20. Very Large Cardinals

This chapter studies properties of large cardinals that generalize measurability. We are particularly interested in the method of elementary embeddings, and introduce two concepts that have become crucial in the theory of large cardinals: supercompact and Woodin cardinals.

Strongly Compact Cardinals

In Chapter 9 we proved that weakly compact cardinals are inaccessible cardinals satisfying the Weak Compactness Theorem for the infinitary language $\mathcal{L}_{\kappa,\omega}$. If we remove the restriction on the size of sets of sentences in the model theoretic characterization of weakly compact cardinals, we obtain a considerably stronger notion. This notion, *strong compactness*, turns out to be much stronger than measurability.

Strongly compact cardinals can be characterized in several different ways. Let us use, as a definition, the property that is a natural generalization of the Ultrafilter Theorem:

Definition 20.1. An uncountable regular cardinal κ is *strongly compact* if for any set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S.

Obviously, every strongly compact cardinal κ is measurable, for any ultrafilter on κ that extends the filter $\{X : |\kappa - X| < \kappa\}$ is nonprincipal.

Let us say that the language $\mathcal{L}_{\kappa,\omega}$ (or $\mathcal{L}_{\kappa,\kappa}$) satisfies the *Compactness Theorem* if whenever Σ is a set of sentences of $\mathcal{L}_{\kappa,\omega}$ ($\mathcal{L}_{\kappa,\kappa}$) such that every $S \subset \Sigma$ with $|S| < \kappa$ has a model, then Σ has a model.

Let A be a set of cardinality greater than or equal to κ. For each $x \in P_\kappa(A)$, let $\hat{x} = \{y \in P_\kappa(A) : x \subset y\}$, and let us consider the filter on $P_\kappa(A)$ generated by the sets $\hat{x}$ for all $x \in P_\kappa(A)$; that is, the filter

$$\{X \subset P_\kappa(A) : X \supset \hat{x} \text{ for some } x \in P_\kappa(A)\}. \tag{20.1}$$

If κ is a regular cardinal, then the filter (20.1) is κ-complete. We call U a *fine measure* on $P_\kappa(A)$ if U is a κ-complete ultrafilter on $P_\kappa(A)$ that extends the filter (20.1); i.e., $\hat{x} \in U$ for all $x \in P_\kappa(A)$.

Lemma 20.2. *The following are equivalent, for any regular cardinal κ:*

(i) *For any set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S.*
(ii) *For any A such that $|A| \geq \kappa$, there exists a fine measure on $P_\kappa(A)$.*
(iii) *The language $\mathcal{L}_{\kappa,\omega}$ satisfies the compactness theorem.*

Proof. (i) $\to$ (ii) is clear.

(ii) $\to$ (iii): Let Σ be a set of sentences of $\mathcal{L}_{\kappa,\omega}$ and assume that every $S \subset \Sigma$ of size less than κ has a model, say $\mathfrak{A}_S$. Let U be a fine measure on $P_\kappa(\Sigma)$, and let us consider the ultraproduct $\mathfrak{A} = \text{Ult}_U\{\mathfrak{A}_S : S \in P_\kappa(\Sigma)\}$. It is routine to verify that Łoś's Theorem holds for the language $\mathcal{L}_{\kappa,\omega}$ provided the ultrafilter is κ-complete; in order to prove the induction step for infinitary connective $\bigwedge_{\xi<\alpha} \varphi_\xi$, one uses the κ-completeness of U. Thus we have, for any sentence σ of $\mathcal{L}_{\kappa,\omega}$,

$$\mathfrak{A} \vDash \sigma \quad \text{if and only if} \quad \{S : \mathfrak{A}_S \vDash \sigma\} \in U. \tag{20.2}$$

Now if $\sigma \in \Sigma$, then $\{\sigma\}^\wedge \in U$ and since $\mathfrak{A}_S \vDash \sigma$ whenever $S \ni \sigma$, (20.2) implies that σ holds in $\mathfrak{A}$. Hence $\mathfrak{A}$ is a model of Σ.

(iii) $\to$ (i): Let S be a set and let F be a κ-complete filter on S. Let us consider the $\mathcal{L}_{\kappa,\omega}$-language which has a unary predicate symbol $\dot{X}$ for each $X \subset S$, and a constant symbol c. Let Σ be the set of $\mathcal{L}_{\kappa,\omega}$ sentences consisting of:

(a) all sentences true in $(S, X)_{X \subset S}$,
(b) $\dot{X}(c)$ for all $X \in F$.

Every set of less than κ sentences in Σ has a model: Take S as the universe, interpret each $\dot{X}$ as X and let c be some element of S that lies in every X whose name is mentioned in the given set of sentences; since F is κ-complete, such c exists.

Hence Σ has a model $\mathfrak{A} = (A, X^{\mathfrak{A}}, c)_{X \subset S}$. Let us define $U \subset P(S)$ as follows:

$$X \in U \quad \text{if and only if} \quad \mathfrak{A} \vDash \dot{X}(c).$$

It is easy to verify that U is a κ-complete ultrafilter and that $U \supset F$: For instance, U is κ-complete because if $\alpha < \kappa$ and $X = \bigcap_{\xi<\alpha} X_\xi$, then $\mathfrak{A}$ satisfies the sentence $\bigwedge_{\xi<\alpha} \dot{X}_\xi(c) \to \dot{X}(c)$. □

Every strongly compact cardinal is measurable, but not every measurable cardinal is strongly compact (although it is consistent that there is exactly one measurable cardinal which is also strongly compact). We shall show that the existence of strongly compact cardinals is a much stronger assumption than the existence of measurable cardinals. We start with the following theorem:

Theorem 20.3 (Vopěnka-Hrbáček). *If there exists a strongly compact cardinal, then there is no set A such that $V = L[A]$.*

Proof. Let us assume that $V = L[A]$ for some set A. Since there is a set of ordinals A' such that $L[A] = L[A']$, we may assume that A is a set of ordinals. Let κ be a strongly compact cardinal, and let $\lambda \geq \kappa$ be a cardinal such that $A \subset \lambda$. There exists a κ-complete ultrafilter U on λ^+ such that $|X| = \lambda^+$ for every $X \in U$ (let U extend the filter $\{X : |\lambda^+ - X| \leq \lambda\}$).

Since U is κ-complete, the ultrapower $\mathrm{Ult}_U(V)$ is well-founded, and thus can be identified with a transitive model M. As usual, if f is a function on λ^+, then $[f]$ denotes the element of M represented by f. Let $j = j_U$ be the elementary embedding of V into M given by U.

Let us now consider another version of ultrapower. Let us consider only those functions on λ^+ that assume at most λ values. For these functions, we still define $f =^* g \pmod U$ and $f \in^* g \pmod U$ in the usual way, and therefore obtain a model of the language of set theory, which we denote $\mathrm{Ult}_U^-(V)$. Łoś's Theorem holds for this version of ultrapower too: If $f, \ldots$ are functions on λ^+ with $|\operatorname{ran}(f)| \leq \lambda$, then

$$\text{(20.3)} \qquad \mathrm{Ult}^- \vDash \varphi(f, \ldots) \quad \text{if and only if} \quad \{\alpha : \varphi(f(\alpha), \ldots)\} \in U.$$

(Check the induction step for $\exists$.) Hence Ult^- is a model of ZFC, elementarily equivalent to V. Also, since U is κ-complete, Ult^- is well-founded and thus is isomorphic to a transitive model N. Every element of N is represented by a function f on λ^+ such that $|\operatorname{ran}(f)| \leq \lambda$. We denote $[f]^-$ the element of N represented by f. We also define an elementary embedding $i : V \to N$ by $i(x) = [c_x]^-$ where c_x is the constant function on λ^+ with value x.

For every function f on λ^+ with $|\operatorname{ran} f| \leq \lambda$, we let

$$\text{(20.4)} \qquad k([f]^-) = [f].$$

It is easy to see that the definition of $k([f]^-)$ does not depend on the choice of f representing $[f]^-$ in N, and that k is an elementary embedding of N into M. In fact, $j = k \circ i$.

If $\gamma < \lambda^+$, then every function from λ^+ into γ has at most λ values, and hence $[f]^- = [f]$ for all $f : \lambda^+ \to \gamma$. If $f : \lambda^+ \to \lambda^+$ has at most λ values, then $f : \lambda^+ \to \gamma$ for some $\gamma < \lambda^+$; it follows that $i(\lambda^+) = \lim_{\gamma \to \lambda^+} i(\gamma)$, and we have $k(\xi) = \xi$ for all $\xi < i(\lambda^+)$.

