19. Iterated Ultrapowers and $L[U]$

In this chapter we investigate inner models for measurable cardinals, using Kunen's technique of iterated ultrapowers.

The Model $L[U]$

Let κ be a measurable cardinal and let U be a κ-complete nonprincipal ultrafilter on κ. Let us consider the model $L[U]$. By Lemma 13.23, $L[U] = L[\bar{U}]$, where $\bar{U} = U \cap L[U]$.

Lemma 19.1. *In $L[U]$, $\bar{U}$ is a κ-complete nonprincipal ultrafilter on κ. Moreover, if U is normal, then $L[U] \vDash \bar{U}$ is normal.*

Proof. A straightforward verification. For instance if U is normal and $f \in L[U]$ is a regressive function on κ, then for some $\gamma < \kappa$, the set $X = \{\alpha : f(\alpha) = \gamma\}$ is in U; since $X \in L[U]$, $L[U] \vDash f$ is constant on some $X \in \bar{U}$. □

We shall eventually prove, among others, that the model $L[U]$ satisfies GCH. For now, we recall Theorem 13.22(iv) by which $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ holds in $L[U]$ for all sufficiently large α. Specifically, using the Condensation Lemma 13.24, we get:

Lemma 19.2. *If $V = L[A]$, and if $A \subset P(\omega_\alpha)$, then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.*

Proof. Let X be a subset of ω_α. Let λ be a cardinal such that $A \in L_\lambda[A]$ and $X \in L_\lambda[A]$. Let M be an elementary submodel of $(L_\lambda[A], \in)$ such that $\omega_\alpha \subset M$, $A \in M$, $X \in M$, and $|M| = \aleph_\alpha$. Let π be the transitive collapse of M, and let $N = \pi(M)$. Since $\omega_\alpha \subset M$, we have $\pi(Z) = Z$ for every $Z \subset \omega_\alpha$ that is in M and in particular $\pi(X) = X$; also, $\pi(A) = \pi(A \cap M) = A \cap N$. Now $N = L_\gamma[A \cap N]$ for some γ, and hence $N = L_\gamma[A]$. Since $|N| = \aleph_\alpha$, we have $\gamma < \omega_{\alpha+1}$ and hence $X \in L_{\omega_{\alpha+1}}[A]$. It follows that every subset of ω_α is in $L_{\omega_{\alpha+1}}[A]$ and therefore $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. □

Theorem 19.3 (Silver). *If $V = L[D]$ where D is a normal measure on a measurable cardinal κ, then the Generalized Continuum Hypothesis holds.*

Proof. If $\lambda \geq \kappa$, then $D \subset P(\lambda)$ and hence $2^\lambda = \lambda^+$ by Lemma 19.2. Thus it suffices to show that $2^\lambda = \lambda^+$ for every infinite cardinal $\lambda < \kappa$. Let $\lambda < \kappa$ and let us assume that there are more than λ^+ subsets of λ; we shall reach a contradiction. If $2^\lambda > \lambda^+$, then there exists a set $X \subset \lambda$ that is the λ^+th subset of λ in the canonical well-ordering $<_{L[D]}$ of $L[D]$. Let α be the least ordinal such that $X \in L_\alpha[D]$. Since the well-ordering $<_{L[D]}$ has the property that each $L_\xi[D]$ is an initial segment of $<_{L[D]}$ every subset of λ preceding X is also in $L_\alpha[D]$ and hence the set $P(\lambda) \cap L_\alpha[D]$ has size at least λ^+.

We shall now apply Lemma 17.36. Let η be a cardinal such that $\eta > \alpha$ and that $D \in L_\eta[D]$, and consider the model $\mathfrak{A} = (A, \in)$ where $A = L_\eta[D]$. We have $\kappa \subset A$, and we consider the set $P = P(\lambda) \cap A$. Since $2^\lambda < \kappa$, we have $|P| < \kappa$. By Lemma 17.36, there is an elementary submodel $\mathfrak{B} = (B, \in) \prec \mathfrak{A}$ such that $\lambda \cup \{D, X, \alpha\} \subset B$, $\kappa \cap B \in D$ and $|P \cap B| \leq \lambda$. Let π be the transitive collapse of $\mathfrak{B}$ onto a transitive set M; we have $M = L_\gamma[\pi(D)]$ for some γ.

Using the normality of D, we show that $\pi(D) = D \cap M$. Clearly, $\pi(\kappa) = \kappa$ because $|\kappa \cap B| = \kappa$. The function π is one-to-one, and for every $\xi < \kappa$, $\pi(\xi) \leq \xi$. Since D is normal, there is a set $Z \in D$ such that $\pi(\xi) = \xi$ for all $\xi \in Z$. Hence if $Y \in B$ is a set in D, then $\pi(Y) \supset \pi(Y \cap Z) = Y \cap Z$, and so $\pi(Y)$ is also in D; similarly, if $Y \in B$ and $\pi(Y) \in D$, then $Y \in D$. It follows that $\pi(D) = D \cap M$.

Hence $M = L_\gamma[D]$. Since $\lambda \subset B$, π maps every subset of λ onto itself, and so $P(\lambda) \cap M = P(\lambda) \cap B$. In particular, we have $\pi(X) = X$ and so $X \in L_\gamma[D]$. By the minimality assumption on α, we have $\alpha \leq \gamma$, and this is a contradiction since on the one hand $|P(\lambda) \cap L_\alpha[D]| \geq \lambda^+$, and on the other hand $|P(\lambda) \cap L_\gamma[D]| \leq \lambda$. □

One proves rather easily that the model $L[D]$ has only one measurable cardinal:

Lemma 19.4. *If $V = L[D]$ and D is a normal measure on κ, then κ is the only measurable cardinal.*

Proof. Let us assume that there is a measurable cardinal $\lambda \neq \kappa$ and let us consider the elementary embedding $j_U : V \to M$ where U is some nonprincipal λ-complete ultrafilter on λ. We shall prove that $M = L[D] = V$ thus getting a contradiction since $U \notin M$ by Lemma 17.9(ii).

Since j is elementary, it is clear that $M = L[j(D)]$. If $\lambda > \kappa$, then $j(D) = D$ and so $M = L[D]$. Thus assume that $\lambda < \kappa$.

Since κ is measurable, the set $Z = \{\alpha < \kappa : \alpha \text{ is inaccessible and } \alpha > \lambda\}$ belongs to D. By Lemma 17.9(v), $j(\kappa) = \kappa$ and $j(\alpha) = \alpha$ for all $\alpha \in Z$. We shall show that $j(D) = D \cap M$. It suffices to show that $j(D) \subset D \cap M$ since $j(D)$ is (in M) an ultrafilter. Let $X \in j(D)$ be represented by $f : \lambda \to D$. Let $Y = \bigcap_{\xi<\lambda} f(\xi)$; we have $Y \in D$, and clearly $j(Y) \subset X$. Now if $\alpha \in Y \cap Z$, then $j(\alpha) = \alpha$ and so $X \supset j(Y) \supset j``(Y \cap Z) = Y \cap Z \in D$ and hence $X \in D$.

Thus $j(D) = D \cap M$, and we have $M = L[j(D)] = L[D \cap M] = L[D]$. □

Iterated Ultrapowers

Let κ be a measurable cardinal and let U be a κ-complete nonprincipal ultrafilter on κ. Using U, we construct an ultrapower of V mod U; and since the ultrapower is well-founded, we identify the ultrapower with its transitive collapse, a transitive model $M = \text{Ult}_U(V)$. Let us denote this transitive model $\text{Ult}_U^{(1)}(V)$ or just $\text{Ult}^{(1)}$. Let $j^{(0)} = j_U$ be the canonical embedding of V in $\text{Ult}^{(1)}$, and let $\kappa^{(1)} = j^{(0)}(\kappa)$ and $U^{(1)} = j^{(0)}(U)$.

In the model $\text{Ult}^{(1)}$, the ordinal $\kappa^{(1)}$ is a measurable cardinal and $U^{(1)}$ is a $\kappa^{(1)}$-complete nonprincipal ultrafilter on $\kappa^{(1)}$. Thus working inside $\text{Ult}^{(1)}$, we can construct an ultrapower mod $U^{(1)}$: $\text{Ult}_{U^{(1)}}(\text{Ult}^{(1)})$. Let us denote this ultrapower $\text{Ult}^{(2)}$, and let $j^{(1)}$ be the canonical embedding of $\text{Ult}^{(1)}$ in $\text{Ult}^{(2)}$ given by this ultrapower. Let $\kappa^{(2)} = j^{(1)}(\kappa^1)$ and $U^{(2)} = j^{(1)}(U^{(1)})$.

We can continue this procedure and obtain transitive models

$$\text{Ult}^{(1)}, \quad \text{Ult}^{(2)}, \quad \ldots, \quad \text{Ult}^{(n)}, \quad \ldots \qquad (n < \omega).$$

[That we can indeed construct such a sequence of classes follows from the observation that for each α, the initial segment $V_\alpha \cap \text{Ult}^{(n)}$ of each ultrapower in the sequence is defined from an initial segment V_β of the universe (where β is something like $\kappa + \alpha + 1$).]

Thus we get a sequence of models $\text{Ult}^{(n)}$, $n < \omega$ (where $\text{Ult}^{(0)} = V$). For any $n < m$, we have an elementary embedding $i_{n,m} : \text{Ult}^{(n)} \to \text{Ult}^{(m)}$ which is the composition of the embeddings $j^{(n)}, j^{(n+1)}, \ldots, j^{(m-1)}$:

$$i_{n,m}(x) = j^{(m-1)} j^{(m-2)} \ldots j^{(n)}(x) \qquad (x \in \text{Ult}^{(n)}).$$

These embeddings form a commutative system; that is,

$$i_{m,k} \circ i_{n,m} = i_{n,k} \qquad (m < n < k).$$

We also let $\kappa^{(n)} = i_{0,n}(\kappa)$, and $U^{(n)} = i_{0,n}(U)$. Note that $\kappa^{(0)} < \kappa^{(1)} < \ldots < \kappa^{(n)} < \ldots$, and $\text{Ult}^{(0)} \supset \text{Ult}^{(1)} \supset \ldots \supset \text{Ult}^{(n)} \supset \ldots$.

Thus we have a directed system of models and elementary embeddings

$$\{\text{Ult}^{(n)}, i_{m,n} : m, n \in \omega\}. \tag{19.1}$$

Even though the models are proper classes, the technique of Lemma 12.2 is still applicable and we can consider the *direct limit*

$$(M, E) = \lim \text{dir}_{n \to \omega} \{\text{Ult}^{(n)}, i_{n,m}\}, \tag{19.2}$$

along with elementary embeddings $i_{n,\omega} : \text{Ult}^{(n)} \to (M, E)$. The direct limit is a model of ZFC and we shall prove below that it is well-founded. Thus we identify it with a transitive model $\text{Ult}^{(\omega)}$. (We shall also prove that $\text{Ult}^{(\omega)} \subset \text{Ult}^{(n)}$ for every n.)

Let $\kappa^{(\omega)} = i_{0,\omega}(\kappa)$ and $U^{(\omega)} = i_{0,\omega}(U)$. Since $\mathrm{Ult}^{(\omega)}$ satisfies that $U^{(\omega)}$ is a $\kappa^{(\omega)}$-complete nonprincipal ultrafilter on $\kappa^{(\omega)}$, we can construct, working inside the model $\mathrm{Ult}^{(\omega)}$, the ultrapower of $\mathrm{Ult}^{(\omega)}$ mod $U^{(\omega)}$ and the corresponding canonical embedding $j^{(\omega)}$.

Let us denote $\mathrm{Ult}^{(\omega+1)}$ the ultrapower of $\mathrm{Ult}^{(\omega)}$ mod $U^{(\omega)}$ and let $i_{\omega,\omega+1}$ be the corresponding canonical elementary embedding. For $n < \omega$, let $i_{n,\omega+1} = i_{\omega,\omega+1} \circ i_{n,\omega}$.

This procedure can be continued, and so we define the *iterated ultrapower* as follows:

$$\begin{aligned}
(\mathrm{Ult}^{(0)}, E^{(0)}) &= (V, \in),\\
(\mathrm{Ult}^{(\alpha+1)}, E^{(\alpha+1)}) &= \mathrm{Ult}_{U^{(\alpha)}}(\mathrm{Ult}^{(\alpha)}, E^{(\alpha)}),\\
(\mathrm{Ult}^{(\lambda)}, E^{(\lambda)}) &= \lim\mathrm{dir}_{\alpha\to\lambda}\{(\mathrm{Ult}^{(\alpha)}, E^{(\alpha)}), i_{\alpha,\beta}\} \qquad (\lambda \text{ limit})
\end{aligned}$$

where $U^{(\alpha)} = i_{0,\alpha}(U)$, for each α. We do not know yet that all the models $\mathrm{Ult}^{(\alpha)}$ are well-founded; but we make a convention that if $\mathrm{Ult}^{(\alpha)}$ is well-founded, then we identify it with its transitive collapse.

If M is a transitive model of set theory and U is (in M) a κ-complete nonprincipal ultrafilter on κ, we can construct, within M, the iterated ultrapowers. Let us denote by $\mathrm{Ult}_U^{(\alpha)}(M)$ the αth iterated ultrapower, constructed in M.

Lemma 19.5 (The Factor Lemma). *Let us assume that* $\mathrm{Ult}^{(\alpha)}$ *is well-founded. Then for each* β, *the iterated ultrapower* $\mathrm{Ult}^{(\beta)}_{U^{(\alpha)}}(\mathrm{Ult}^{(\alpha)})$ *taken in* $\mathrm{Ult}^{(\alpha)}$ *is isomorphic to the iterated ultrapower* $\mathrm{Ult}^{(\alpha+\beta)}$.

