Theorem (as well as Lemma 17.36) are due to Rowbottom [1971].

The main results on Erdős cardinals are due to Rowbottom, Reinhardt, and Silver. Rowbottom proved that if $\eta_{\omega_1}$ exists, then there are only countably many constructible reals (see [1971]); Theorem 17.33 is due to Reinhardt and Silver [1965], and Theorem 17.34 is due to Silver [1970a].

Exercise 17.10: Ketonen [1973].

Ineffable cardinals were introduced by Jensen; Exercises 17.26 and 17.27 are due to Kunen and Jensen.

Exercise 17.29: Reinhardt and Silver [1965].