Similarly, $i(A) = j(A)$, and we have $M = L[j(A)] = L[i(A)] = N$.

Now we reach a contradiction by observing that $j(\lambda^+) > i(\lambda^+)$: Since the diagonal function $d(\alpha) = \alpha$ represents in M an ordinal greater than each $j(\gamma)$, $\gamma < \lambda^+$, we have $j(\lambda^+) > \lim_{\gamma \to \lambda^+} j(\gamma)$. While N thinks that $i(\lambda^+)$ is the successor of $i(\lambda)$, M thinks that $j(\lambda^+)$ is the successor of $j(\lambda)$ (and $j(\lambda) = i(\lambda)$). Thus $M \neq N$, a contradiction. □

The following theorem shows that the consistency strength of strong compactness exceeds the strength of measurability:

Theorem 20.4 (Kunen). *If there exists a strongly compact cardinal then there exists an inner model with two measurable cardinals.*

Kunen proved a stronger version (and the proof can be so modified): For every ordinal ϑ there exists an inner model with ϑ measurable cardinals. This was improved by Mitchell who showed that the existence of a strongly compact cardinal leads to an inner model that has a measurable cardinal κ of Mitchell order κ^{++}.

We begin with a combinatorial lemma:

Lemma 20.5. *Let κ be an inaccessible cardinal. There exists a family $\mathcal{G}$ of functions $g : \kappa \to \kappa$ such that $|\mathcal{G}| = 2^\kappa$, and whenever $\mathcal{H} \subset \mathcal{G}$ is a subfamily of size $< \kappa$ and $\{\beta_g : g \in \mathcal{H}\}$ is any collection of ordinals $< \kappa$, then there exists an α such that $g(\alpha) = \beta_g$ for all $g \in \mathcal{H}$.*

Proof. Let $\mathcal{A}$ be a family of almost disjoint subsets of κ (i.e., $|A| = \kappa$ for each $A \in \mathcal{A}$ and $|A \cap B| < \kappa$ for any distinct $A, B \in \mathcal{A}$), such that $|\mathcal{A}| = 2^\kappa$. For each $A \in \mathcal{A}$, let f_A be a mapping of A onto κ such that for each $\beta < \kappa$, the set $\{a \in A : f_A(a) = \beta\}$ has size κ. Let s_α, $\alpha < \kappa$, enumerate all subsets $s \subset \kappa$ of size $< \kappa$.

For each $A \in \mathcal{A}$, let $g_A : \kappa \to \kappa$ be defined as follows: If $s_\alpha \cap A = \{x\}$, then $g_A(\alpha) = f_A(x)$; $g_A(\alpha) = 0$ otherwise. Let $\mathcal{G} = \{g_A : A \in \mathcal{A}\}$.

If $A \neq B \in \mathcal{A}$, then it is easy to find s_α such that $g_A(\alpha) \neq 0$ and $g_B(\alpha) = 0$; hence $|\mathcal{G}| = 2^\kappa$. If $\mathcal{H} \subset \mathcal{A}$ has size $< \kappa$ and if $\{\beta_A : A \in \mathcal{H}\}$ are given, then for each $A \in \mathcal{H}$ we choose $x_A \in A$ such that $x_A \notin B$ for any other $B \in \mathcal{H}$ and that $f_A(x_A) = \beta_A$. Then if α is such that $s_\alpha = \{x_A : A \in \mathcal{A}\}$, we have $g_A(\alpha) = \beta_A$ for every $A \in \mathcal{H}$. □

Lemma 20.6. *Let κ be a strongly compact cardinal. For every $\delta < (2^\kappa)^+$ there exists a κ-complete ultrafilter U on κ such that $j_U(\kappa) > \delta$.*

Proof. Let $\delta < (2^\kappa)^+$. Let $\mathcal{G}$ be a family of functions $g : \kappa \to \kappa$ of size $|\delta|$ with the property stated in Lemma 20.5; let us enumerate $\mathcal{G} = \{g_\alpha : \alpha \le \delta\}$.

For any $\alpha < \beta \le \delta$, let $X_{\alpha,\beta} = \{\xi : g_\alpha(\xi) < g_\beta(\xi)\}$. Using the property of $\mathcal{G}$ from Lemma 20.5, we can see that any collection of less than κ of the $X_{\alpha,\beta}$ has a nonempty intersection and hence $F = \{X : X \supset X_{\alpha,\beta}$ for some $\alpha < \beta \le \delta\}$ is a κ-complete filter on κ. There exists a κ-complete ultrafilter U extending F. It is clear that if $\alpha < \beta \le \delta$, then $g_\alpha < g_\beta \bmod U$, and hence $j_U(\kappa) > \delta$. □

Combining Lemma 20.6 with Lemmas 19.23 and 19.24, we already have a strong consequence of strong compactness.

We shall apply the technique of iterated ultrapowers to construct an inner model with two measurable cardinals.

Let D be a normal measure on κ, and let $i_{0,\alpha}$ denote, for each α, the elementary embedding $i_{0,\alpha} : V \to \text{Ult}^{(\alpha)}$; let $\kappa^{(\alpha)} = i_{0,\alpha}(\kappa)$ and $D^{(\alpha)} = i_{0,\alpha}(D)$.

First recall (19.5): If λ is a limit ordinal, then $X \in \text{Ult}^{(\lambda)}$ belongs to $D^{(\lambda)}$ if and only if $X \supset \{\kappa^{(\gamma)} : \alpha \le \gamma < \lambda\}$ for some $\alpha < \lambda$. Let

(20.5) $$C = \{\nu : \nu \text{ is a strong limit cardinal, } \nu > 2^\kappa, \text{ and } \operatorname{cf} \nu > \kappa\}.$$

By Lemma 19.15, if $\nu \in C$ then $\kappa^{(\nu)} = \nu$, and $i_{0,\alpha}(\nu) = \nu$ for all $\alpha < \nu$. Thus if $\gamma_0 < \gamma_1 < \ldots < \gamma_n < \ldots$ are elements of the class C, and if $\lambda = \lim_{n\to\infty} \gamma_n$, then $\kappa^{(\lambda)} = \lambda$, and $X \in \mathrm{Ult}^{(\lambda)}$ belongs to $D^{(\lambda)}$ just in case $X \supset \{\gamma_n : n_0 \le n\}$ for some n_0.

If A is a set of ordinals of order-type ω, $A = \{\gamma_n : n \in \omega\}$, we define a filter $F(A)$ on $\lambda = \sup A$ as follows:

$$X \in F(A) \quad \text{if and only if} \quad \exists n_0\, (\forall n \ge n_0)\, \gamma_n \in X. \tag{20.6}$$

The above discussion leads us to this: If $A \subset C$ has order-type ω, and if $\lambda = \sup A$, then for every $X \in \mathrm{Ult}^{(\lambda)}$, $X \in D^{(\lambda)}$ if and only if $X \in F(A)$. In other words,

$$D^{(\lambda)} = F(A) \cap \mathrm{Ult}^{(\lambda)}. \tag{20.7}$$

Hence $F(A) \cap \mathrm{Ult}^{(\lambda)} \in \mathrm{Ult}^{(\lambda)}$; and so, $L[F(A)] = L[D^{(\lambda)}]$. Thus $F(A) \cap L[F(A)] = D^{(\lambda)} \cap L[D^{(\lambda)}]$, and we have

$$L[F(A)] \vDash F(A) \cap L[F(A)] \text{ is a normal measure on } \lambda. \tag{20.8}$$

The only assumption needed to derive (20.8) is that κ is measurable and A is a subset of the class C. We shall now use Lemma 20.6 and a similar construction to obtain a model with two measurable cardinals.

Suppose that $A = \{\gamma_n : n \in \omega\}$ is as above, and that $A' = \{\gamma'_n : n \in \omega\}$ is another subset of C of order-type ω, such that $\gamma'_0 > \lambda = \sup A$; let $\lambda' = \sup A'$. Let $F = F(A)$ and $F' = F(A')$. Our intention is to choose A and A' such that the model $L[F, F']$ has two measurable cardinals, namely λ and λ', and that $F \cap L[F, F']$ and $F' \cap L[F, F']$ are normal measures on λ and λ', respectively.