Moreover, there is for each β *an isomorphism* $e_\beta^{(\alpha)}$ *such that if for all* ξ *and* η, $i_{\xi,\eta}^{(\alpha)}$ *denotes the elementary embedding of* $\mathrm{Ult}^{(\xi)}_{U^{(\alpha)}}(\mathrm{Ult}^{(\alpha)})$ *into* $\mathrm{Ult}^{(\eta)}_{U^{(\alpha)}}(\mathrm{Ult}^{(\alpha)})$, *then the following diagram commutes*:

$$\begin{array}{ccc}
\mathrm{Ult}^{(\xi)}_{U^{(\alpha)}}(\mathrm{Ult}^{(\alpha)}) & \xrightarrow{\;i_{\xi,\eta}^{(\alpha)}\;} & \mathrm{Ult}^{(\eta)}_{U^{(\alpha)}}(\mathrm{Ult}^{(\alpha)})\\
{\scriptstyle e_\xi^{(\alpha)}}\downarrow & & \downarrow{\scriptstyle e_\eta^{(\alpha)}}\\
\mathrm{Ult}_U^{(\alpha+\xi)} & \xrightarrow[\;i_{\alpha+\xi,\alpha+\eta}^{(\alpha)}\;]{} & \mathrm{Ult}_U^{(\alpha+\eta)}
\end{array}$$

Proof. The proof is by induction on β. If $\beta = 0$, then the 0th iterated ultrapower in $\mathrm{Ult}^{(\alpha)}$ is $\mathrm{Ult}^{(\alpha)}$; and we let $e_0^{(\alpha)}$ be the identity mapping. If $\mathrm{Ult}^{(\beta)}_{U^{(\alpha)}}$ and $\mathrm{Ult}_U^{(\alpha+\beta)}$ are isomorphic and $e_\beta^{(\alpha)}$ is the isomorphism, then $\mathrm{Ult}^{(\beta+1)}_{U^{(\alpha)}}$ and $\mathrm{Ult}_U^{(\alpha+\beta+1)}$ are ultrapowers of $\mathrm{Ult}^{(\beta)}_{U^{(\alpha)}}$ and $\mathrm{Ult}_U^{(\alpha+\beta)}$, respectively, mod $i_{0,\beta}^{(\alpha)}(U^{(\alpha)})$ and mod $i_{0,\alpha+\beta}(U)$, respectively; and since $i_{0,\alpha+\beta}(U) = e_\beta^{(\alpha)}(i_{0,\beta}^{(\alpha)}(U^{(\alpha)}))$, the isomorphism $e_\beta^{(\alpha)}$ induces an isomorphism $e_{\beta+1}^{(\alpha)}$ between $\mathrm{Ult}^{(\beta+1)}_{U^{(\alpha)}}$ and $\mathrm{Ult}_U^{(\alpha+\beta+1)}$.

If λ is a limit ordinal, then $\mathrm{Ult}^{(\lambda)}_{U(\alpha)}$ and $\mathrm{Ult}^{(\alpha+\lambda)}_U$ are (in $\mathrm{Ult}^{(\alpha)}$) the direct limits of $\{\mathrm{Ult}^{(\beta)}_{U(\alpha)}, i^{(\alpha)}_{\beta,\gamma} : \beta,\gamma < \lambda\}$ and $\{\mathrm{Ult}^{(\alpha+\beta)}_U, i_{\alpha+\beta,\alpha+\gamma} : \beta,\gamma < \lambda\}$, respectively. It is clear that the isomorphisms $e^{(\alpha)}_\beta$, $\beta < \lambda$, induce an isomorphism $e^{(\alpha)}_\lambda$ between $\mathrm{Ult}^{(\lambda)}_{U(\alpha)}$ and $\mathrm{Ult}^{(\alpha+\lambda)}_U$. □

Corollary 19.6. *For every limit ordinal λ, if $\mathrm{Ult}^{(\lambda)}$ is well-founded then $\mathrm{Ult}^{(\lambda)} \subset \mathrm{Ult}^{(\alpha)}$ for all $\alpha < \lambda$.*

Proof. $\mathrm{Ult}^{(\lambda)}$ is a class in $\mathrm{Ult}^{(\alpha)}$; it is the iterated ultrapower $\mathrm{Ult}^{(\beta)}_{U(\alpha)}(\mathrm{Ult}^{(\alpha)})$ where $\alpha + \beta = \lambda$. □

Next we show that the iterated ultrapowers $\mathrm{Ult}^{(\alpha)}_U$ are all well-founded.

Theorem 19.7 (Gaifman). *Let U be a κ-complete nonprincipal ultrafilter on κ. Then for every α, the αth iterated ultrapower $\mathrm{Ult}^{(\alpha)}$ is well-founded.*

Proof. Clearly, if $\mathrm{Ult}^{(\alpha)}$ is well-founded, then $\mathrm{Ult}^{(\alpha+1)}$ is well-founded. Thus if γ is the least γ such that $\mathrm{Ult}^{(\gamma)}$ is not well-founded, then γ is a limit ordinal. The ordinals of the model $\mathrm{Ult}^{(\gamma)}$ are not well-ordered; let ξ be the least ordinal such that the ordinals of Ult^{γ} below $i_{0,\gamma}(\xi)$ are not well-ordered.

Let $x_0, x_1, x_2, \ldots$ be a descending sequence of ordinals in the model $\mathrm{Ult}^{(\gamma)}$ such that x_0 is less than $i_{0,\gamma}(\xi)$. Since $\mathrm{Ult}^{(\gamma)}$ is the direct limit of $\mathrm{Ult}^{(\alpha)}$, $\alpha < \gamma$, there is an $\alpha < \gamma$ and an ordinal ν (less than $i_{0,\alpha}(\xi)$) such that $x_0 = i_{\alpha,\gamma}(\nu)$. Let β be such that $\alpha + \beta = \gamma$.

By our assumptions, the following is true (in V):

(19.3) $(\forall \gamma' \le \gamma)(\forall \xi' < \xi)$ the ordinals below $i_{0,\gamma'}(\xi')$ in $\mathrm{Ult}^{(\gamma')}$ are well-ordered.

When we apply the elementary embedding $i_{0,\alpha}$ to (19.3), we get:

(19.4) $\mathrm{Ult}^{(\alpha)} \vDash (\forall \gamma' \le i_{0,\alpha}(\gamma))(\forall \xi' < i_{0,\alpha}(\xi))$ the ordinals below $i^{(\alpha)}_{0,\gamma'}(\xi')$ in $\mathrm{Ult}^{(\gamma')}_{U(\alpha)}$ are well-ordered.

Now $\beta \le \gamma \le i_{0,\alpha}(\gamma)$, and $\nu < i_{0,\alpha}(\xi)$. Hence if we let $\gamma' = \beta$ and $\xi' = \nu$ in (19.4), we get

$$\mathrm{Ult}^{(\alpha)} \vDash \text{the ordinals below } i^{(\alpha)}_{0,\beta}(\nu) \text{ in } \mathrm{Ult}^{(\beta)}_{U(\alpha)} \text{ are well-ordered.}$$

By the Factor Lemma, $\mathrm{Ult}^{(\beta)}_{U(\alpha)}$ is (isomorphic to) $\mathrm{Ult}^{(\alpha+\beta)}$, and $i^{(\alpha)}_{0,\beta}(\nu)$ is $i_{\alpha,\alpha+\beta}(\nu)$. Since $\alpha + \beta = \gamma$ and $i_{\alpha,\gamma}(\nu) = x_0$, and since being well-ordered is absolute (for the transitive model $\mathrm{Ult}^{(\alpha)}$), we have:

$$\text{The ordinals below } x_0 \text{ in } \mathrm{Ult}^{(\gamma)} \text{ are well-ordered.}$$

But this is a contradiction since $x_1, x_2, x_3, \ldots$ is a descending sequence of ordinals below x_0 in $\mathrm{Ult}^{(\gamma)}$. □

Thus for any given κ-complete nonprincipal ultrafilter U on κ we have a transfinite sequence of transitive models, the iterated ultrapowers $\mathrm{Ult}_U^{(\alpha)}(V)$, and the elementary embeddings $i_{\alpha,\beta} : \mathrm{Ult}^{(\alpha)} \to \mathrm{Ult}^{(\beta)}$. Let $\kappa^{(\alpha)} = i_{0,\alpha}(\kappa)$ for each α; we shall show that the sequence $\kappa^{(\alpha)}$, $\alpha \in \mathit{Ord}$, is a normal sequence.

Lemma 19.8.

(i) *If $\gamma < \kappa^{(\alpha)}$, then $i_{\alpha,\beta}(\gamma) = \gamma$ for all $\beta \geq \alpha$.*

(ii) *If $X \subset \kappa^{(\alpha)}$ and $X \in \mathrm{Ult}^{(\alpha)}$, then $X \subset i_{\alpha,\beta}(X)$ for all $\beta \geq \alpha$; in fact $X = \kappa^{(\alpha)} \cap i_{\alpha,\beta}(X)$.*

Proof. By the Factor Lemma, it suffices to give the proof for $\alpha = 0$.

(i) As we know, $i_{0,1}(\gamma) = \gamma$ for all $\gamma < \kappa$. By induction on β, if $i_{0,\beta}(\gamma) = \gamma$, then $i_{0,\beta+1}(\gamma) = i_{\beta,\beta+1}(\gamma) = \gamma$ because $\gamma < \kappa^{(\beta)}$; if λ is limit and $i_{0,\beta}(\xi) = \xi$ for all $\xi \leq \gamma$ and $\beta < \lambda$, then $i_{0,\lambda}(\gamma) = \gamma$.

(ii) Follows from (i). □

Lemma 19.9. *The sequence $\langle \kappa^{(\alpha)} : \alpha \in \mathit{Ord} \rangle$ is normal; i.e., increasing and continuous.*

Proof. For each α, $\kappa^{(\alpha+1)} = i_{\alpha,\alpha+1}(\kappa^{(\alpha)}) > \kappa^{(\alpha)}$. To show that the sequence is continuous, let λ be a limit ordinal; we want to show that $\kappa^{(\lambda)} = \lim_{\alpha \to \lambda} \kappa^{(\alpha)}$. If $\gamma < \kappa^{(\lambda)}$, then $\gamma = i_{\alpha,\lambda}(\delta)$ for some $\alpha < \lambda$ and $\delta < \kappa^{(\alpha)}$. Hence $\gamma = \delta$ and so $\gamma < \kappa^{(\alpha)}$. □

Lemma 19.10. *Let D be a normal measure on κ, and let for each α, $\mathrm{Ult}^{(\alpha)}$ be the αth iterated ultrapower* mod D, *$\kappa^{(\alpha)} = i_{0,\alpha}(\kappa)$, and $D^{(\alpha)} = i_{0,\alpha}(D)$. Let λ be an infinite limit ordinal. Then for each $X \in \mathrm{Ult}^{(\lambda)}$, $X \subset \kappa^{(\lambda)}$,*

$$X \in D^{(\lambda)} \quad \text{if and only if} \quad (\exists \alpha < \lambda)\, X \supset \{\kappa^{(\gamma)} : \alpha \leq \gamma < \lambda\}. \tag{19.5}$$

Proof. Since for no X can both X and its complement contain a final segment of the sequence $\langle \kappa^{(\gamma)} : \gamma < \lambda \rangle$, it suffices to show that if $X \in D^{(\lambda)}$, then there is an α such that $\kappa^{(\gamma)} \in X$ for all $\gamma \geq \alpha$.

There exists an $\alpha < \lambda$ such that $X = i_{\alpha,\lambda}(Y)$ for some $Y \in D^{(\alpha)}$. Let us show that $\kappa^{(\gamma)} \in X$ for all γ, $\alpha \leq \gamma < \lambda$. Let $\gamma \geq \alpha$ and let $Z = i_{\alpha,\gamma}(Y)$. Then $Z \in D^{(\gamma)}$ and since $D^{(\gamma)}$ is a normal measure on $\kappa^{(\gamma)}$ in $\mathrm{Ult}^{(\gamma)}$, we have $\kappa^{(\gamma)} \in i_{\gamma,\gamma+1}(Z)$. However, $i_{\gamma,\gamma+1}(Z) \subset i_{\gamma+1,\lambda}(i_{\gamma,\gamma+1}(Z)) = X$ and hence $\kappa^{(\gamma)} \in X$. □

Representation of Iterated Ultrapowers

We shall now give an alternative description of each of the models $\mathrm{Ult}^{(\alpha)}$ by means of a single ultrapower of the universe, using an ultrafilter on a certain Boolean algebra of subsets of κ^{α}. This will enable us to obtain more precise information about the embeddings $i_{0,\alpha} : V \to \mathrm{Ult}^{(\alpha)}$.

We shall deal first with the finite iterations. Let U be a κ-complete nonprincipal ultrafilter on κ. Let us use the symbol $\forall^*\alpha$ for "almost all $\alpha < \kappa$:"

$$\forall^*\alpha\,\varphi(\alpha) \quad \text{if and only if} \quad \{\alpha < \kappa : \varphi(\alpha)\} \in U.$$

If $X \subset \kappa^n$ and $\alpha < \kappa$, let

$$X_{(\alpha)} = \{\langle\alpha_1, \ldots, \alpha_{n-1}\rangle : \langle\alpha, \alpha_1, \ldots, \alpha_{n-1}\rangle \in X\}.$$

We define ultrafilters U_n on κ^n, by induction on n:

$$\begin{aligned} U_1 &= U, \\ U_{n+1} &= \{X \subset \kappa^{n+1} : \forall^*\alpha\, X_{(\alpha)} \in U_n\}. \end{aligned}$$

Each U_n is a nonprincipal κ-complete ultrafilter on κ^n, and if $Z \in U$, then $Z^n \in U_n$. It is easy to see that for all $X \subset \kappa^n$,

$$X \in U_n \quad \text{if and only if} \quad \forall^*\alpha_0\, \forall^*\alpha_1\, \ldots\, \forall^*\alpha_{n-1}\, \langle\alpha_0, \ldots, \alpha_{n-1}\rangle \in X.$$

Note that U_n concentrates on *increasing* n-sequences:

$$\{\langle\alpha_0, \ldots, \alpha_{n-1}\rangle \in \kappa^n : \alpha_0 < \ldots < \alpha_{n-1}\} \in U_n$$

(because $\forall\alpha_0\,(\forall\alpha_1 > \alpha_0)\ldots(\forall\alpha_{n-1} > \alpha_{n-2})\,\alpha_0 < \ldots < \alpha_{n-1}$).

Lemma 19.11. *For every n,*

$$\mathrm{Ult}_{U_n}(V) = \mathrm{Ult}^{(n)}(V)$$

and $j_{U_n} = i_{0,n}$.

Here j_{U_n} is the canonical embedding $j : V \to \mathrm{Ult}_{U_n}(V)$.

Proof. By induction on n. The case $n = 1$ is trivial. Let us assume that the lemma is true for n and let us consider $\mathrm{Ult}_{U_{n+1}}$. Let f be a function on κ^{n+1}. For each $t = \langle\alpha_0, \ldots, \alpha_{n-1}\rangle \in \kappa^n$, let $f_{(t)}$ be the function on κ defined by $f_{(t)}(\xi) = f(\alpha_0, \ldots, \alpha_{n-1}, \xi)$ and let F be a function on κ^n such that $F(t) = f_{(t)}$ for all $t \in \kappa^n$. In $\mathrm{Ult}_{U_n} = \mathrm{Ult}^{(n)}$, the function F represents a function on $j_{U_n}(\kappa) = \kappa^{(n)}$: Let $\tilde{f} = [F]_{U_n}$. This way we assign to each function f on κ^{n+1} a function $\tilde{f} \in \mathrm{Ult}^{(n)}$ on $\kappa^{(n)}$.

Conversely, if $h \in \mathrm{Ult}^{(n)}$ is a function on $\kappa^{(n)}$, there is an f on κ^{n+1} such that $h = \tilde{f}$: There exists some F on κ^n such that $h = [F]_{U_n}$ and that for each $t \in \kappa^n$, $F(t)$ is a function on κ; thus we let $f(\alpha_0, \ldots, \alpha_n)$ be the value of $F(\alpha_0, \ldots, \alpha_{n-1})$ at α_n.

We shall show that the correspondence $[f]_{U_{n+1}} \mapsto [\tilde{f}]_{U^{(n)}}$ is an isomorphism between $\text{Ult}_{U_{n+1}}$ and $\text{Ult}^{(n+1)} = \text{Ult}_{U^{(n)}}(\text{Ult}^{(n)})$. We have

$$\begin{aligned}
[f]_{U_{n+1}} = [g]_{U_{n+1}} &\leftrightarrow \forall^*\alpha_0 \ \ldots\ \forall^*\alpha_{n-1}\ \forall^*\xi\ \ f(\alpha_0,\ldots,\alpha_{n-1},\xi) = g(\alpha_0,\ldots,\\
&\qquad \alpha_{n-1},\xi)\\
&\leftrightarrow \forall^* t\,\{\xi<\kappa : f_{(t)}(\xi) = g_{(t)}(\xi)\} \in U\\
&\leftrightarrow \text{Ult}_{U_n} \vDash \{\xi < j_{U_n}(\kappa) : \tilde{f}(\xi) = \tilde{g}(\xi)\} \in j_{U_n}(U)\\
&\leftrightarrow \text{Ult}^{(n)} \vDash \{\xi < \kappa^{(n)} : \tilde{f}(\xi) = \tilde{g}(\xi)\} \in U^{(n)}\\
&\leftrightarrow [\tilde{f}]_{U^{(n)}} = [\tilde{g}]_{U^{(n)}}
\end{aligned}$$

and similarly for $\in$ in place of $=$.

Thus $\text{Ult}_{U_{n+1}} = \text{Ult}^{(n+1)}$. To show that $j_{U_{n+1}} = i_{0,n+1}$, let $f = c_x$ be the constant function on κ^{n+1} with value x. It follows that $\tilde{f}$ is the constant function on $\kappa^{(n)}$ with value $i_{0,n}(x)$, and therefore

$$j_{U_{n+1}}(x) = [c_x]_{U_{n+1}} = i_{n,n+1}(i_{0,n}(x)) = i_{0,n+1}(x). \qquad \square$$

The infinite iterations are described with the help of ultrafilters U_E on κ^E, where E ranges over finite sets of ordinal numbers. If E is a finite set of ordinals, then the order isomorphism π between $n = |E|$ and E induces, in a natural way, an ultrafilter U_E corresponding to U_n:

$$U_E = \{\pi(X) : X \subset \kappa^n\}$$

where $\pi(\langle\alpha_0,\ldots,\alpha_{n-1}\rangle) = t \in \kappa^E$ with $t(\pi(k)) = \alpha_k$ for all $k = 0,\ldots,n-1$.

If S is any set of ordinals and $E \subset S$ is a finite set, we define a mapping $\text{in}_{E,S}$ (an *inclusion map*) of $P(\kappa^E)$ into $P(\kappa^S)$ as follows:

$$\text{in}_{E,S}(X) = \{t \in \kappa^S : t{\restriction}E \in X\} \qquad (\text{all } X \subset \kappa^E).$$

Lemma 19.12. *If $E \subset F$ are finite sets of ordinals, then for each $X \subset \kappa^E$,*

$$X \in U_E \quad \textit{if and only if} \quad \text{in}_{E,F}(X) \in U_F.$$

Proof. By induction on (m,n) where $m = |E|$ and $n = |F|$. Let $E \subset F$ be finite sets of ordinals. Let a be the least element of F, and let us assume that $a \in E$ (if $a \notin E$, then the proof is similar). Let $E' = E - \{a\}$ and $F' = F - \{a\}$.

If $X \subset \kappa^E$, let us define for each $\alpha < \kappa$, the set $X_{(\alpha)} \subset \kappa^{E'}$ as follows: $X_{(\alpha)} = \{t{\restriction}E' : t \in X \text{ and } t(a) = \alpha\}$; for $Z \subset \kappa^F$, let us define $Z_{(\alpha)} \subset \kappa^{F'}$ similarly (for all $\alpha < \kappa$). It should be clear that

$$\text{(19.6)} \qquad X \in U_E \leftrightarrow \forall^*\alpha\, X_{(\alpha)} \in U_{E'} \quad \text{and} \quad Z \in U_F \leftrightarrow \forall^*\alpha\, Z_{(\alpha)} \in U_{F'}.$$

Now we observe that if $Z = \text{in}_{E,F}(X)$, then $Z_{(\alpha)} = \text{in}_{E',F'}(X_{(\alpha)})$, and the lemma for E, F follows from (19.6) and the induction hypothesis. $\square$

Let us now consider an ordinal number α. If $E \subset \alpha$ is a finite set, let us say that a set $Z \subset \kappa^\alpha$ has *support* E if $Z = \text{in}_{E,\alpha}(X)$ for some $X \subset \kappa^E$. Note that if Z has support E and $E \subset F$, then Z also has support F. Let B_α denote the collection of all subsets of κ^α that have finite support. $(B_\alpha, \subset)$ is a Boolean algebra.

Let U_α be the following ultrafilter on B_α: For each $Z \in B_\alpha$, if $Z = \text{in}_{E,\alpha}(X)$ where $X \subset \kappa^E$, let

$$Z \in U_\alpha \quad \text{if and only if} \quad X \in U_E.$$

By Lemma 19.12, the definition of U_α does not depend on the choice of support E of Z.

We shall now construct an ultrapower mod U_α. If f is a function on κ^α, let us say that f has a *finite support* $E \subset \alpha$ if $f(t) = f(s)$ whenever $t, s \in \kappa^\alpha$ are such that $t{\restriction}E = s{\restriction}E$. In other words, there is g on κ^E such that $f(t) = g(t{\restriction}E)$ for all $t \in \kappa^\alpha$. Let us consider only functions f on κ^α with finite support and define

$$\begin{aligned} f =_\alpha g \quad &\text{if and only if} \quad \{t : f(t) = g(t)\} \in U_\alpha, \\ f\ E_\alpha\ g \quad &\text{if and only if} \quad \{t : f(t) \in g(t)\} \in U_\alpha. \end{aligned} \tag{19.7}$$

The sets on the right-hand side of (19.7) have finite support, namely $E \cup F$ where E and F are, respectively, supports of f and g.

Let $(\text{Ult}_{U_\alpha}(V), E_\alpha)$ be the model whose elements are equivalence classes mod $=_\alpha$ of functions on κ^α with finite support.

We are now in a position to state the main lemma.

Lemma 19.13 (The Representation Lemma). *For every α, the model $(\text{Ult}_{U_\alpha}(V), E_\alpha)$ is (isomorphic to) the αth iterated ultrapower $\text{Ult}_U^{(\alpha)}(V)$, and the canonical embedding $j_{U_\alpha} : V \to \text{Ult}_{U_\alpha}$ is equal to $i_{0,\alpha}$. Moreover, if $\alpha \le \beta$ and $[f]_{U_\alpha} \in \text{Ult}^{(\alpha)}$, then $i_{\alpha,\beta}([f]_{U_\alpha}) = [g]_{U_\beta}$ where g is defined by $g(t) = f(t{\restriction}\alpha)$ for all $t \in \kappa^\beta$.*

Proof. By induction on α. The induction step from α to $\alpha + 1$ follows closely the proof of Lemma 19.11; thus let us describe only how to assign to $[f]_{U_{\alpha+1}}$ the corresponding $[\tilde{f}]_{U^{(\alpha)}}$ in $\text{Ult}^{(\alpha+1)}$. Let f be a function on $\kappa^{\alpha+1}$ with support $E \cup \{\alpha\}$ where $E \subset \alpha$. For each $t \in \kappa^\alpha$ let $f_{(t)}(\xi) = f(t^\frown\xi)$ for all $\xi < \kappa$, and let F be a function on κ^α (with support E) such that $F(t) = f_{(t)}$ for all $t \in \kappa^\alpha$. Let $\tilde{f} = [F]_{U_\alpha}$; $\tilde{f}$ is in $\text{Ult}^{(\alpha)}$ and is a function on $\kappa^{(\alpha)}$.

When λ is a limit ordinal, a routine verification shows that Ult_{U_λ} is the direct limit of $\{\text{Ult}_{U_\alpha}, i_{\alpha,\beta} : \alpha, \beta < \lambda\}$ and that the embeddings $i_{\alpha,\lambda}$ commute with the $i_{\alpha,\beta}$. ∎

Uniqueness of the Model $L[D]$

Theorem 19.14 (Kunen).

(i) *If $V = L[D]$ and D is a normal measure on κ, then κ is the only measurable cardinal and D is the only normal measure on κ.*

(ii) *For every ordinal κ, there is at most one $D \subset P(\kappa)$ such that $D \in L[D]$ and*

$$L[D] \vDash D \text{ is a normal measure on } \kappa.$$

(iii) *If $\kappa_1 < \kappa_2$ are ordinals and if D_1, D_2 are such that*

$$L[D_i] \vDash D_i \text{ is a normal measure on } \kappa_i \qquad (i = 1,\ 2)$$

then $L[D_2]$ is an iterated ultrapower of $L[D_1]$; i.e., there is α such that $L[D_2] = \mathrm{Ult}_{D_1}^{(\alpha)}(L[D_1])$, and $D_2 = i_{0,\alpha}(D_1)$.

The proof of Theorem 19.14 uses iterated ultrapowers. The following lemma uses the representation of iterated ultrapowers.

Lemma 19.15. *Let U be a κ-complete nonprincipal ultrafilter on κ and let, for each α, $i_{0,\alpha} : V \to \mathrm{Ult}^{(\alpha)}$ be the embedding of V in its αth iterated ultrapower.*

(i) *If α is a cardinal and $\alpha > 2^\kappa$, then $i_{0,\alpha}(\kappa) = \alpha$.*

(ii) *If λ is a strong limit cardinal, $\lambda > \alpha$, and if $\operatorname{cf}\lambda > \kappa$, then $i_{0,\alpha}(\lambda) = \lambda$.*

Proof. It follows from the Representation Lemma that for all ξ, η, the ordinals below $i_{0,\xi}(\eta)$ are represented by functions with finite support from κ^ξ into η and hence $|i_{0,\xi}(\eta)| \le |\xi| \cdot |\eta|^\kappa$.

(i) We have $i_{0,\alpha}(\kappa) = \lim_{\xi\to\alpha} i_{0,\xi}(\kappa)$, and for each $\xi < \alpha$, $|i_{0,\xi}(\kappa)| \le |\xi| \cdot 2^\kappa < \alpha$. Hence $i_{0,\kappa}(\kappa) = \alpha$.