The argument leading to (20.8) can again be used to show that $F \cap L[F, F']$ is a normal measure on λ in $L[F, F']$. This is because we have again

$$D^{(\lambda)} = F \cap \mathrm{Ult}^{(\lambda)};$$

moreover, $i_{0,\lambda}(\gamma'_n) = \gamma'_n$ for each n, and hence $i_{0,\lambda}(A') = A'$ and we have

$$i_{0,\lambda}(F') = F' \cap \mathrm{Ult}^{(\lambda)}. \tag{20.9}$$

Therefore

$$L[F, F'] = L[D^{(\lambda)}, F'] = L[D^{(\lambda)}, i_{0,\lambda}(F')]$$

and

$$F \cap L[F, F'] = D^{(\lambda)} \cap L[D^{(\lambda)}, i_{0,\lambda}(F')], \tag{20.10}$$

which gives

$$L[F, F'] \vDash F \cap L[F, F'] \text{ is a normal measure on } \lambda. \tag{20.11}$$

In order to find A, A' so that F' also gives a normal measure in $L[F, F']$, let us make the following observation: Let us think for a moment that $A \subset \kappa$ and

$A' \subset C$. Then $i_{0,\lambda'}(A) = A$ and $D^{(\lambda')} = F' \cap \text{Ult}^{(\lambda')}$, and the same argument as above shows that

$$L[F, F'] \vDash F' \cap L[F, F'] \text{ is a normal measure on } \lambda'. \tag{20.12}$$

We shall use this observation below.

Let us define the following classes of cardinals (compare with (18.29)):

$$C_0 = C, \qquad C_{\alpha+1} = \{\nu \in C_\alpha : |C_\alpha \cap \nu| = \nu\}, \\ C_\gamma = \bigcap_{\alpha<\gamma} C_\alpha \quad (\gamma \text{ limit}). \tag{20.13}$$

Each C_α is nonempty; in fact each C_α is unbounded and δ-closed for all δ of cofinality $> \kappa$.

Now we let

$$\gamma_n = \text{the least element of } C_n, \\ A = \{\gamma_n : n \in \omega\}, \qquad \lambda = \lim_{n\to\infty} \gamma_n \tag{20.14}$$

and let $A' = \{\gamma'_n : n \in \omega\}$ be a subset of $C_{\omega+1}$.

Let us consider the model $L[A, A']$, and let for each $n \le \omega$

$$\begin{aligned} M_n &= \text{the Skolem hull of } C_n \text{ in } L[A, A'] \\ &= \text{the class of all } x \in L[A, A'] \text{ such that} \\ &\quad L[A, A'] \vDash x = t[\nu_1, \ldots, \nu_k, \gamma_0, \ldots, \gamma_k, \gamma'_0, \ldots, \gamma'_k, A, A'] \\ &\quad \text{where } t \text{ is a Skolem term and } \nu_1, \ldots, \nu_k \in C_n. \end{aligned} \tag{20.15}$$

(Let us not worry about the problem whether (20.15) is expressible in the language of set theory; it can be shown that it is, similarly as in the case of ordinal definable sets. Alternatively, we can consider the model $L_\theta[A, A']$ where θ is some large enough cardinal in $C_{\omega+1}$.)

Each M_n is an elementary submodel of $L[A, A']$; let π_n be the transitive collapse of M_n; then $\pi_n(M_n) = L[\pi_n(A), \pi_n(A')]$ and $j_n = \pi_n^{-1}$ is an elementary embedding

$$j_n : L[\pi_n(A), \pi_n(A')] \to L[A, A'].$$

Lemma 20.7. *For each $n < \omega$, $\pi_n(\gamma_n) < (2^\kappa)^+$.*

Proof. By induction on n. First let $n = 0$. Let $\alpha < \gamma_0$ be in M_0. Then $\alpha = t(\nu_1, \ldots, \nu_k, A, A')$ for some Skolem term t and some $\nu_1, \ldots, \nu_k \in C_0$. Let $i_{0,\alpha}$ be the elementary embedding into $\text{Ult}_U^{(\alpha)}$ for some U on κ. Since γ_0 is the least element of C_0, we have $\alpha < \nu$ for all $\nu \in C_0$ and hence $i_{0,\alpha}(\nu) = \nu$ for all $\nu \in C_0$. Hence also $i_{0,\alpha}(A) = A$ and $i_{0,\alpha}(A') = A'$ and it follows that $i_{0,\alpha}(\alpha) = \alpha$. Now $i_{0,\alpha}(\alpha) = \alpha$ is possible only if $\alpha < \kappa$. Hence each $\alpha < \gamma_0$ in M_0 is less than κ and therefore $\pi_0(\gamma_0) \le \kappa$.

Now let us assume that $\pi_n(\gamma_n) < (2^\kappa)^+$ and let us show that $\pi_{n+1}(\gamma_{n+1}) < (2^\kappa)^+$. By Lemma 20.6 there exists a U such that $j_U(\kappa) > \pi_n(\gamma_n)$. We shall show that $\pi_n(\alpha) < j_U(\kappa)$ for all $\alpha < \gamma_{n+1}$ in M_{n+1}; since $\pi_{n+1}(\alpha) \le \pi_n(\alpha)$ it follows that $\pi_{n+1}(\gamma_{n+1}) = \sup\{\pi_{n+1}(\alpha) : \alpha < \gamma_{n+1} \text{ and } \alpha \in M_{n+1}\} \le j_U(\kappa) < (2^\kappa)^+$.

First notice that it follows from the definition of C_{n+1} in (20.13) that $\pi_n(\nu) = \nu$ for all $\nu \in C_{n+1}$. Note also that $\gamma_m \in C_{n+1}$ for all $m \ge n+1$, and $A' \subset C_{n+1}$.

Let $\alpha < \gamma_{n+1}$ be in M_{n+1}. Then (in $L[A, A']$),

$$\alpha = t(\gamma_0, \ldots, \gamma_n, \nu_1, \ldots, \nu_k, A, A')$$

where t is some Skolem term and $\nu_1, \ldots, \nu_k \in C_{n+1}$. Hence (in $L[\pi_n(A), \pi_n(A')]$)

$$\pi_n(\alpha) = t(\pi_n(\gamma_0), \ldots, \pi_n(\gamma_n), \nu_1, \ldots, \nu_k, \pi_n(A), A').$$

Now we argue inside the model $\text{Ult}_U(V)$ (which contains both $\pi_n(A)$ and A'): Consider the αth iterated ultrapower (modulo some measure on $j_U(\kappa)$). Since $\pi_n(\gamma_0), \ldots, \pi_n(\gamma_n)$ are all less than $j_U(\kappa)$, we have $i_{0,\alpha}(\pi_n(\gamma_i)) = \pi_n(\gamma_i)$ for all $i = 0, \ldots, n$. We also have $i_{0,\alpha}(\nu) = \nu$ for each $\nu \in C_{n+1}$ (because $\alpha < \nu$ for each $\nu \in C_{n+1}$ and $C_{n+1} \subset C$). It follows that $i_{0,\alpha}(\pi_n(\alpha)) = \pi_n(\alpha)$. Now (because $\pi_n(\alpha) \le \alpha$) this is only possible if $\pi_n(\alpha) < j_U(\kappa)$. □

We can now complete the proof of Theorem 20.4. Let us consider the model M_ω, the Skolem hull in $L[A, A']$ of C_ω. Let π_ω be the transitive collapse of M_ω and $B = \pi_\omega(A)$. Since $A' \subset C_{\omega+1}$, we have $\pi_\omega(A') = A'$, and $j_\omega = \pi_\omega^{-1}$ is an elementary embedding

$$j_\omega : L[B, A'] \to L[A, A'].$$

By Lemma 20.7, $\pi_\omega(\gamma_n) \le \pi_n(\gamma_n) < (2^\kappa)^+$ for all n, and hence $\pi_\omega(\lambda) < (2^\kappa)^+$. Let U be a κ-complete ultrafilter on κ such that $j_U(\kappa) > \pi_\omega(\lambda)$.

In Ult_U, B is a subset of $j_U(\kappa)$ and A' is a subset of the class C. Thus we can apply (20.12) and get

$$\text{Ult}_U \vDash (L[F(B), F(A')] \vDash F(A') \cap L[F(B), F(A')] \text{ is a normal measure on } \lambda').$$

Hence

$$L[B, A'] \vDash (L[F(B), F(A')] \vDash F(A') \cap L[F(B), F(A')] \text{ is a normal measure on } \lambda'),$$

and applying j_ω, we get

$$L[A, A'] \vDash (L[F(A), F(A')] \vDash F(A') \cap L[F(A), F(A')] \text{ is a normal measure on } \lambda').$$

Therefore $F' \cap L[F, F']$ is (in $L[F, F']$) a normal measure on λ'. This completes the proof of Theorem 20.4. □

The following theorem provides further evidence of the effect of large cardinals on cardinal arithmetic.