(ii) Since $\operatorname{cf}\lambda > \kappa$, every function $f : \kappa^\alpha \to \lambda$ with finite support is bounded below λ: There exists $\gamma < \lambda$ such that $f(t) < \gamma$ for all $t \in \kappa^\alpha$. Hence $i_{0,\alpha}(\lambda) = \lim_{\gamma\to\lambda} i_{0,\alpha}(\gamma)$. Since λ is strong limit, we have $|i_{0,\alpha}(\gamma)| < \lambda$ for all $\gamma < \lambda$ and hence $i_{0,\lambda} = \lambda$. □

It is clear from the proof that in (ii) it is enough to assume that $\gamma^\kappa < \lambda$ for all cardinals $\gamma < \lambda$, instead of that λ is a strong limit cardinal.

Let $U \subset P(\kappa)$. If θ is a cardinal and $U \in L_\theta[U]$, then by absoluteness of relative constructibility, every elementary submodel of $(L_\theta[U], \in)$ that contains U and all ordinals $< \kappa$, is isomorphic to $M = L_\alpha[U]$ for some α. (If π is the transitive collapse of the submodel, then $\pi(U) = U \cap M \in M$, and $M = L_\alpha[U]$.) Let θ be a cardinal such that $U \in L_\theta[U]$ and let us consider the model $(L_\theta[U], \in, U)$ where U is regarded as a constant. This model has a definable well-ordering, hence definable Skolem functions, and so we can talk about Skolem hulls of subsets of $L_\theta[U]$.

Lemma 19.16. *Assume that in $L[D]$, D is a normal measure on κ. Let A be a set of ordinals of size at least κ^+ and let θ be a cardinal such that $D \in L_\theta[D]$ and $A \subset L_\theta[D]$. Let $M \prec (L_\theta[D], \in, D)$ be the Skolem hull of $\kappa \cup A$. Then M contains all subsets of κ in $L[D]$.*

For every $X \subset \kappa$ in $L[D]$ there is a Skolem term t such that for some $\alpha_1, \dots, \alpha_n < \kappa$ and $\gamma_1, \dots, \gamma_m \in A$,

$$L_\theta[D] \vDash X = t[\alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_m, D].$$

Proof. Let π be the transitive collapse of M. We have $\pi(M) = L_\alpha[D]$ for some α, and since $A \subset M$, we have necessarily $\alpha \geq \kappa^+$. By Lemma 19.2, every $X \subset \kappa$ in $L[D]$ is in $L_{\kappa^+}[D]$, and since π is the identity on κ, we have $X \in M$ for all $X \subset \kappa$ in $L[D]$. ☐

The following is the key lemma in the proof of uniqueness of $L[D]$:

Lemma 19.17. *Let $D \subset P(\kappa)$ be such that $D \in L[D]$ and*

$$L[D] \vDash D \text{ is a normal measure on } \kappa.$$

For each α, let $\mathrm{Ult}_D^{(\alpha)}(L[D])$ denote the αth iterated ultrapower, constructed inside $L[D]$. Let $i_{0,\alpha}$ be the corresponding elementary embedding. Let λ be a regular cardinal greater than κ^+, and let F be the closed unbounded filter on λ. Then:

(i) $i_{0,\lambda}(D) = F \cap \mathrm{Ult}_D^{(\lambda)}(L[D])$;
(ii) $\mathrm{Ult}_D^{(\lambda)}(L[D]) = L[F]$.

Proof. First, we have $i_{0,\lambda}(\kappa) = \lambda$ by Lemma 19.15(i) because $\lambda > \kappa^+ \geq (\kappa^+)^{L[D]} = (2^\kappa)^{L[D]}$. Let $D^{(\lambda)} = i_{0,\lambda}(D)$ and let $M = \mathrm{Ult}_D^{(\lambda)}(L[D])$. If $X \in D^{(\lambda)}$, then by (19.5), X contains a closed unbounded subset and hence $X \in F$. Since $D^{(\lambda)}$ is an ultrafilter in M and F is a filter, it follows that $D^{(\lambda)} = F \cap M$.

As for (ii) we have

$$M = \mathrm{Ult}^{(\lambda)}(L[D]) = L[D^{(\lambda)}] = L[F \cap M] = L[F]. \qquad ☐$$

We shall now prove parts (i) and (ii) of Kunen's Theorem. We already know by Lemma 19.4 that in $L[D]$, κ is the only measurable cardinal. Thus (i) and (ii) follow from this lemma:

Lemma 19.18. *Let $D_1, D_2 \subset P(\kappa)$ be such that $D_1 \in L[D_1]$, $D_2 \in L[D_2]$ and*

$$L[D_i] \vDash D_i \text{ is a normal measure on } \kappa \qquad (i = 1,\ 2).$$

Then $D_1 = D_2$.

Proof. Let $D_1, D_2 \subset P(\kappa)$ be such that $L[D_i] \vDash D_i$ is a normal measure on κ, for $i = 1, 2$; we want to show that $D_1 = D_2$. By symmetry, it suffices to show that if $X \subset \kappa$ is in D_1, then $X \in D_2$.

Let λ be a regular cardinal greater than κ^+ and let F be the closed unbounded filter on λ. Let us consider the λth iterated ultrapowers $M_i = \mathrm{Ult}^{(\lambda)}_{D_i}(L[D_i])$ $(i = 1, 2)$, and the corresponding embeddings $i^1_{0,\lambda}$, $i^2_{0,\lambda}$.

By Lemma 19.17, $M_1 = M_2 = L[F]$, and $i^1_{0,\lambda}(D_1) = i^2_{0,\lambda}(D_2) = F \cap L[F]$. Let $G = F \cap L[F]$.

Let A be a set of ordinals, $|A| = \kappa^+$, such that all $\gamma \in A$ are greater than λ and that $i^1_{0,\lambda}(\gamma) = i^2_{0,\lambda}(\gamma)$ for all $\gamma \in A$; such a set exists by Lemma 19.15(ii). Let θ be a cardinal greater than all $\gamma \in A$ such that $i^1_{0,\lambda}(\theta) = i^2_{0,\lambda}(\theta) = \theta$.

Now let X be a subset of κ such that $X \in D_1$. By Lemma 19.16, X belongs to the Skolem hull of $\kappa \cup A$ in $(L_\theta[D_1], \in, D_1)$. Thus there is a Skolem term t such that for some $\alpha_1, \ldots, \alpha_n < \kappa$ and $\gamma_1, \ldots, \gamma_m \in A$,

$$L_\theta[D_1] \vDash X = t[\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_m, D_1]. \tag{19.8}$$

Let $Y \in L_\theta[D_2]$ be such that

$$L_\theta[D_2] \vDash Y = t[\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_m, D_2]. \tag{19.9}$$

We shall show that $Y \in D_2$ and $Y = X$, hence $X \in D_2$.

First we observe that $i^1_{0,\lambda}(X) = i^2_{0,\lambda}(Y)$: Let $Z_1 = i^1_{0,\lambda}(X)$ and $Z_2 = i^2_{0,\lambda}(Y)$. We have $i^1_{0,\lambda}(\alpha) = \alpha$, $i^1_{0,\lambda}(\gamma) = \gamma$, $i^1_{0,\lambda}(\theta) = \theta$, and $i^1_{0,\lambda}(D_1) = G$; and thus when we apply $i^1_{0,\lambda}$ to (19.8), we get

$$L_\theta[G] \vDash Z_1 = t[\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_m, G]. \tag{19.10}$$

Similarly, when we apply $i^2_{0,\lambda}$ to (19.8), we get (19.10) with Z_2 instead of Z_1. Thus $Z_1 = Z_2$.

Now, by Lemma 19.8(ii), we have $X = Z_1 \cap \kappa$ and $Y = Z_2 \cap \kappa$. Hence $X = Y$.

Finally, since $i^2_{0,\lambda}(Y) \in F$, it follows that $i^2_{0,\lambda}(Y) \in i^2_{0,\lambda}(D_2)$ and hence $Y \in D_2$. Thus $X \in D_2$ and this completes the proof of $D_1 = D_2$. □

The key lemma in the proof of Theorem 19.14(iii) is the following:

Lemma 19.19. *Let κ, D be such that $L[D] \vDash D$ is a normal measure on κ, and let γ be an ordinal such that $\kappa < \gamma < i_{0,1}(\kappa)$, where $i_{0,1}$ is the embedding of $L[D]$ in $\mathrm{Ult}_D(L[D])$. Then there is no $U \subset P(\gamma)$ such that $L[U] \vDash U$ is a normal measure on γ.*

Proof. Let us assume that on the contrary there is such a U. Let j be the canonical embedding of $L[U]$ in $\mathrm{Ult}_U(L[U])$. Let $\lambda = |\gamma|^{++}$, and let F be the closed unbounded filter on λ. Let $G = F \cap L[F]$.

Since $L[U] \vDash \mathrm{GCH}$, we have $j(\lambda) = \lambda$ (see the remark following Lemma 19.15). In $L[U]$, G is the λth iterate of U, and in $L[j(U)]$, G is the $j(\lambda)$th iterate of $j(U)$; hence $j(G) = G$.

Let $f : \kappa \to \kappa$ be a function in $L[D]$ such that f represents γ in $\mathrm{Ult}_D(L[D])$. Since D is normal, the diagonal $d(\alpha) = \alpha$ represents κ, and thus we have $(i_{0,1}(f))(\kappa) = \gamma$. Let $i_{0,\lambda}$ be the embedding of $L[D]$ in $\mathrm{Ult}_D^{(\lambda)}(L[D]) = L[G]$. It is clear that $(i_{0,\lambda}(f))(\kappa) = \gamma$.

Now let A be a set of ordinals such that $|A| = \kappa^+$, that all $\xi \in A$ are greater than λ, and that $i_{0,\lambda}(\xi) = \xi$ and $j(\xi) = \xi$ for all $\xi \in A$. Let θ be a cardinal greater than all $\xi \in A$, such that $i_{0,\lambda}(\theta) = \theta$ and $j(\theta) = \theta$.

By Lemma 19.16, the function f is definable in $L_\theta[D]$ from $A \cup \kappa \cup \{D\}$; thus $i_{0,\lambda}(f)$ is definable in $L_\theta[G]$ from $A \cup \kappa \cup \{G\}$. Hence γ is definable in $L_\theta[G]$ from $A \cup \kappa \cup \{G\} \cup \{\kappa\}$, and so there is a Skolem term t such that

$$L_\theta[G] \models \gamma = t[\alpha_1, \ldots, \alpha_n, \xi_1, \ldots, \xi_m, G, \kappa]. \tag{19.11}$$

for some $\alpha_1, \ldots, \alpha_n < \kappa$ and $\xi_1, \ldots, \xi_m \in A$.

Now we apply the elementary embedding j to (19.11); and since $j(\theta) = \theta$, $j(G) = G$, $j(\xi) = \xi$ for $\xi \in A$, and $j(\alpha) = \alpha$ for all $\alpha < \gamma$ (hence $j(\kappa) = \kappa$), we have

$$L_\theta[G] \models j(\gamma) = t[\alpha_1, \ldots, \alpha_n, \xi_1, \ldots, \xi_m, G, \kappa].$$

which is a contradiction because $j(\gamma) > \gamma$. □

Proof of Theorem 19.14(iii). Let $\kappa_1 < \kappa_2$ and let D_1, D_2 be such that $L[D_i] \models D$ is a normal measure on κ_i ($i = 1, 2$). Let $i_{0,\alpha}$ denote the embedding of $L[D_1]$ in $\mathrm{Ult}_{D_1}^{(\alpha)}(L[D_1])$ and let α be the unique α such that $i_{0,\alpha}(\kappa_1) \le \kappa_2 < i_{0,\alpha+1}(\kappa_1)$. By Lemma 19.19 (if we let $\kappa = i_{0,\alpha}(\kappa_1)$, $D = i_{0,\alpha}(D_1)$, and $\gamma = \kappa_2$), it is necessary that $\kappa_2 = i_{0,\alpha}(\kappa_1)$. Now the statement follows from the uniqueness of $i_{0,\alpha}(D_1)$. □

Thus we have proved that the model $V = L[D]$ (where D is a normal measure on κ) is unique, has only one measurable cardinal and only one normal measure on κ, and it satisfies the Generalized Continuum Hypothesis. The next lemma completes the characterization of $L[D]$ by showing that for every κ-complete nonprincipal ultrafilter U on κ, $L[U]$ is equal to $L[D]$.

Lemma 19.20. *Let U be a nonprincipal κ-complete ultrafilter on κ. Then $L[U] = L[D]$ where D is the normal measure on κ in $L[D]$.*

Proof. By the absoluteness of $L[D]$, we have $L[D] \subset L[U]$ because $L[U]$ satisfies that κ is measurable. Thus it suffices to prove that $U \cap L[D] \in L[D]$. Let $j = j_U$ be the canonical embedding $j : V \to \mathrm{Ult}_U(V)$, and let $\gamma = j(\kappa)$. Let $d(\alpha) = \alpha$ be the diagonal function and let δ be the ordinal represented in $\mathrm{Ult}_U(V)$ by d; thus

$$X \in U \quad \text{if and only if} \quad \delta \in j(X) \tag{19.12}$$

for all $X \subset \kappa$.

Since $L[j(D)] \vDash j(D)$ is a normal measure on γ, there exists an α such that $\gamma = i_{0,\alpha}(\kappa)$, $j(D) = i_{0,\alpha}(D)$, and $L[j(D)] = \mathrm{Ult}_D^{(\alpha)}(L[D])$. We shall show that for every $X \subset \kappa$ in $L[D]$,

$$j(X) = i_{0,\alpha}(X). \tag{19.13}$$

This, together with (19.12), gives

$$U \cap L[D] = \{X \in L[D] : X \subset \kappa \text{ and } \delta \in i_{0,\alpha}(X)\} \tag{19.14}$$

and therefore $U \cap L[D] \in L[D]$.

The proof of (19.13) uses Lemma 19.16 again. We let A be a set of size κ^+ of ordinals greater than α such that $i_{0,\alpha}(\xi) = j(\xi) = \xi$ for all $\xi \in A$, and let θ be a cardinal greater than all $\xi \in A$, such that $i_{0,\lambda}(\theta) = j(\theta) = \theta$.

If $X \subset \kappa$ is in $L[D]$, then there is a Skolem term t such that

$$L_\theta[D] \vDash X = t[\alpha_1, \ldots, \alpha_n, \xi_1, \ldots, \xi_m, D].$$

for some $\alpha_1, \ldots, \alpha_n < \kappa$ and $\xi_1, \ldots, \xi_m \in A$. Since $i_{0,\alpha}$ and j agree on $\kappa \cup A \cup \{\theta\}$, and $i_{0,\alpha}(D) = j(D)$, it follows that $i_{0,\alpha}(X) = j(X)$. □

The proof of Lemma 19.20 gives additional information about κ-complete ultrafilters in $L[D]$. Let us assume that $V = L[D]$ and let U be a nonprincipal κ-complete ultrafilter on κ. By (19.14), we have

$$U = \{X \subset \kappa : \delta \in i_{0,\alpha}(X)\} \tag{19.15}$$

where α is such that $j(\kappa) = i_{0,\alpha}(\kappa)$, and $\delta < j(\kappa)$. Note that for any $\beta \geq \alpha$, we also have $U = \{X \subset \kappa : \delta \in i_{0,\beta}(X)\}$. Now a simple observation gives the following characterization of κ-complete ultrafilters on κ in $L[D]$:

Lemma 19.21. *Assume $V = L[D]$. If U is a nonprincipal κ-ultrafilter on κ, then there exists some $\delta < i_{0,\omega}(\kappa)$ such that*

$$U = \{X \subset \kappa : \delta \in i_{0,\omega}(X)\}.$$

Proof. Let $j = j_U$ be the canonical embedding of $V = L[D]$ in Ult_U. We have $j(\kappa) = i_{0,\alpha}(\kappa)$ for some α. We shall show that α is a finite number; then the lemma follows by (19.15).

First we note that because $V = L[D] = L[U]$, we have $\mathrm{Ult}_D^{(\alpha)} = \mathrm{Ult}_U = L[i_{0,\alpha}(D)] = L[j(U)]$. Now if $\alpha \geq \omega$, then in $\mathrm{Ult}_D^{(\alpha)}$, $i_{0,\omega}(\kappa)$ is an inaccessible cardinal (because it is measurable in $\mathrm{Ult}_D^{(\omega)}$), while in Ult_U, $i_{0,\omega}(\kappa)$ has cofinality ω (because it has cofinality ω in V and Ult_U contains all ω-sequences of ordinals). Hence $\alpha < \omega$. □

Corollary 19.22. *If $V = L[D]$, there are exactly κ^+ nonprincipal κ-complete ultrafilters on κ.*

Proof. If κ is measurable, then it is easy to obtain 2^κ nonprincipal κ-complete ultrafilters on κ (because there are 2^κ subsets of κ of size κ such that $|X \cap Y| < \kappa$ for any two of them). By Lemma 19.21, if $V = L[D]$, there are at most $|i_{0,\omega}(\kappa)| = \kappa^+$ of them. □

Indiscernibles for $L[D]$

If there exist two measurable cardinals, $\kappa < \lambda$, then it is possible to prove analogous theorems for the model $L[D]$ as we did in Chapter 18 for L under the assumption of one measurable cardinal. More specifically, one can prove the existence of a closed unbounded set $I \subset \kappa$ and a closed unbounded class J of ordinals bigger than κ, such that $I \cup J$ contains all uncountable cardinals except κ, that every $X \in L[D]$ is definable in D from $I \cup J$, and that the elements of $I \cup J$ are indiscernibles for $L[D]$ in the following sense: The truth value of

$$L[D] \vDash \varphi[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m]$$

is independent of the choice of $\alpha_1 < \ldots < \alpha_n \in I$ and $\beta_1 < \ldots < \beta_m \in J$. In analogy with Silver indiscernibles, the above situations can be described by means of a certain set of formulas $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$, which is called $0^\dagger$ (*zero-dagger*).

If $0^\dagger$ exists, then one can prove the consistency of the theory ZFC + "there exists a measurable cardinal;" and hence one cannot prove the relative consistency of "$0^\dagger$ exists" with ZFC + "there exists a measurable cardinal."

We shall not give details of the theory of indiscernibles for $L[D]$. Instead, let us present an argument showing that if there exist two measurable cardinals, $\kappa < \lambda$, then there is a proper class of cardinals that are inaccessible in $L[D]$.

Let U be a normal measure on λ and let for each α, $i_{0,\alpha}$ be the elementary embedding of V in $\text{Ult}_U^{(\alpha)}$; let $i_{\alpha,\beta} : \text{Ult}^{(\alpha)} \to \text{Ult}^{(\beta)}$. Let C be the class of all cardinals α such that $\text{cf}\,\alpha > \lambda$ and $\gamma^\lambda < \alpha$ for all $\gamma < \alpha$. By Lemma 19.15, if $\alpha \in C$, then $i_{0,\alpha}(\kappa) = \alpha$ and $i_{0,\alpha}(\beta) = \beta$ for all $\beta \in C$ greater than α. Hence if $\alpha, \beta \in C$, then $i_{\alpha,\beta}(\alpha) = \beta$ and $i_{\alpha,\beta}(\gamma) = \gamma$ for all $\gamma \in C$ that are greater than β or less than α.

Now if D is a normal measure on κ, then because $\kappa < \lambda$, we have $i_{\alpha,\beta}(D) = D$ for all $\alpha, \beta \in C$. Thus each $i_{\alpha,\beta}$ $(\alpha, \beta \in C)$, restricted to $L[D]$, is an elementary embedding of $L[D]$ in $L[D]$ such that $i_{\alpha,\beta}(\alpha) = \beta$ and $i_{\alpha,\beta}(\gamma) = \gamma$ for all $\gamma \in C$ below α or above β. Using these embeddings $i_{\alpha,\beta}$ (as in the proof of Lemma 18.26), one shows that the elements of C are indiscernibles for the model $L[D]$.

Since some elements of C are regular cardinals, and some are limit cardinals, it follows that all elements of C are inaccessible cardinals in $L[D]$.

In the above argument, it was not necessary that κ be a measurable cardinal, only that κ be measurable in $L[D]$. Thus we have proved:

Lemma 19.23. *Let κ be a measurable cardinal, and assume that:*

(19.16) *For some $\gamma < \kappa$, there exists a $D \subset P(\gamma)$ such that $L[D] \vDash D$ is a normal measure on γ.*

Then there are arbitrarily large successor cardinals that are inaccessible in $L[D]$.

We have proved in Lemma 19.21 that if U is a nonprincipal κ-complete ultrafilter on κ, then $j_U(\kappa) < i_{0,\omega}(\kappa)$, where $i_{0,\omega}$ is the embedding of $L[D]$ in $\mathrm{Ult}_D^{(\omega)}(L[D])$. We can prove a stronger statement:

Lemma 19.24. *If there is a κ-complete nonprincipal ultrafilter U on κ such that $j_U(\kappa) \geq i_{0,\omega}(\kappa)$, then* (19.16) *holds.*

Proof. Let us work in the model $M = \mathrm{Ult}_U(V)$. The cardinal $j(\kappa)$ is measurable while $i_{0,\omega}(\kappa)$ has cofinality ω, and so $i_{0,\omega}(\kappa) < j(\kappa)$. Let F be the collection of all subsets X of $i_{0,\omega}(\kappa)$ such that $X \supset \{i_{0,n}(\kappa) : n \geq n_0\}$ for some n_0. Using Lemma 19.10, we proceed as in the proof of Lemma 19.17 to show that

$$L[F] \vDash F \cap L[F] \text{ is a normal measure on } i_{0,\omega}(\kappa).$$

Thus (19.16) holds in M for $j(\kappa)$. Since j is an elementary embedding, (19.16) holds in V for κ. □

Corollary 19.25. *If κ is a measurable cardinal and $2^\kappa > \kappa^+$, then* (19.16) *holds. Consequently, it is impossible to prove the consistency of "κ is measurable and $2^\kappa > \kappa^+$" relative to* ZFC + "*κ is a measurable cardinal.*"

Proof. On the one hand, $|i_{0,\omega}(\kappa)| = (\kappa^+)^{L[D]} \leq \kappa^+$; on the other hand, if U is any κ-complete ultrafilter on κ, we have $j_U(\kappa) > 2^\kappa > \kappa^+$. □

General Iterations

We shall now describe two generalizations of iterated ultrapowers. The first deals with iteration of ultrapowers of transitive models by ultrafilters that are not necessarily members of the model.

Let M be a transitive model of set theory. In fact, it is not necessary for the theory of iterated ultrapowers to assume that M satisfies all axioms of ZFC. It is enough to assume that M is a model of ZFC$^-$, set theory without the Power Set Axiom. Thus M can be a set (e.g., $(L_\alpha, \in)$ is a model of ZFC$^-$ when α is a regular uncountable cardinal in L).

Let κ be a cardinal in M, and let U be an M-ultrafilter on κ (Definition 18.21).

Definition 19.26. An M-ultrafilter U on κ is *iterable* if

(19.17) $\qquad \{\alpha < \kappa : X_\alpha \in U\} \in M \quad$ whenever $\quad \langle X_\alpha : \alpha < \kappa\rangle \in M.$

We shall consider *normal iterable* M-ultrafilters, i.e., M-ultrafilters that are nonprincipal, κ-complete, normal (as in Definition 18.21) and iterable.

Let U be a normal iterable M-ultrafilter on κ. Using functions in M, we form an ultrapower $\mathrm{Ult}_U(M)$, which may or may not be well-founded. Let $j = j_U$ be the canonical elementary embedding $j : M \to \mathrm{Ult}_U(M)$.

Lemma 19.27. *If* $\mathrm{Ult}_U(M)$ *is well-founded, and* N *is the transitive collapse of the ultrapower, then*

(i) $P^M(\kappa) = P^N(\kappa)$.
(ii) $j``U$ *is a normal iterable* N*-ultrafilter on* $j(\kappa)$.

Proof. (i) It is a routine verification by induction that $j(\alpha) = \alpha$ for all $\alpha < \kappa$. For every $X \in P^M(\kappa)$, we have $X = j(X) \cap \kappa$, and therefore $X \in P^N(\kappa)$, verifying $P^M(\kappa) \subset P^N(\kappa)$.

If $Y \in P^N(\kappa)$, let $f \in M$ be such that $Y \in [f]_U$. Then $Y \in P^M(\kappa)$ follows (by (19.17)) because for all $\alpha < \kappa$,

$$\alpha \in Y \quad \text{if and only if} \quad \{\xi < \kappa : \alpha \in f(\xi)\} \in U.$$

(ii) Let $W = j``U$. To verify that N and W satisfy (19.17), let $\langle X_\alpha : \alpha < j(\kappa)\rangle \in N$ be represented in the ultrapower by $f \in M$. We may assume that for each α, $X_\alpha \subset j(\kappa)$, and that $f(\xi) = \langle X^\xi_\eta : \eta < \kappa\rangle$ for each $\xi < \kappa$. By (19.17), we have $\{(\xi,\eta) : X^\xi_\eta \in U\} \in M$. Thus if we define $g(\xi) = \{\eta < \kappa : X^\xi_\eta \in U\}$, we have $g \in M$. Now it is routine to show that $[g]_U = \{\alpha < j(\kappa) : X_\alpha \in W\}$. □

If j is an elementary embedding $j : M \to N$ with critical point κ, and if $P^M(\kappa) = P^N(\kappa)$, then the M-ultrafilter $\{X : \kappa \in j(X)\}$ is iterable; see Exercise 19.8.

Let U be a normal iterable M-ultrafilter on κ. If the ultrapower $\mathrm{Ult}_U(M)$ is well-founded, let M_1 be its transitive collapse, let $j : M \to M_1$ be the canonical elementary embedding, and let $U^{(1)} = j``U$; $U^{(1)}$ is a normal iterable M_1-ultrafilter on $\kappa^{(1)} = j(\kappa)$. We can now proceed with the iteration as when $M = V$ and $U \in M$, as long as the iterated ultrapowers are well-founded. At limit stages we take direct limits, and use the following lemma that is quite routine to verify:

Lemma 19.28. *Let* α *be a limit ordinal, and let for each* $\beta < \alpha$, $U^{(\beta)}$ *be a normal iterable* M_β*-ultrafilter on* $\kappa^{(\beta)}$*, and assume that the direct limit of* $\{(M_\beta, \in, U^{(\beta)}), i_{\beta,\gamma} : \beta, \gamma < \alpha\}$ *is well-founded. If* $(M_\alpha, \in, U^{(\alpha)})$ *is the transitive direct limit then* $U^{(\alpha)}$ *is a normal iterable* M_α*-ultrafilter on* $\kappa^{(\alpha)} = \lim_{\beta\to\alpha} \kappa^{(\beta)}$. □

The Representation Lemma 19.13 holds true in the present context as well. The M-ultrafilters U_α are defined as before, starting with M-ultrafilters U_n on $P^M(\kappa^n)$:

(19.18) $$X \in U_{n+1} \quad \text{if and only if} \quad \{\xi < \kappa : X_{(\xi)} \in U_n\} \in U$$

where $X_{(\xi)} = \{\langle \xi_1, \ldots, \xi_n\rangle : \langle \xi, \xi_1, \ldots, \xi_n\rangle \in X\}$. By induction on n one proves that each U_n is an iterable M-ultrafilter on κ^n.