Theorem 20.8 (Solovay). *If κ is a strongly compact cardinal, then the Singular Cardinal Hypothesis holds above κ. That is, if $\lambda > \kappa$ is a singular cardinal, then $2^{\operatorname{cf}\lambda} < \lambda$ implies $\lambda^{\operatorname{cf}\lambda} = \lambda^+$. (Consequently, if $\lambda > \kappa$ is a singular strong limit cardinal, then $2^\lambda = \lambda^+$.)*

We shall prove the theorem in a sequence of lemmas. An ultrafilter on λ is *uniform* if every set in the ultrafilter has size λ.

Lemma 20.9. *If κ is a strongly compact cardinal and $\lambda > \kappa$ is a regular cardinal, then there exists a κ-complete uniform ultrafilter D on λ with the property that almost all* (mod D) *ordinals $\alpha < \lambda$ have cofinality less than κ.*

Proof. Let U be a fine measure on $P_\kappa(\lambda)$. Since U is fine, every $\alpha < \lambda$ belongs to almost all (mod U) $x \in P_\kappa(\lambda)$. Let us consider the ultrapower $\operatorname{Ult}_U(V)$ and let f be the least ordinal function in Ult_U greater than all the constant functions c_γ, $\gamma < \lambda$:

$$[f] = \lim_{\gamma\to\lambda} j_U(\gamma). \tag{20.16}$$

We note first that $f(x) < \lambda$ for almost all x: Let $g : P_\kappa(\lambda) \to \lambda$ be the function $g(x) = \sup x$. If $\gamma < \lambda$, then $\gamma \le g(x)$ for almost all x and hence $j(\gamma) \le [g]$; thus $[f] \le [g] \le j(\lambda)$.

Let D be the ultrafilter on λ defined as follows:

$$X \in D \quad \text{if and only if} \quad f_{-1}(X) \in U \qquad (X \subset \lambda). \tag{20.17}$$

It is clear that D is κ-complete, and since f is greater than the constant function, D is nonprincipal. For the same reason, the diagonal function $d(\alpha) = \alpha$ is greater (in Ult_D) than all the constant functions c_γ, $\gamma < \lambda$, and since λ is regular, D is uniform. In order to show that almost all (mod D) $\alpha < \lambda$ have cofinality $< \kappa$, it suffices by (20.17), to show that $\operatorname{cf}(f(x)) < \kappa$ for almost all x (mod U).

That will follow immediately once we show that for almost all x (mod U),

$$f(x) = \sup\{\alpha \in x : \alpha < f(x)\}. \tag{20.18}$$

We clearly have $\ge$ in (20.18). To prove $\le$, consider the function $h(x) = \sup\{\alpha \in x : \alpha < f(x)\}$. For each $\gamma < \lambda$, γ is in almost every x and hence $\gamma \le h(x)$ almost everywhere. Thus $[h] \ge j_U(\gamma)$ for all $\gamma < \lambda$ and so $f(x) \le h(x)$ almost everywhere. □

Lemma 20.10. *If κ is strongly compact and $\lambda > \kappa$ is a regular cardinal, then there exist a κ-complete nonprincipal ultrafilter D on λ and a collection $\{M_\alpha : \alpha < \lambda\}$ such that*

(20.19) (i) *$|M_\alpha| < \kappa$ for all $\alpha < \lambda$,*
(ii) *for every $\gamma < \lambda$, γ belongs to M_α for almost all α* (mod D).

(An ultrafilter D that has a family $\{M_\alpha : \alpha < \lambda\}$ with property (20.19) is called (κ, λ)-*regular*.)

Proof. Let D be the ultrafilter on λ constructed in Lemma 20.9. It follows from the construction of D that $[d]_D = \lim_{\gamma\to\lambda} j_D(\gamma)$. For almost all α (mod D), there exists an $A_\alpha \subset \alpha$ of size less than κ and cofinal in α. If $\operatorname{cf}\alpha \geq \kappa$, let $A_\alpha = \emptyset$. Let A be the set of ordinals represented in $\operatorname{Ult}_D(V)$ by the function $\langle A_\alpha : \alpha < \lambda\rangle$. The set A is cofinal in the ordinal represented by the diagonal function d; and since $[d] = \lim_{\gamma\to\lambda} j_D(\gamma)$, it follows that for each $\eta < \lambda$ there is $\eta' > \eta$ such that $A \cap \{\xi : j_D(\eta) \leq \xi < j_D(\eta')\}$ is nonempty.

We construct a sequence $\langle \eta_\gamma : \gamma < \lambda\rangle$ of ordinals $< \lambda$ as follows: Let $\eta_0 = 0$ and $\eta_\gamma = \lim_{\delta\to\gamma} \eta_\delta$ if γ is limit; let $\eta_{\gamma+1}$ be some ordinal such that there exists $\xi \in A$ such that $j_D(\eta_\gamma) \leq \xi < j_D(\eta_{\gamma+1})$.

In other words, if we denote I_γ the interval $\{\xi : \eta_\gamma \leq \xi < \eta_{\gamma+1}\}$, then for every γ, the interval I_γ has nonempty intersection with almost every A_α. Thus if we let

$$M_\alpha = \{\gamma < \lambda : I_\gamma \cap A_\alpha \neq \emptyset\}$$

for each $\alpha < \lambda$, then $\{M_\alpha : \alpha < \lambda\}$ has property (20.19)(ii). To see that M_α has property (i) as well, notice that $|A_\alpha| < \kappa$ for all α and that since the I_γ are mutually disjoint, each A_α intersects less than κ of them. □

Lemma 20.11. *If κ is strongly compact and $\lambda > \kappa$ is a regular cardinal, then there exists a collection $\{M_\alpha : \alpha < \lambda\} \subset P_\kappa(\lambda)$ such that*

$$P_\kappa(\lambda) = \bigcup_{\alpha<\lambda} P(M_\alpha). \tag{20.20}$$

Consequently, $\lambda^{<\kappa} = \lambda$.

Proof. Let $\{M_\alpha : \alpha < \lambda\}$ be as in Lemma 20.10. If x is a subset of λ of size less than κ, then by (20.19)(ii) and by κ-completeness of D, $x \subset M_\alpha$ for almost all α. Hence $x \in P(M_\alpha)$ for some $\alpha < \lambda$. This proves (20.20); since κ is inaccessible, it follows that $|P_\kappa(\lambda)| = \lambda$. □

Proof of Theorem 20.8. Let κ be a strongly compact cardinal. If $\lambda > \kappa$ is an arbitrary cardinal, then we have, by Lemma 20.11

$$\lambda^{<\kappa} \leq (\lambda^+)^{<\kappa} = \lambda^+.$$

In particular, we have $\lambda^{\aleph_0} \leq \lambda^+$ for every $\lambda > \kappa$. This implies that the Singular Cardinal Hypothesis holds for every $\lambda > \kappa$. □

Supercompact Cardinals

We proved in Lemma 20.2 that a strongly compact cardinal κ is characterized by the property that every $P_\kappa(A)$ has a fine measure. If we require the fine measure to satisfy a normality condition, then we obtain a stronger notion—a *supercompact* cardinal. Ultrapowers by normal measures on $P_\kappa(A)$ induce elementary embeddings that can be used to derive strong consequences of supercompact cardinals. For instance, Theorems 20.3 and 20.4 become almost trivial if the existence of a strongly compact cardinal is replaced by the existence of a supercompact cardinal. It is consistent to assume that a strongly compact cardinal is not supercompact, or that every strongly compact cardinal is supercompact, but it is not known whether supercompact cardinals are consistent relative to strongly compact cardinals.

Definition 20.12. A fine measure U on $P_\kappa(A)$ is *normal* if whenever $f : P_\kappa(A) \to A$ is such that $f(x) \in x$ for almost all x, then f is constant on a set in U. A cardinal κ is *supercompact* if for every A such that $|A| \geq \kappa$, there exists a normal measure on $P_\kappa(A)$.

Let $\lambda \geq \kappa$ be a cardinal and let us consider the ultrapower $\text{Ult}_U(V)$ by a normal measure U on $P_\kappa(\lambda)$; let $j = j_U$ be the corresponding elementary embedding. Clearly, a set $X \subset P_\kappa(\lambda)$ belongs to U if and only if $[d] \in j(X)$, where d, the *diagonal function*, is the function $d(x) = x$.

Lemma 20.13. *If U is a normal measure on $P_\kappa(\lambda)$, then $[d] = \{j(\gamma) : \gamma < \lambda\} = j``\lambda$, and hence for every $X \subset P_\kappa(\lambda)$,*

$$X \in U \quad \textit{if and only if} \quad j``\lambda \in j(X). \tag{20.21}$$

Proof. On the one hand, if $\gamma < \lambda$, then $\gamma \in x$ for almost all x and hence $j(\gamma) \in [d]$. On the other hand, if $[f] \in [d]$, then $f(x) \in x$ for almost all x and by normality, there is $\gamma < \lambda$ such that $[f] = j(\gamma)$. □

It follows from (20.21) that if f and g are functions on $P_\kappa(\lambda)$, then

$$[f] = [g] \quad \text{if and only if} \quad (jf)(j``\lambda) = (jg)(j``\lambda).$$

and

$$[f] \in [g] \quad \text{if and only if} \quad (jf)(j``\lambda) \in (jg)(j``\lambda).$$

Consequently,

$$[f] = (jf)(j``\lambda) \tag{20.22}$$

for every function f on $P_\kappa(\lambda)$.

For each $x \in P_\kappa(\lambda)$, let us denote

$$\begin{aligned} \kappa_x &= x \cap \kappa, \quad \text{and} \\ \lambda_x &= \text{the order-type of } x. \end{aligned} \tag{20.23}$$

Note that the order-type of $j``\lambda$ is λ and hence by (20.22), λ is represented in the ultrapower by the function $x \mapsto \lambda_x$. Also, since $\lambda_x < \kappa$ for all x, we have $j(\kappa) > \lambda$. By the κ-completeness of U, we have $j(\gamma) = \gamma$ for all $\gamma < \kappa$; and since κ is moved by j, it follows that $j``\lambda \cap j(\kappa) = \kappa$ and therefore κ is represented by the function $x \mapsto \kappa_x$.

This gives the following characterization of supercompact cardinals:

Lemma 20.14. *Let $\lambda \geq \kappa$. A normal measure on $P_\kappa(\lambda)$ exists if and only if there exists an elementary embedding $j : V \to M$ such that*

(20.24)
(i) *$j(\gamma) = \gamma$ for all $\gamma < \kappa$;*
(ii) *$j(\kappa) > \lambda$;*
(iii) *$M^\lambda \subset M$; i.e., every sequence $\langle a_\alpha : \alpha < \lambda \rangle$ of elements of M is a member of M.*

A cardinal κ is called λ-*supercompact* if it satisfies (20.24).

Proof. (a) Let U be a normal measure on $P_\kappa(\lambda)$. We let $M = \text{Ult}_U(V)$ and let j be the canonical elementary embedding $j : V \to \text{Ult}$. We have already proved (i) and (ii). To prove (iii), it suffices to show that whenever $\langle a_\alpha : \alpha < \lambda \rangle$ is such that $a_\alpha \in M$ for all $\alpha < \lambda$, then the set $\{a_\alpha : \alpha < \lambda\}$ belongs to M. Let f_α, $\alpha < \lambda$, be functions representing elements of M: $[f_\alpha] \in M$. We consider the function f on $P_\kappa(\lambda)$ defined as follows: $f(x) = \{f_\alpha(x) : \alpha \in x\}$; we claim that $[f] = \{a_\alpha : \alpha < \lambda\}$.

On the one hand, if $\alpha < \lambda$, then $\alpha \in x$ for almost all x and hence $[f_\alpha] \in [f]$. On the other hand, if $[g] \in [f]$, then for almost all x, $g(x) = f_\alpha(x)$ for some $\alpha \in x$. By normality, there exists some $\gamma < \lambda$ such that $g(x) = f_\gamma(x)$ for almost all x, and hence $[g] = a_\gamma$.

(b) Let $j : V \to M$ be an elementary embedding that satisfies (i), (ii), and (iii). By (iii), the set $\{j(\gamma) : \gamma < \lambda\}$ belongs to M and so the following defines an ultrafilter on $P_\kappa(\lambda)$:

$$X \in U \quad \text{if and only if} \quad j``\lambda \in j(X). \tag{20.25}$$

A standard argument shows that U is a κ-complete ultrafilter. U is a fine measure because for every $\alpha \in \lambda$, $\{x : \alpha \in x\}$ is in U. Finally, U is normal: If $f(x) \in x$ for almost all x, then $(jf)(j``\lambda) \in j``\lambda$. Hence $(jf)(j``\lambda) = j(\gamma)$ for some $\gamma < \lambda$, and so $f(x) = \gamma$ for almost all x. □

We have seen several examples how large cardinals restrict the behavior of the continuum function (e.g., if κ is measurable and $2^\kappa > \kappa^+$, then $2^\alpha > \alpha^+$ for cofinally many $\alpha < \kappa$). This is more so for supercompact cardinals:

Lemma 20.15. *If κ is λ-supercompact and $2^\alpha = \alpha^+$ for every $\alpha < \kappa$, then $2^\alpha = \alpha^+$ for every $\alpha \leq \lambda$.*

Proof. Let $j : V \to M$ witness that κ is λ-supercompact. If $\alpha \leq \lambda$, then because $\lambda < j(\kappa)$ and by elementarity, $(2^\alpha)^M = (\alpha^+)^M$. Now $M^\lambda \subset M$ implies that $P^M(\alpha) = P(\alpha)$ and so $2^\alpha \leq (2^\alpha)^M = (\alpha^+)^M = \alpha^+$. □

See Exercises 20.5–20.7 for a more general statement.

Lemma 20.16. *If κ is supercompact, then there exists a normal measure D on κ such that almost every $\alpha < \kappa$ (mod D) is measurable. In particular, κ is the κth measurable cardinal.*

Proof. Let $\lambda = 2^\kappa$ and let $j : V \to M$ witness the λ-supercompactness of κ. Let D be defined by $D = \{X : \kappa \in j(X)\}$, and let $j_D : V \to \text{Ult}_D$ be the corresponding elementary embedding. Let $k : \text{Ult}_D \to M$ be the elementary embedding defined in Lemma 17.4:

$$k([f]_D) = (jf)(\kappa).$$

Note that $k(\kappa) = \kappa$.

Now, $P(\kappa) \subset M$ and every subset of M of size λ is in M; hence every $U \subset P(\kappa)$ is in M and it follows that in M, κ is a measurable cardinal. Since k is elementary and $k(\kappa) = \kappa$, we have $\text{Ult}_D \vDash \kappa$ is a measurable cardinal, and the lemma follows. □

In contrast to Lemma 20.16, it is consistent that the least strongly compact cardinal is the least measurable. The following lemma and corollary also show that strongly compactness and supercompactness are not equivalent.

Lemma 20.17. *Let κ be a measurable cardinal such that there are κ strongly compact cardinals below κ. Then κ is strongly compact.*

Proof. Let F be a nonprincipal κ-complete ultrafilter on κ such that $C \in F$ where $C = \{\alpha < \kappa : \alpha \text{ is strongly compact}\}$. Let A be such that $|A| \geq \kappa$; we shall show that there is a fine measure on $P_\kappa(A)$.

For each $\alpha \in C$, let U_α be a fine measure on $P_\alpha(A)$, and let us define $U \subset P_\kappa(A)$ as follows:

$$X \in U \quad \text{if and only if} \quad \{\alpha \in C : X \cap P_\alpha(A) \in U_\alpha\} \in F.$$

It is easy to verify that U is a fine measure on $P_\kappa(A)$. □

Corollary 20.18. *If there exists a measurable cardinal that is a limit of strongly compact cardinals, then the least such cardinal is strongly compact but not supercompact.*

Proof. Let κ be the least measurable limit of compact cardinals. By Lemma 20.17, κ is strongly compact. Let us assume that κ is supercompact. Let $\lambda = 2^\kappa$ and let $j : V \to M$ be an elementary embedding such that κ is the least ordinal moved, and that $M^\lambda \subset M$. If $\alpha < \kappa$ is strongly compact, then $M \vDash j(\alpha)$ is strongly compact, but $j(\alpha) = \alpha$ and therefore $M \vDash \kappa$ is a limit of strongly compact cardinals. Since every $U \subset P(\kappa)$ is in M, κ is measurable in M and hence in M, κ is a measurable limit of strongly compact cardinals. This is a contradiction because M thinks that $j(\kappa)$ is the least measurable limit of strongly compact cardinals. □

The assumption of Corollary 20.18 holds if there are extendible cardinals (defined in the next section). There is also a consistency proof showing that not every strongly compact cardinal is supercompact. (And another consistency proof gives a model that has exactly one strongly compact cardinal and the cardinal is supercompact.)