To define the ultrafilters U_α and the ultrapowers $\mathrm{Ult}_{U_\alpha}(M)$, we restrict ourselves, as before, to sets $Z \subset \kappa^\alpha$ and functions f on κ^α with finite support, with the additional restriction imposed by M: If $E = \{\alpha_1, \ldots, \alpha_n\}$ with $\alpha_1 < \ldots < \alpha_n$ is the support of Z or f then the restriction of Z or f to κ^E is such that its isomorph $\bar{Z} \subset \kappa^n$ or $\bar{f} : \kappa^n \to M$ is an element of M.

In general, iterated ultrapowers of M by an M-ultrafilter need not be well-founded. If, however, all countable iterations are well-founded then all iterations are well-founded (Exercise 19.9). An important sufficient condition for well-foundedness of iterated ultrapowers is the following (external σ-completeness):

(19.19) For any $\{X_n\}_{n\in\omega} \subset U$, $\bigcap_{n=0}^{\infty} X_n$ is nonempty

(see Exercise 19.10).

The other generalization deals with iterated ultrapowers of an inner model where each successor step $\alpha + 1$ of the iteration is obtained as an ultrapower of M_α by an arbitrary measure in M_α.

Definition 19.29. An *iterated ultrapower* of an inner model M is a sequence $\langle M_\gamma : \gamma \le \lambda \rangle$ constructed as follows:

(i) $M_0 = M$.
(ii) $M_{\gamma+1} = \text{Ult}_{U^{(\gamma)}}(M_\gamma)$ where $U^{(\gamma)} \in M_\gamma$ is a κ_γ-complete ultrafilter on κ_γ, and the ultrapower is constructed in M_γ; $i_{\gamma,\gamma+1} : M_\gamma \to M_{\gamma+1}$ is the canonical embedding, and for all $\alpha < \gamma$, $i_{\alpha,\gamma+1} = i_{\gamma,\gamma+1} \circ i_{\alpha,\gamma}$.
(iii) If γ is a limit ordinal, then M_γ is the direct limit of $\{M_\alpha, i_{\alpha,\beta} : \alpha \le \beta < \gamma\}$.

Theorem 19.30 (Mitchell). *Let M be an inner model of* ZFC. *Every iterated ultrapower of M is well-founded.*

Proof. First we outline the proof of the theorem for $M = V$. The idea is to represent each iterated ultrapower M_γ as an ultrapower by an ultrafilter U_γ. The ultrafilters U_γ are defined by induction on γ. For each γ we define an ordinal function k_γ (that represents κ_γ in the ultraproduct by U_γ), the set D_γ (the domain of k_γ), the algebra P_γ of subsets of D_γ, the class F_γ of functions on D_γ and the ultrafilter U_γ on P_γ.

The domain D_γ of k_γ is the set

$$\{p \in Ord^\gamma : \forall \alpha < \gamma \ p(\alpha) < k_\alpha(p{\restriction}\alpha)\}.$$

The algebra P_γ and the class F_γ are

$$P_\gamma = \{X \subset D_\gamma : X \text{ has finite support}\},$$
$$F_\gamma = \{f \in V^{D_\gamma} : f \text{ has finite support}\}.$$

If γ is a limit ordinal, we let $U_\gamma = \bigcup_{\alpha<\gamma} U_\alpha$. If $\gamma = \alpha + 1$, then assume that M_α is transitive and isomorphic to $\text{Ult}_{U_\alpha}(V)$. Let $k_\alpha \in F_\alpha$ be a function that represents κ_α, and let $g \in F_\alpha$ be a function that represents $U^{(\alpha)}$, in the ultrapower M_α, i.e., $[k_\alpha]_{U_\alpha} = \kappa_\alpha$, $[g]_{U_\alpha} = U^{(\alpha)}$. Thus for U_α-almost all $p \in D_\alpha$, $g_\alpha(p)$ is an ultrafilter on $k_\alpha(p)$. For $X \in P_{\alpha+1}$ we let

$$X \in U_{\alpha+1} \quad \text{if and only if} \quad \{p \in D_\alpha : X_{(p)} \in g(p)\} \in U_\alpha$$

where $X_{(p)} = \{\xi < k_\alpha(p) : p \cup \{(\alpha, \xi)\} \in X\}$. It is now routine to verify that $\mathrm{Ult}_{U_{\alpha+1}}(V)$ is isomorphic to $\mathrm{Ult}_{U^{(\alpha)}}(M_\alpha)$.

The proof that each $\mathrm{Ult}_{U_\alpha}(V)$ is well-founded uses the argument presented in Exercise 19.10.

Now if M is an arbitrary inner model, and $\langle M_\gamma : \gamma \le \lambda\rangle$ is an iterated ultrapower that is not necessarily defined inside M, we use an absoluteness argument. We can still use the representation of M_γ by $\mathrm{Ult}_{U_\alpha}(M)$; in this case the functions $p \in Ord^\gamma$, the sets $X \subset D_\gamma$ and the functions $f \in V^{D_\gamma}$ are all assumed to be members of M.

If $E \subset \gamma$ is a finite set, then P_E and F_E denote the subsets of P_γ and F_γ, respectively, of those sets or functions whose support is E. Let U_E be the restriction of U_γ to P_E, and let M_E be the ultrapower of M mod U_E (using functions in F_E). For $E \subset E'$, let $i_{E,E'}$ be the canonical elementary embedding of M_E in $M_{E'}$ and let $i_{E,\gamma}$ be the embedding of M_E in M_γ.

If some iterated ultrapower of M is not well-founded, then, as in Exercise 19.9, one can show that there is a countable λ such that an iterated ultrapower $\langle M_\gamma : \gamma \le \lambda\rangle$ is not well-founded. Let κ be the supremum of all the κ_γ, $\gamma \le \lambda$, in this iteration. Let $\{a_n\}_{n<\omega}$ be a decreasing sequence of ordinals in M_λ, and let $E_0 \subset E_1 \subset \ldots \subset E_n \subset \ldots$ be a sequence of finite subsets of λ such that $\bigcup_{n=0}^\infty E_n = \lambda$, and that each E_n is a support for (a function representing) a_n. For each n, let $b_n \in M_{E_n}$ be such that $a_n = i_{E_n,\lambda}(b_n)$. Let η be sufficiently large so that $b_n \in V_\eta^M$ for all n. Thus there exists a sequence $\{(E_n, M_n, b_n)\}_{n=0}^\infty$ such that $E_0 \subset E_1 \subset \ldots \subset E_n \subset \ldots$ are finite subsets of λ, that each M_n is an iterated ultrapower of M indexed by E_n, b_n is an ordinal in $M_n = \mathrm{Ult}_{E_n}(M)$, and for each n, $M_{n+1} \vDash i_{E_n,E_{n+1}}(b_n) > b_{n+1}$.

As each M_n is a finite iteration, it is clear that it is a class in M. Consider, in M, the set of all triples (E, N, b) such that E is a finite subset of λ, N is a finite ultrapower iteration indexed by E and using measures on ordinals $\le \kappa$, and b is an ordinal in N represented by a function in V_η. Let $(E', N', b') < (E, N, b)$ if $E' \supset E$ and if $N' \vDash i_{E,E'}(b) > b'$. We have established that this relation $<$ is not well-founded (in the universe). Thus by absoluteness of well-foundedness, this relation is not well-founded in M. However, that means that there is an iterated ultrapower constructed in M that is not well-founded, contrary to the result of the first part of this proof. □

The Mitchell Order

Definition 19.31. Let κ be a measurable cardinal. If U_1 and U_2 are normal measures on κ, let

$$U_1 < U_2 \quad \text{if and only if} \quad U_1 \in \mathrm{Ult}_{U_2}(V).$$

The relation $U_1 < U_2$ is called the *Mitchell order.*

The Mitchell order is transitive, and by Lemma 17.9(ii) is irreflexive. Moreover, it is well-founded:

Lemma 19.32. *The Mitchell order is well-founded.*

Proof. Toward a contradiction, let κ be the least measurable cardinal on which the Mitchell order is not well-founded, and let $U_0 > U_1 > \ldots > U_n > \ldots$ be a descending sequence of normal measures on κ. Let $M = \text{Ult}_{U_0}(V)$ and let $j : V \to M$ be the canonical elementary embedding. As $\kappa < j(\kappa)$, and $j(\kappa)$ is the least measurable cardinal in M on which the Mitchell order is not well-founded, we reach a contradiction once we show that $U_1 > U_2 > \ldots > U_n > \ldots$ is a descending sequence in M.

The measures U_n, $n \geq 1$, are in M, and so is the sequence $\{U_n\}_{n=1}^{\infty}$, so we need to verify that $U_{n+1} < U_n$ still holds in M. Since $U_{n+1} \in \text{Ult}_{U_n}(V)$, U_{n+1} is represented in the ultrapower by a function $f = \langle u_\alpha : \alpha < \kappa \rangle$. As $P^M(\kappa) = P(\kappa)$ and $M^\kappa \subset M$, the function f is in M, and represents U_{n+1} in the ultrapower $\text{Ult}^M_{U_n}(M)$. Hence $M \vDash U_{n+1} < U_n$. $\square$

Definition 19.33. If U is a normal measure on κ, let $o(U)$, the *order* of U, denote the rank of U in $<$. Let $o(\kappa)$, the *order* of κ, denote the height of $<$.

Lemma 19.34. *Let o be the function $\langle o(\alpha) : \alpha < \kappa \rangle$. If U is a normal measure on κ then $o(U) = [o]_U$.*

Proof. Clearly, $[o]_U = o^M(\kappa)$ where $M = \text{Ult}_U(V)$. The set $\{U' : U' < U\}$ is the set of all normal measures in M, and since $<$ is absolute for M (see Lemma 19.32), the order of U in V is the order of κ in M. $\square$

Thus $o(U) > 0$ if and only if U-almost all $\alpha < \kappa$ are measurable. If κ is a measurable cardinal of order ≥ 2 then κ has a normal measure that concentrates on measurable cardinals $\alpha < \kappa$. Thus the consistency strength of $o(\kappa) \geq 2$ is more than measurability. Measurable cardinals of higher order provide a hierarchy of large cardinal axioms. A consequence of Lemma 19.34 is that $|o(U)| \leq 2^\kappa$ and therefore $o(\kappa) \leq (2^\kappa)^+$. In particular, if GCH holds, then $o(\kappa) \leq \kappa^{++}$ for every measurable cardinal κ.

There exist canonical inner models for measurable cardinals of higher order, analogous to the model $L[U]$. We shall now outline the theory of these inner models.

The key technical device is the technique of *coiteration.* It is the method used in the proof of Lemma 19.35 below. Let $\mathcal{U}$ be a set of normal measures (on possibly different cardinals). $\mathcal{U}$ is *closed* if for every measure $U \in \mathcal{U}$ on κ, every normal measure on κ in $j_U(\mathcal{U})$ is in $\mathcal{U}$. If $\mathcal{U}$ is a closed set of normal measures and $U, W \in \mathcal{U}$, let $U <_{\mathcal{U}} W$ mean that $U \in j_W(\mathcal{U})$. As $<_{\mathcal{U}}$ is a suborder of the Mitchell order it is well-founded and we define $o^{\mathcal{U}}(U)$

and $o^{\mathcal{U}}(\kappa)$ accordingly. The *length* of $\mathcal{U}$, $l(\mathcal{U})$, is the least ϑ such that $\kappa < \vartheta$ for all κ with $o^{\mathcal{U}}(\kappa) > 0$.

Let M and N be inner models of ZFC, and let $\mathcal{U} \in M$ and $\mathcal{W} \in N$ be closed sets of normal measures in M and N, respectively. We say that $\mathcal{U}$ is an *initial segment* of $\mathcal{W}$ if

(19.20) (i) $l(\mathcal{U}) \le l(\mathcal{W})$,
(ii) for every $\alpha < l(\mathcal{U})$, $o^{\mathcal{U}}(\alpha) = o^{\mathcal{W}}(\alpha)$,
(iii) for every $\kappa < l(\mathcal{U})$, if $U \in M$ and $W \in N$ are on κ and $o^{\mathcal{U}}(U) = o^{\mathcal{W}}(W)$, then $U \cap M \cap N = W \cap M \cap N$.

Lemma 19.35. *Let M and N be inner models of* ZFC *and let $\mathcal{U}$ and $\mathcal{W}$ be closed sets of measures in M and N, respectively. Then there exist iterated ultrapowers $i_{0,\lambda} : M \to M_\lambda$ and $j_{0,\lambda} : N \to N_\lambda$, using measures in $\mathcal{U}$ and $\mathcal{W}$, respectively, such that either $i_{0,\lambda}(\mathcal{U})$ is an initial segment of $j_{0,\lambda}(\mathcal{W})$, or vice versa.*

Proof. By induction on γ, we define iterated ultrapowers M_γ and N_γ, and the embeddings $i_{\beta,\gamma} : M_\beta \to M_\gamma$ and $j_{\beta,\gamma} : N_\beta \to N_\gamma$. We let $M_0 = M$ and $N_0 = N$, and if λ is a limit ordinal, M_λ and N_λ are direct limits of $\{M_\gamma, i_{\beta,\gamma} : \beta, \gamma < \lambda\}$ and $\{N_\gamma, j_{\beta,\gamma} : \beta, \gamma < \lambda\}$, respectively.