The construction of a normal measure from an elementary embedding in (20.25) yields a commutative diagram analogous to (17.3). Let $j : V \to M$ be an elementary embedding with critical point κ such that $j(\kappa) > \lambda$ and $M^\lambda \subset M$, cf. (20.24). Let

$$U = \{X \in P_\kappa(\lambda) : j``\lambda \in j(X)\}$$

be the normal measure defined from j. Let $\text{Ult} = \text{Ult}_U(V)$, and $j_U : V \to \text{Ult}$.

For each $[f] \in \text{Ult}$, let

$$k([f]) = (jf)(j``\lambda). \tag{20.26}$$

As in Lemma 17.4, one verifies that $k : \text{Ult} \to M$ is an elementary embedding, and $j = k \circ j_U$.

We claim that

$$k(\alpha) = \alpha \quad \text{for all } \alpha \leq \lambda. \tag{20.27}$$

To prove (20.27), let $\alpha \leq \lambda$, and let us denote, for each $x \in P_\kappa(\lambda)$,

$$\alpha_x = \text{the order-type of } x \cap \alpha$$

(compare with (20.23)). Since the order-type of $j_U``\lambda \cap j_U(\alpha)$ is $\alpha$, it follows from (20.22) that the function $f(x) = \alpha_x$ represents $\alpha$ in the ultrapower: $[f] = (j_U f)(j``\lambda) =$ the order-type of $j_U``\lambda \cap j_U(\alpha) = \alpha$. Now (20.27) follows:

$$k(\alpha) = k([f]) = (jf)(j``\lambda) = \text{the order-type of } j``\lambda \cap j(\alpha) = \alpha.$$

Lemma 20.19.

(i) *If $\lambda \geq \kappa$ and if κ is μ-supercompact, where $\mu = 2^{\lambda^{<\kappa}}$, then for every $\mathcal{X} \subset P(P_\kappa(\lambda))$ there exists a normal measure on $P_\kappa(\lambda)$ such that $\mathcal{X} \in \text{Ult}_U(V)$.*

(ii) *If κ is 2^κ-supercompact, then for every $\mathcal{X} \subset P(\kappa)$ there exists a normal measure D on κ such that $\mathcal{X} \in \text{Ult}_D(V)$.*

Proof. (i) Assume on the contrary that there exists some $\mathcal{X} \subset P(P_\kappa(\lambda))$ such that $\varphi(\mathcal{X}, \kappa, \lambda)$ where φ is the statement

$$\mathcal{X} \notin \text{Ult}_U \text{ for every normal measure } \mathcal{U} \text{ on } P_\kappa(\lambda). \tag{20.28}$$

Let $j : V \to M$ be a witness to the μ-supercompactness of κ. As $M^\mu = M$, the ultrapowers by normal measures on $P_\kappa(\lambda)$ are correctly computed in M, and so $M \vDash \exists \mathcal{X}\, \varphi(\mathcal{X}, \kappa, \lambda)$.

Let $U = \{X \in P_\kappa(\lambda) : j``\lambda \in j(X)\}$ and let $k : \text{Ult}_U \to M$ be such that $j = k \circ j_U$. By (20.27), $k(\kappa) = \kappa$ and $k(\lambda) = \lambda$, and since $k : \text{Ult} \to M$ is elementary, we have $\text{Ult} \vDash \exists \mathcal{X}\, \varphi(\mathcal{X}, \kappa, \lambda)$. Let $\mathcal{X} \in \text{Ult}$ be such that $\text{Ult} \vDash \varphi(\mathcal{X}, \kappa, \lambda)$. By (20.27) again, $k(\alpha) = \alpha$ for all $\alpha \le \lambda$, and it follows that $k(\mathcal{X}) = \mathcal{X}$. By elementarity again, $M \vDash \varphi(k(\mathcal{X}), k(\kappa), k(\lambda))$ and so $M \vDash \varphi(\mathcal{X}, \kappa, \lambda)$. This contradicts (20.28) because $\mathcal{X} \in \text{Ult}_U$.

(ii) Similar, using (17.3). □

Corollary 20.20.

(i) *If κ is supercompact then there are 2^{2^κ} normal measures on κ.*

(ii) *If κ is supercompact then for every $\lambda \ge \kappa$ there are $2^{2^{\lambda^{<\kappa}}}$ normal measures on $P_\kappa(\lambda)$.*

(iii) *If κ is supercompact then the Mitchell order of κ is $(2^\kappa)^+ \ge \kappa^{++}$.*

Proof. (i) If D is a normal measure on κ and $\mathcal{X} \subset P(\kappa)$ is in Ult_D, then $\mathcal{X}$ is represented by a function f on κ such that $f(\alpha) \subset P(\alpha)$ for all $\alpha < \kappa$. Since the number of such functions is 2^κ, it follows that Ult_D contains only 2^κ subsets of $P(\kappa)$. However, by Lemma 20.19(ii), each $\mathcal{X} \subset P(\kappa)$ is contained in some ultrapower Ult_D where D is a normal measure on κ, and therefore there must exist 2^{2^κ} normal measures on κ.

(ii) Similar, using Lemma 20.19(i).

(iii) There is an increasing chain of length $(2^\kappa)^+$ of normal measures on κ in the Mitchell order: Given at most 2^κ such measures, one can code them as some $\mathcal{X} \subset P(\kappa)$. By Lemma 20.19(ii) there exists a normal measure U on κ such that $\mathcal{X} \in \text{Ult}_U$. □

We conclude this section with the following theorem reminiscent of the Diamond Principle.

Theorem 20.21 (Laver). *Let κ be a supercompact cardinal. There exists a function $f : \kappa \to V_\kappa$ such that for every set x and every $\lambda \ge \kappa$ such that $\lambda \ge |\text{TC}(x)|$ there exists a normal measure U on $P_\kappa(\lambda)$ such that $j_U(f)(\kappa) = x$.*

(Such an f is called a *Laver function.*)

Proof. Assume that the theorem is false. For each $f : \kappa \to V_\kappa$, let λ_f be the least cardinal $\lambda_f \ge \kappa$ for which there exists an x with $|\text{TC}(x)| \le \lambda_x$ such that $j_U(f)(\kappa) \ne x$ for every normal measure U on $P_\kappa(\lambda_f)$. Let ν be greater than all the λ_f and let $j : V \to M$ be a witness to the ν-supercompactness of κ.

Let $\varphi(g, \delta)$ be the statement that for some cardinal α, $g : \alpha \to V_\alpha$ and δ is the least cardinal $\delta \ge \alpha$ for which there exists an x with $|\text{TC}(x)| \le \delta$ such that there is no normal measure U on $P_\alpha(\delta)$ with $(j_U g)(\alpha) = x$. (Let λ_g denote this δ.) Since $M^\nu \subset M$, we have $M \vDash \varphi(f, \lambda_f)$, for all $f : \kappa \to V_\kappa$.

Let A be the set of all $\alpha < \kappa$ such that $\varphi(g, \lambda_g)$ holds for all $g : \alpha \to V_\alpha$. Clearly, $\kappa \in j(A)$.

Now we define $f : \kappa \to V_\kappa$ inductively as follows. If $\alpha \in A$, we let $f(\alpha) = x_\alpha$ where x_α witnesses $\varphi(f{\restriction}\alpha, \lambda_{f\restriction\alpha})$; otherwise, $f(\alpha) = \emptyset$.

Let $x = (jf)(\kappa)$. It follows from the construction of f that x witnesses $\varphi(f, \lambda_f)$ in M, and hence in V. Let $U = \{X \in P_\kappa(\lambda) : j``\lambda \in j(X)\}$; we shall reach a contradiction by showing that $(j_U f)(\kappa) = x$. Let $k : \text{Ult}_U \to M$ be the elementary embedding from (20.26) such that $j = k \circ j_U$. By (20.27), $k(x) = x$, and therefore

$$(j_U f)(\kappa) = k^{-1}((jf)(\kappa)) = k^{-1}(x) = x. \qquad \square$$

Beyond Supercompactness

Elementary embeddings can be used to define large cardinals that are stronger than supercompact.

Definition 20.22. A cardinal κ is *extendible* if for every $\alpha > \kappa$ there exist an ordinal β and an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ.

Lemma 20.23. *Let $\lambda \geq \kappa$ be a regular cardinal and let κ be λ-supercompact. Let $\alpha < \kappa$. If α is γ-supercompact for all $\gamma < \kappa$, then α is λ-supercompact.*

Proof. Let U be a normal measure on $P_\kappa(\lambda)$, and let us consider $j_U : V \to \text{Ult}_U$. Since $j(\alpha) = \alpha$, we have Ult $\vDash$ (α is γ-supercompact for all $\gamma < j(\kappa)$); in particular, Ult $\vDash$ α is λ-supercompact. Hence there is D such that Ult $\vDash$ D is a normal measure on $P_\alpha(\lambda)$. Now, $|P_\alpha(\lambda)| = \lambda$ and $\text{Ult}^\lambda \subset \text{Ult}$, and hence every subset of $P_\alpha(\lambda)$ is in Ult. It follows that D is a normal measure on $P_\alpha(\lambda)$. $\square$

Theorem 20.24.

(i) *If κ is extendible, then κ is supercompact.*

(ii) *If κ is extendible, then there is a normal measure D on κ such that $\{\alpha < \kappa : \alpha \text{ is supercompact}\} \in D$.*

Proof. (i) Let $\alpha > \kappa$ be a limit cardinal with the property that if $V_\alpha \vDash$ (κ is λ-supercompact for all λ), then κ is supercompact. (Such an α exists by the Reflection Principle.) Thus it suffices to show that κ is λ-supercompact for all regular $\lambda < \alpha$.

Let $j : V_\alpha \to V_\beta$ be such that κ is the critical point. Consider the sequence $\kappa_0 = \kappa$, $\kappa_1 = j(\kappa)$, ..., $\kappa_{n+1} = j(\kappa_n)$, ..., as long as $j(\kappa_n)$ is defined. First we note that by Exercise 17.8 either there is some n such that $\kappa_n < \alpha \leq j(\kappa_n)$ or $\alpha = \lim_{n\to\infty} \kappa_n$. Therefore, it is sufficient to prove, by induction on n, that κ is λ-supercompact for each regular $\lambda < \kappa_n$ (if $\lambda < \alpha$).

Clearly, κ is λ-supercompact for each $\lambda < \kappa_1$. Thus let $n \geq 1$ and let us assume that κ is λ-supercompact for all $\lambda < \kappa_n$. Applying j, we get: $V_\beta \vDash$ ($j(\kappa)$ is λ-supercompact for all regular $\lambda < \kappa_{n+1}$). Now we also have

$V_\beta \vDash (\kappa$ is γ-supercompact for all $\gamma < j(\kappa))$ and we can apply Lemma 20.23 (in V_β) to conclude that $V_\beta \vDash (\kappa$ is λ-supercompact for all regular $\lambda < \kappa_{n+1})$. This completes the induction step.

(ii) Let α be some limit ordinal greater than κ and let $j : V_\alpha \to V_\beta$ be such that κ is the critical point. Let $D = \{X \subset \kappa : \kappa \in j(X)\}$. By (i), κ is supercompact, and so $V_\beta \vDash (\kappa$ is γ-supercompact for all $\gamma < j(\kappa))$. Hence $A \in D$, where $A = \{\alpha < \kappa : \alpha$ is γ-supercompact for all $\gamma < \kappa\}$. By Lemma 20.23, every $\alpha \in A$ is supercompact. □

Let us consider now the following axiom schema called *Vopěnka's Principle* (VP):

(20.29) Let C be a proper class of models of the same language. Then there exist two members $\mathfrak{A}$, $\mathfrak{B}$ of the class C such that $\mathfrak{A}$ can be elementarily embedded in $\mathfrak{B}$.

Lemma 20.25. *If Vopěnka's Principle holds, then there exists an extendible cardinal.*

Proof. Let A be the class of all limit ordinals α such that $\operatorname{cf}\alpha = \omega$ and that for every $\kappa < \alpha$, if $V_\alpha \vDash (\kappa$ is extendible), then κ is extendible; and for $\kappa < \gamma < \alpha$, if there is an elementary embedding $j : V_\gamma \to V_\delta$ with critical point κ, then $V_\alpha \vDash$ (there is an elementary embedding). Using the Reflection Principle, we see that A is a proper class. Let C consist of the models $(V_{\alpha+1}, \in)$, for $\alpha \in A$.

By Vopěnka's Principle, there exist $\alpha, \beta \in A$ and an elementary embedding $j : V_{\alpha+1} \to V_{\beta+1}$. Since $j(\alpha) = \beta$, j moves some ordinal; its critical point is measurable and so it is not α (which has cofinality ω). Let κ be the critical point.

Now $V_\alpha \vDash (\kappa$ is extendible) because for every $\gamma < \alpha$, $j{\restriction}V_\gamma$ reflects to a witness to extendibility. By definition of A, κ is extendible. □

A similar argument shows that Vopěnka's Principle implies existence of arbitrarily large extendible cardinals.

Definition 20.26. A cardinal κ is *huge* if there exists an elementary embedding $j : V \to M$ with critical point κ such that $M^{j(\kappa)} \subset M$.

"Huge" is expressible in ZF: see Exercise 20.11.

Lemma 20.27. *If κ is a huge cardinal, then Vopěnka's Principle is consistent*: $(V_\kappa, \in)$ *is a model of* VP.

Proof. We shall show that if C is a set of models and $\operatorname{rank}(C) = \kappa$, then there exist two members $\mathfrak{A}, \mathfrak{B} \in C$ and an elementary embedding $h : \mathfrak{A} \to \mathfrak{B}$.

Let $j : V \to M$ be such that κ is the least cardinal moved and that $M^{j(\kappa)} \subset M$. Since $\operatorname{rank}(C) = \kappa$, there exists an $\mathfrak{A}_0 \in j(C)$ such that $\mathfrak{A}_0 \notin C$. It follows that $j(\mathfrak{A}_0) \neq \mathfrak{A}_0$.

Let $e_0 = j{\restriction}\mathfrak{A}_0$; it is easy to see that e_0 is an elementary embedding of $\mathfrak{A}_0$ into $j(\mathfrak{A}_0)$, and since $|\mathfrak{A}_0| < j(\kappa)$, we have $e_0 \in M$. Hence

$$M \vDash \text{there exists an } \mathfrak{A} \in j(C),\ \mathfrak{A} \neq j(\mathfrak{A}_0), \text{ and there exists an elementary } e : \mathfrak{A} \to j(\mathfrak{A}_0);$$

and so there exists some $\mathfrak{A} \in C$, $\mathfrak{A} \neq \mathfrak{A}_0$, and there exists an elementary $e : \mathfrak{A} \to \mathfrak{A}_0$. Let $\mathfrak{A}$, e be such; clearly,

$$M \vDash e \text{ is an elementary embedding of } \mathfrak{A} \text{ into } \mathfrak{A}_0,$$

and because $\operatorname{rank}(\mathfrak{A}) < \kappa$, we have $\mathfrak{A} = j(\mathfrak{A})$, and hence $\mathfrak{A} \in j(C)$, and so

$$M \vDash \text{there exist distinct } \mathfrak{A}, \mathfrak{B} \in j(C), \text{ and there exists an elementary } h : \mathfrak{A} \to \mathfrak{B}.$$

It follows that there exist distinct $\mathfrak{A}, \mathfrak{B} \in C$, and an elementary embedding $h : \mathfrak{A} \to \mathfrak{B}$. □

While the least huge cardinal is greater than the least measurable cardinal (see Exercise 20.13), it is smaller than the least supercompact cardinal (if both exist) even though the consistency of "there exists a huge cardinal" is stronger than the consistency of "there exists a supercompact cardinal." See Exercise 20.12.

Finally, consider the following axiom:

(20.30) There exists a nontrivial elementary embedding $j : V_\lambda \to V_\lambda$ where λ is a limit ordinal.

Let κ be the critical point of an elementary embedding $j : V_\lambda \to V_\lambda$. The necessarily $\lambda \geq \kappa_n$ for each n (where κ_n are as in Theorem 17.7), and it follows from Exercise 17.8 that $\lambda = \lim_{n\to\infty} \kappa_n$. It is easily seen that κ is huge (by Exercise 20.11), in fact n-huge for all n; see Exercise 20.15.