If at stage γ, $\mathcal{U}_\gamma = i_{0,\gamma}(\mathcal{U})$ and $\mathcal{W}_\gamma = j_{0,\gamma}(\mathcal{W})$ are not initial segments of one another, then there exist ordinals α_γ and δ_γ such that $\alpha_\gamma < l(\mathcal{U}_\gamma)$, $\alpha_\gamma < \mathcal{W}_\gamma$, $\mathcal{U}_\gamma{\restriction}\alpha_\gamma$ and $\mathcal{W}_\gamma{\restriction}\alpha_\gamma$ agree on $M_\gamma \cap N_\gamma$, the measures on α_γ of order $< \delta_\gamma$ in $\mathcal{U}_\gamma$ and in $\mathcal{W}_\gamma$ agree on $M_\gamma \cap N_\gamma$, and

(19.21) either (i) $\delta_\gamma = o^{\mathcal{U}_\gamma}(\alpha) < o^{\mathcal{W}_\gamma}(\alpha)$, or
(ii) $\delta_\gamma = o^{\mathcal{W}_\gamma}(\alpha_\gamma) < o^{\mathcal{U}_\gamma}(\alpha_\gamma)$, or
(iii) $\delta_\gamma < o^{\mathcal{W}_\gamma}(\alpha_\gamma)$, $\delta_\gamma < o^{\mathcal{U}_\gamma}(\alpha_\gamma)$ and for some $U_\gamma \in \mathcal{U}_\gamma$ and $W_\gamma \in \mathcal{W}_\gamma$ of order δ_γ there exists an $X_\gamma \in M_\gamma \cap N_\gamma$ such that $X_\gamma \in U_\gamma$ but $X_\gamma \notin W_\gamma$.

If (i) occurs, let $i_{\gamma,\gamma+1}$ be the identity and $j_{\gamma,\gamma+1} : N_\gamma \to N_{\gamma+1} = \mathrm{Ult}_W(N_\gamma)$ where W is any $W \in \mathcal{W}_\gamma$ such that $o^{\mathcal{W}_\gamma}(W) = \delta_\gamma$. Similarly, if (ii) occurs, then $j_{\gamma,\gamma+1}$ is the identity and $M_{\gamma+1}$ is an ultrapower. If (iii) occurs, let $i_{\gamma,\gamma+1} : M_\gamma \to M_{\gamma+1} = \mathrm{Ult}_{U_\gamma}(M_\gamma)$ and $j_{\gamma,\gamma+1} : N_\gamma \to N_{\gamma+1} = \mathrm{Ult}_{W_\gamma}(N_\gamma)$.

Note that if $\beta < \gamma$ then $\alpha_\beta \le \alpha_\gamma$. Moreover, in cases (i) and (ii) we have $\alpha_{\gamma+1} > \alpha_\gamma$ as $o^{\mathcal{U}_{\gamma+1}}(\alpha_\gamma) = o^{\mathcal{W}_{\gamma+1}}(\alpha_\gamma) = \delta_\gamma$, and the measures of order $< \delta_\gamma$ agree.

We will show that the process eventually stops. Thus assume the contrary.

For every limit ordinal γ, M_γ is a direct limit, and so there exists some $\beta = \beta(\gamma) < \gamma$ such that α_γ is in the range of $i_{\beta,\gamma}$, $\alpha_\gamma = i_{\beta,\gamma}(\alpha)$ for some $\alpha = \alpha(\gamma) < l(\mathcal{U}_\beta)$. There is a stationary class Γ_1 of ordinals such that $\beta(\gamma)$ is the same β for all $\gamma \in \Gamma_1$. Also, there is a stationary class $\Gamma_2 \subset \Gamma_1$ such that $\alpha(\gamma)$ is the same $\alpha < l(\mathcal{U}_\beta)$ for all $\gamma \in \Gamma_2$. It follows that if $\beta < \gamma$ are in Γ_2 then $i_{\beta,\gamma}(\alpha_\beta) = \alpha_\gamma$. Similarly there is a stationary class $\Gamma_3 \subset \Gamma_2$ such that $j_{\beta,\gamma}(\alpha_\beta) = \alpha_\gamma$ whenever $\beta < \gamma$ are in Γ_3. Continuing in this manner, we find a stationary class $\Gamma \subset \Gamma_3$ such that for all $\beta < \gamma$ in Γ, $i_{\beta,\gamma}(\alpha_\beta, \delta_\beta) = j_{\beta,\gamma}(\alpha_\beta, \delta_\beta) = (\alpha_\gamma, \delta_\gamma)$, and (in Case (iii)) $i_{\beta,\gamma}(X_\beta) = j_{\beta,\gamma}(X_\beta)$.

Let $\beta \in \Gamma$ and assume that (19.21)(i) occurs. Let $\gamma \in \Gamma$ be greater than β. Since $i_{\beta,\beta+1}$ is the identity, and $\text{crit}(U_\xi) = \alpha_\xi \geq \alpha_{\beta+1} > \alpha_\beta$ for all $\xi > \beta$, we have $i_{\beta,\gamma}(\alpha_\beta) = \alpha_\beta$, while $j_{\beta,\gamma}(\alpha_\beta) \geq j_{\beta,\beta+1}(\alpha_\beta) > \alpha_\beta$, contrary to $i_{\beta,\gamma}(\alpha_\beta) = j_{\beta,\gamma}(\alpha_\beta)$. Thus (i) does not occur, and similarly, (ii) leads to a contradiction.

Case (iii) gives a contradiction as follows: Let $\gamma > \beta$ be in Γ. Since $X_\beta \in U_\beta$, we have $\alpha_\beta \in i_{\beta,\gamma}(X_\beta)$, and since $X_\beta \notin W_\beta$, we have $\alpha_\beta \notin j_{\beta,\gamma}(X_\beta)$. This contradicts $i_{\beta,\gamma}(X_\beta) = j_{\beta,\gamma}(X_\beta)$, and therefore the process must eventually stop. □

The Models $L[\mathcal{U}]$

If A_α, $\alpha < \theta$, is a sequence of sets, let us define the model

$$L\langle A_\alpha : \alpha < \theta \rangle \tag{19.22}$$

as the model $L[A]$ where $A = \{(\alpha, X) : X \in A_\alpha\}$. Under this definition, $L\langle A_\alpha : \alpha < \theta\rangle = L[\langle B_\alpha : \alpha < \theta\rangle]$, where $B_\alpha = A_\alpha \cap L\langle A_\alpha : \alpha < \theta\rangle$ for all $\alpha < \theta$.

If κ_α, $\alpha < \theta$, is a sequence of measurable cardinals, and for each α, U_α is a κ_α-complete nonprincipal ultrafilter on κ_α, then in $L\langle U_\alpha : \alpha < \theta\rangle$, each $U_\alpha \cap L\langle U_\alpha : \alpha < \theta\rangle$ is again a κ_α-complete nonprincipal ultrafilter on κ_α.

More generally, let $\mathcal{U}$ be a set of normal measures indexed by pairs of ordinals (α, β) such that $U_{\alpha,\beta}$ is a measure on α. Then $L[\mathcal{U}]$ denotes the model $L\langle U_{\alpha,\beta} : \alpha, \beta\rangle$.

The technique described in the preceding section can be used to generalize many results about the model $L[U]$ to obtain canonical inner models for measurable cardinals of higher order. We shall illustrate the method by constructing a model with exactly two normal measures on a measurable cardinal of order 2.

Definition 19.36. A *canonical inner model* for a measurable cardinal κ of order 2 is a model

$$L[\mathcal{U}] = L\langle U_\alpha, U^0, U^1\rangle_{\alpha \in A} \tag{19.23}$$

such that in $L[\mathcal{U}]$

(19.24) (i) U^1 is a normal measure on κ of order 1.
(ii) U^0 is a normal measure on κ of order 0 and $U^0 < U^1$.
(iii) $A \in U^1$, each U_α is a normal measure on α of order 0, and $\langle U_\alpha : \alpha \in A\rangle$ represents U^0 in the ultrapower by U^1.

If $o(\kappa) \geq 2$ then a canonical model $L[\mathcal{U}]$ is obtained as follows: Let $A \subset \kappa$ be the set of all measurable cardinals below κ, let U^1 be a normal measure on κ of order 1, let U^0 be a normal measure on κ such that $U^0 < U^1$, and let U_α, $\alpha \in A$, be normal measures such that $[\langle U_\alpha : \alpha \in A\rangle]_{U^1} = U^0$. Then $L\langle U_\alpha, U^0, U^1\rangle_{\alpha\in A}$ is a canonical inner model (with $\mathcal{U} = \langle U_\alpha \cap L[\mathcal{U}], U^0 \cap L[\mathcal{U}], U^1 \cap L[\mathcal{U}]\rangle_{\alpha\in A}$).

The canonical model is unique (for the particular choice of the set A), in the sense that if $\mathcal{W} = \langle W_\alpha, W^0, W^1\rangle_{\alpha\in A}$ is any other sequence that satisfies (19.24), then $L[\mathcal{U}] = L[\mathcal{W}]$. We prove below a more general result.

Theorem 19.37 (Mitchell). *Let $A \subset \kappa$, and let $\mathcal{U} = \langle U_\alpha, U^0, U^1\rangle_{\alpha\in A}$ and $\mathcal{W} = \langle W_\alpha, W^0, W^1\rangle_{\alpha\in A}$ be such that for each $\alpha \in A$, U_α and W_α are normal measures on α of order* 0, *U^0 and W^0 are normal measures on κ of order* 0, *and U^1 and W^1 are normal measures on κ of order* 1. *Then $L[\mathcal{U}] = L[\mathcal{W}]$ and*

$$(19.25) \qquad \begin{aligned} &\text{(i)}\ U_\alpha \cap L[\mathcal{U}] = W_\alpha \cap L[\mathcal{W}] && (\textit{all } \alpha \in A),\\ &\text{(ii)}\ U^\varepsilon \cap L[\mathcal{U}] = W^\varepsilon \cap L[\mathcal{W}] && (\varepsilon = 0,\ 1). \end{aligned}$$

Proof. We use Lemma 19.35. Let $\mathcal{D}$ be the following set of measures: The U_α's, the W_α's, U^1, U^0, W^1, W^0, and all the normal measures on κ in $j_{U^1}(\mathcal{U} \cup \mathcal{W})$ and $j_{W^1}(\mathcal{U} \cup \mathcal{W})$ (so that $\mathcal{D}$ is a closed set of measures).

By Lemma 19.35 (applied to $\mathcal{D}$) there exist iterated ultrapowers $i = i_{0,\lambda} : V \to M$ and $j = j_{0,\lambda} : V \to N$ such that $i(\mathcal{D})$ is an initial segment of $j(\mathcal{D})$.

We have $l(i(\mathcal{D})) = i(\kappa) + 1$, and by (19.20), $o^{i(\mathcal{D})}(i(\kappa)) = o^{j(\mathcal{D})}(j(\kappa)) = 2$ and for all $\alpha < i(\kappa)$, $o^{i(\mathcal{D})}(\alpha) = o^{j(\mathcal{D})}(\alpha) = 1$ if $\alpha \in i(A)$ and $o^{i(\mathcal{D})}(\alpha) = o^{j(\mathcal{D})}(\alpha) = 0$ if $\alpha \notin i(A)$. It follows that $i(\kappa) = j(\kappa)$ and $i(A) = j(A)$.

By (19.20)(iii), if $D \in i(\mathcal{D})$ and $E \in j(\mathcal{D})$ are normal measures on some $\alpha \in i(A)$ then $D \cap M \cap N = E \cap M \cap N$; the same is true if $D \in i(\mathcal{D})$ and $E \in j(\mathcal{D})$ are measures on $i(\kappa)$ and $o^{i(\mathcal{D})}(D) = o^{i(\mathcal{D})}(E)$. It follows that $L[i(\mathcal{U})] = L[j(\mathcal{U})] = L[j(\mathcal{W})] = L[i(\mathcal{W})] \subset M \cap N$, $i(U^\varepsilon) \cap L[i(\mathcal{U})] = j(U^\varepsilon) \cap L[i(\mathcal{U})] = j(W^\varepsilon) \cap L[i(\mathcal{U})] = i(W^\varepsilon) \cap L[i(\mathcal{U})]$ ($\varepsilon = 0,\ 1$), and for every $\alpha \in i(A)$, $(i\mathcal{U})_\alpha = (i\mathcal{W})_\alpha$, where $\langle (i\mathcal{U})_\alpha, iU^0, iU^1\rangle_{\alpha\in iA} = i\mathcal{U} = i(\langle U_\alpha, U^0, U^1\rangle_{\alpha\in A})$. (By induction on γ, one shows that $L_\gamma[i(\mathcal{U})] = L_\gamma[i(\mathcal{W})]$).

Now (19.25) follows since i is an elementary embedding, and $i : L[\mathcal{U}] \to L[i(\mathcal{U})]$, $i : L[\mathcal{W}] \to L[i(\mathcal{W})]$. $\square$

The analog of Theorem 19.14(i) for $L[\mathcal{U}]$ is the following:

Theorem 19.38 (Mitchell). *In $L[\mathcal{U}]$, κ and $\alpha \in A$ are the only measurable cardinals, and U_α, U^0 and U^1 are the only normal measures.*

Proof. For every ordinal $\gamma \leq \kappa$, let $\mathcal{U}\restriction\gamma = \langle U_\alpha : \alpha \in A\cap\gamma\rangle$; if $\gamma > \kappa$, $\mathcal{U}\restriction\gamma = \mathcal{U}$. Toward a contradiction, let γ be the least ordinal such that in $L[\mathcal{U}\restriction\gamma]$ there are normal measures other than those in $\mathcal{U}\restriction\gamma$, and let $\mathcal{D} = \mathcal{U}\restriction\gamma$. Let α be the least cardinal in $L[\mathcal{D}]$ that carries a normal measure not in $\mathcal{D}$, and let D be such a measure of least Mitchell order.

If $\alpha \notin A \cup \{\kappa\}$, let $M = L[\mathcal{D}]$, $N = \mathrm{Ult}_D(L[\mathcal{D}]) = L[j_D(\mathcal{D})]$, and apply Lemma 19.35 to M, N and (closed) sets of measures $\mathcal{D}$ and $j_D(\mathcal{D})$. There are iterated ultrapowers $i = i_{0,\lambda} : M \to M_\lambda$ and $j = j_{0,\lambda} : N \to N_\lambda$ such that $i(\mathcal{D})$ is an initial segment of $j(j_D(\mathcal{D}))$ or vice versa. Now because of the choice of $\mathcal{D}$ as a minimal counterexample in $L[\mathcal{D}]$ to the theorem, no proper initial segment of either $i(\mathcal{D})$ or $j(j_D(\mathcal{D}))$ can be a counterexample, and consequently, $M_\lambda = N_\lambda = L[i(\mathcal{D})] = L[j(j_D(\mathcal{D}))]$. As $j_D(\mathcal{D})\restriction(\alpha+1) = \mathcal{D}\restriction(\alpha+1)$, we have $i(\alpha) = j(\alpha) = \alpha$, which contradicts the fact that $D \in M$ but $D \notin N$.

The same argument works if $\alpha \in A$ and $U_\alpha < D$, or if $\alpha = \kappa$ and $U^1 < D$.

If $\alpha \in A$ and $o(D) = 0$, let $U = U_\alpha$; if $\alpha = \kappa$ and $o(D) = \varepsilon$ ($\varepsilon = 0, 1$), let $U = U^\varepsilon$. Let $M = \mathrm{Ult}_U(L[\mathcal{D}]) = L[j_U(\mathcal{D})]$ and $N = \mathrm{Ult}_D(L[\mathcal{D}]) = L[j_D(\mathcal{D})]$. By Lemma 19.35 there are iterated ultrapowers $i : M \to M_\lambda$ and $j : N \to N_\lambda$ such that $i(j_U(\mathcal{D}))$ is an initial segment of $j(j_D(\mathcal{D}))$ or vice versa. Using the minimality argument again, we get $M_\lambda = N_\lambda = L[i(j_U(\mathcal{D}))] = L(\mathcal{E})$ where $\mathcal{E} = i(j_U(\mathcal{D})) = j(j_D(\mathcal{D}))$. Again, $j_U(\mathcal{D})\restriction(\alpha+1) = j_D(\mathcal{D})\restriction(\alpha+1)$, so $i(\alpha) = j(\alpha) = \alpha$.

To reach a contradiction we show that $X \in U$ if and only if $X \in D$, for every $X \subset \alpha$ in $L[\mathcal{D}]$. We proceed as in the proof of Lemma 19.18. If $X \in P^{L[\mathcal{D}]}(\alpha)$ then X is definable in $L[\mathcal{D}]$ from $\mathcal{D}$ and ordinals that are not moved by j_U, j_D, i or j. As in (19.8)–(19.10) it follows that $X = Z \cap \alpha$ where $Z = i(X)$, that $Z = j(Z \cap \alpha) = j(X)$ and that $X \in U$ if and only if $\alpha \in i(X)$ if and only if $\alpha \in j(X)$ if and only if $X \in D$. □

Theorems 19.37 and 19.38 admit a generalization to yield canonical inner models for measurable cardinals of higher order. We shall state the following result without proof:

Theorem 19.39 (Mitchell). *There exists an inner model $L[\mathcal{U}]$ such that*

(i) *for every α, $o^{L[\mathcal{U}]}(\alpha) = o^{\mathcal{U}}(\alpha) = \min\{o(\alpha), (\alpha^{++})^{L[\mathcal{U}]}\}$;*
(ii) *$\mathcal{U} = \langle U_{\alpha,\beta} : \beta < o^{\mathcal{U}}(\alpha)\rangle$;*
(iii) *each $U_{\alpha,\beta}$ is in $L[\mathcal{U}]$ a normal measure of order β;*
(iv) *every normal measure in $L[\mathcal{U}]$ is $U_{\alpha,\beta}$ for some α and β;*
(v) $L[\mathcal{U}] \vDash \mathrm{GCH}$. □

Exercises

19.1. Let κ be a measurable cardinal and $j : V \to M$ be the corresponding elementary embedding. Let $M_0 = V$, $M_1 = M$, and for each $n < \omega$, $M_{n+1} = j(M_n)$ and $i_{n,n+1} = j\restriction M_n$. The direct limit of $\{M_n, i_{n,m} : n, m < \omega\}$ is not well-founded.

[$i_{0,\omega}(\kappa)$, $i_{1,\omega}(\kappa)$, $\ldots$, $i_{n,\omega}(\kappa)$, $\ldots$ is a descending sequence of ordinals in the model.]

19.2. Show that if $m \le n$, then for each f on κ^m, $i_{m,n}([f]_{U_m}) = [g]_{U_n}$ where g is the function on κ^n defined by $g(\alpha_0, \ldots, \alpha_{n-1}) = f(\alpha_0, \ldots, \alpha_{m-1})$.

19.3. Prove this version of Łoś's Theorem (for functions with finite support): $(\text{Ult}_{U_\alpha}, E_\alpha) \vDash \varphi([f_1], \ldots, [f_n])$ if and only if $\{t : \varphi(f_1(t), \ldots, f_n(t))\} \in U_\alpha$.

Let D_n be the measure on κ^n defined from a normal measure D, and let $\kappa^{(n)} = i_{0,n}(\kappa)$ where $i_{0,n} : V \to \text{Ult}_D^{(n)}$.

19.4. The ordinal $\kappa^{(n-1)}$ is represented in Ult_{D_n} by the function $d_n(\alpha_1, \ldots, \alpha_n) = \alpha_n$.

19.5. $A \in D_n$ if and only if $(\kappa, \kappa^{(1)}, \ldots, \kappa^{(n-1)}) \in j_{D_n}(A)$.

19.6. If $A \in D_n$ then there exists a $B \in D$ such that $[B]^n \subset A$.

[Let $n = 3$. Let $B_1 = \{\alpha_1 : (\alpha_1, \kappa, \kappa^{(1)}) \in i_{0,2}(A)\}$, $B_2 = \{\alpha_2 \in B_1 : (\forall \alpha_1 \in B_1 \cap \alpha_2)\, (\alpha_1, \alpha_2, \kappa) \in i_{0,1}(A)\}$ and $B = B_3 = \{\alpha_3 \in B_2 : (\forall \alpha_2 \in B_2 \cap \alpha_3)(\forall \alpha_1 \in B_2 \cap \alpha_2)\, (\alpha_1, \alpha_2, \alpha_3) \in A\}$.]

Compare with Theorem 10.22.

19.7. Assume $V = L[D]$. If U is a κ-complete nonprincipal ultrafilter on κ and if $U \neq D$, then there is a monotone function $f : \kappa \to \kappa$ such that $\kappa \le [f]_U < [d]_U$. (Hence U does not extend the closed unbounded filter.)

[U satisfies (19.15) for some δ; if $\delta = \kappa^{(n)}$ for some n, then $U = D$. Let n be such that $\kappa^{(n-1)} < \delta < \kappa^{(n)}$; let $g : \kappa^n \to \kappa$ represents δ in Ult_{D_n}. Let $f(\xi) =$ least α such that $g(\alpha_1, \ldots, \alpha_{n-1}, \alpha) \ge \xi$ for some $\alpha_1 < \ldots < \alpha_{n-1} < \alpha$. The function f is monotone. To show that $[f]_U < [d]_U$, we argue as follows: For almost all (mod D_n) $\alpha_1, \ldots, \alpha_n$, $g(\alpha_1, \ldots, \alpha_n) > \alpha_n$; hence for almost all $\alpha_1, \ldots, \alpha_n$, $f(g(\alpha_1, \ldots, \alpha_n)) < g(\alpha_1, \ldots, \alpha_n)$. Hence $(j_{D_n}(f))(\delta) < \delta$, and hence for almost all ξ (mod U), $f(\xi) < \xi$. Thus $[f]_U < [d]_U$.]

19.8. If M and N are transitive models of ZFC^-, if $j : M \to N$ is an elementary embedding with critical point κ, and if $P^M(\kappa) = P^N(\kappa)$, then $\{X \in P^M(\kappa) : \kappa \in j(X)\}$ is a normal iterable M-ultrafilter.

19.9. If $\text{Ult}_{U_\alpha}(M)$ is well-founded for all $\alpha < \omega_1$, then $\text{Ult}_{U_\alpha}(M)$ is well-founded for all α.

[Assume that $\text{Ult}_{U_\alpha}(M)$ is not well-founded and let $f_0, f_1, \ldots, f_n, \ldots$ constitute a counterexample. Each f_n has a finite support E_n. Let β be the order-type of $\bigcup_{n=0}^\infty E_n$; we have $\beta < \omega_1$. Produce a counterexample in $\text{Ult}_{U_\beta}(M)$.]

19.10. If arbitrary countable intersections of elements of U are nonempty, then $\text{Ult}_{U_\alpha}(M)$ is well-founded for all α.

[Let $f_0, f_1, \ldots, f_n, \ldots$ be a counterexample, let $X_n = \{t \in \kappa^\alpha : f_n(t) \ni f_{n+1}(t)\}$. To reach a contradiction, find $t \in \bigcap_{n=0}^\infty X_n$. Construct t by induction such that for each $\nu < \alpha$ if $\alpha = \nu + \eta$, then $t{\restriction}\nu$ has the property that for all n, $\{s \in \kappa^n : (t{\restriction}\nu)^\frown s \in X_n\} \in U_\eta$: Given $t{\restriction}\nu$, there is $t(\nu)$ such that the condition is satisfied for $t{\restriction}(\nu + 1)$. Then $t \in \bigcap_{n=0}^\infty X_n$.]

19.11. Assume that every constructible subset of ω_1 either contains or is disjoint from a closed unbounded set. Let F be the closed unbounded filter on ω_1. Then $D = F \cap L$ is an iterable L-ultrafilter and $\text{Ult}_{D_\alpha}(L)$ is well-founded (and hence equal to L) for all α.

19.12. If $L[\mathcal{U}]$, $\mathcal{U} = \langle U_\alpha, U^0, U^1 \rangle_{\alpha \in A}$, is a canonical inner model for a measurable cardinal of order 2, if $B \in U^1$ is a subset of A, and if $\mathcal{W} = \langle W_\alpha, W^0, W^1 \rangle_{\alpha \in B}$, $W_\alpha = U_\alpha \cap L[\mathcal{W}]$, $W^\varepsilon = U^\varepsilon \cap L[\mathcal{W}]$, then $L[\mathcal{W}]$ is also a canonical inner model.

19.13. If there exist two different normal measures of order 1 on κ, then there exist canonical inner models $L[\mathcal{U}]$ and $L[\mathcal{W}]$ such that $\mathcal{U} = \langle U_\alpha, U^0, U^1 \rangle_{\alpha \in A}$, $\mathcal{W} = \langle W_\beta, W^0, W^1 \rangle_{\beta \in B}$ and such that $A = U^1$ and $B = W^1$ are disjoint subsets of κ.

Historical Notes

Most of the results in the first part of Chapter 19 are due to Kunen, who in [1970] developed the method of iterated ultraproducts invented by Gaifman (cf. [1964] and [1974]). Kunen found the representation of iterated ultraproducts (Lemma 19.13) and generalized the construction for M-ultrafilters. Kunen applied the method to obtain the main results of the model $L[D]$ (Theorem 19.14).

Theorem 19.3 (the proof of the GCH in $L[D]$) is due to Silver [1971d].

The description of κ-complete ultrafilters on κ in $L[D]$ (Lemma 19.21) is due to Kunen [1970] and Paris [1969]. Lemma 19.4 was first proved by Solovay. Theorem 19.7 is due to Gaifman; cf. [1974]. The proof of well-foundedness in Exercise 19.10 is due to Kunen. Lemmas 19.20 and 19.24 are results of Kunen [1970]. $0^\dagger$ was formulated by Solovay.

Kunen generalized the basic results on $L[D]$ to the model $L\langle D_\alpha : \alpha < \theta\rangle$ constructed from a sequence of measures (with $\theta <$ the least measurable cardinal in the sequence). Mitchell [1974] and [1983] generalized the theory of $L[D]$ to inner models for sequences of measures. The definition of $o(\kappa)$, Theorem 19.30 (well-foundedness of iterated ultrapowers) as well as the results on $L[\mathcal{U}]$ are all due to Mitchell.

The results in Exercises 19.9, 19.10 and 19.11 are due to Kunen [1970].

Exercise 19.7: Jech [1972/73].