In view of Kunen's Theorem 17.7, axiom (20.30) (and its variants) is the strongest possible large cardinal axiom.

Extenders and Strong Cardinals

In this section we show how elementary embeddings can be analyzed using direct limits of ultrapowers. An elementary embedding can be approximated by a system of measures called *extenders*. The theory of extenders plays a crucial role in the inner model theory. While this theory is too weak to describe supercompactness, it is strong enough to describe a weak version of it that is considerably stronger than measurability.

Definition 20.28. A cardinal κ is a *strong cardinal* if for every set x there exists an elementary embedding $j : V \to M$ with critical point κ such that $x \in M$.

Clearly, every supercompact cardinal is strong. Strong cardinals have more consistency strength than measurable cardinals, and allow some of the techniques associated with supercompactness; see Exercises 20.16–20.19.

It follows from the theory of extenders below that "strongness" is expressible in ZF. As with supercompactness, one can also define local versions of strongness: κ is λ-*strong*, where $\lambda \geq \kappa$, if there exists some $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $V_\lambda \subset M$. A cardinal κ is strong if and only if it is λ-strong for all $\lambda \geq \kappa$.

Let $j : V \to M$ be an elementary embedding with critical point κ and let $\kappa \leq \lambda \leq j(\kappa)$. We shall define the (κ, λ)-*extender* derived from j.

For every finite subset $a \subset \lambda$, let E_a be the measure on $[\kappa]^{<\omega}$ defined as follows:

$$X \in E_a \quad \text{if and only if} \quad a \in j(X); \tag{20.31}$$

note that E_a concentrates on $[\kappa]^{|a|}$. The (κ, λ)-*extender* derived from j is the collection

$$E = \{E_a : a \in [\lambda]^{<\omega}\}; \tag{20.32}$$

κ is the *critical point* of E and λ is the *length* of the extender.

Let $a \in [\lambda]^{<\omega}$. The measure E_a on $[\kappa]^{<\omega}$ is κ-complete; let Ult_{E_a} denote the ultrapower of V by E_a and let $j_a : V \to \mathrm{Ult}_{E_a}$ be the corresponding elementary embedding. If for each equivalence class $[f]$ of a function f on $[\kappa]^{<\omega}$ we let

$$k_a([f]) = j(f)(a), \tag{20.33}$$

then k_a is an elementary embedding $k_a : \mathrm{Ult}_{E_a} \to M$ and $k_a \circ j_a = j$.

The measures E_a, $a \in [\lambda]^{<\omega}$, are *coherent*, in the following sense: Let $a \subset b$, where $b = \{\alpha_1, \ldots, \alpha_n\}$ with $\alpha_1 < \ldots < \alpha_n$. Then $\pi_{b,a} : [\lambda]^{|b|} \to [\lambda]^{|a|}$ is defined by

$$\pi_{b,a}(\{\xi_1, \ldots, \xi_n\}) = \{\xi_{i_1}, \ldots, \xi_{i_m}\}, \qquad (\xi_1 < \ldots < \xi_n) \tag{20.34}$$

where $a = \{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$, and

$$X \in E_a \quad \text{if and only if} \quad \{s : \pi_{b,a}(s) \in X\} \in E_b. \tag{20.35}$$

(Compare with Lemma 19.12.)

It follows that $i_{a,b} : \mathrm{Ult}_{E_a} \to \mathrm{Ult}_{E_b}$ defined by

$$i_{a,b}([f]_{E_a}) = [f \circ \pi_{b,a}]_{E_b}$$

is an elementary embedding, and

$$\{\mathrm{Ult}_{E_a}, i_{a,b} : a \subset b \in [\lambda]^{<\omega}\} \tag{20.36}$$

is a directed system. The direct limit Ult_E of (20.36) is well-founded: Note that the embeddings k_a have a direct limit $k : \mathrm{Ult}_E \to M$ such that $k \circ j_E = j$ where j_E is the elementary embedding $j_E : V \to \mathrm{Ult}_E$.

There is another description of the direct limit Ult_E: The elements of Ult_E are equivalence classes $[a,f]_E$ where $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^{|a|} \to V$. Here (a,f) and (b,g) are equivalent if $\{s \in [\kappa]^{|a\cup b|} : \tilde{f}(s) = \tilde{g}(s)\} \in E_{a\cup b}$, where $\tilde{f} = f \circ \pi_{a\cup b,a}$ and $\tilde{g} = g \circ \pi_{a\cup b,b}$. The embedding $j_E : V \to \mathrm{Ult}_E$ is defined by $j_E(x) = [\emptyset, c_x]$ where c_x is the constant function with value x. The embedding $k : \mathrm{Ult}_E \to M$ is defined by

$$k([a,f]) = j(f)(a). \tag{20.37}$$

Now $k \circ j_E = j$ follows.

Lemma 20.29.

(i) *$k(\alpha) = \alpha$ for all $\alpha < \lambda$.*
(ii) *j_E has critical point κ and $j_E(\kappa) \geq \lambda$.*
(iii) $\mathrm{Ult}_E = \{j_E(f)(a) : a \in [\lambda]^{<\omega},\ f : [\kappa]^{<\omega} \to V\}$.

Proof. For each $a \in [\lambda]^{<\omega}$, let $j_{a,\infty} : \mathrm{Ult}_{E_a} \to \mathrm{Ult}_E$ be the direct limit embedding such that $j_{a,\infty} \circ j_a = j_E$; then $k \circ j_{a,\infty} = k_a$. If $x \in \mathrm{Ult}_E$ then $x = j_{a,\infty}([f])$ for some $[f] \in \mathrm{Ult}_{E_a}$, and

$$k(x) = k(j_{a,\infty}([f])) = k_a([f]) = j(f)(a)$$

(see also (20.37)). Hence

$$k``\mathrm{Ult}_E = \{j(f)(a) : a \in [\lambda]^{<\omega},\ f \in [\kappa]^{<\omega} \to V\}. \tag{20.38}$$

(i) By letting f be the identity function, we get from (20.38) that $a \in k``\mathrm{Ult}_E$, for each $a \in [\lambda]^{<\omega}$. Hence $\lambda \subset k``\mathrm{Ult}_E$, and therefore $k(\alpha) = \alpha$ for all $\alpha < \lambda$.

(ii) This follows from (i), because $j = k \circ j_E$.

(iii) Since $k(a) = a$ for every $a \in [\lambda]^{<\omega}$, it follows from (20.38) that for every $x \in \mathrm{Ult}_E$, $k(x) = j(f)(a) = k(j_E(f))(k(a)) = k(j_E(f)(a))$ for some a and f, and hence $x = j_E(f)(a)$. □

Hence j_E is an elementary embedding, $j_E : V \to \mathrm{Ult}_E$, with critical point κ. Since $j = k \circ j_E$ and since $k(a) = a$ for all $a \in [\lambda]^{<\omega}$, it follows that for all $X \in [\kappa]^{|a|}$, $a \in j_E(X)$ if and only if $a \in j(X)$. Hence E is the extender derived from j_E.

Extenders can be defined directly, without reference to an embedding j. The following, somewhat technical, properties guarantee that the (κ,λ)-extender is derived from the direct limit embedding j_E: Let $\kappa \leq \lambda$, and let $E = \{E_a : a \in [\lambda]^{<\omega}\}$. E is a (κ, λ)-extender if

(20.39) (i) Each E_a is a κ-complete measure on $[\kappa]^{|a|}$, and
(a) at least one E_a is not κ^+-complete,
(b) for each $\alpha \in \kappa$, at least one E_a contains the set $\{s \in [\kappa]^{|a|} : \alpha \in s\}$.
(ii) (Coherence) The E_a's are coherent, i.e., satisfy (20.35).
(iii) (Normality) If $\{s \in [\kappa]^{|a|} : f(s) \in \max s\} \in E_a$, then for some $b \supset a$, $\{t \in [\kappa]^{|b|} : (f \circ \pi_{b,a})(t) \in t\} \in E_b$.
(iv) The limit ultrapower Ult_E is well-founded.

We leave out the verification that the extender derived from some j satisfies (20.39), and that the properties (20.39) suffice to prove (ii) and (iii) of Lemma 20.29, and that E is derived from j_E.

An immediate consequence of the above technique is the following characterization of strong cardinals:

Lemma 20.30. *A cardinal κ is strong if and only if for every $\lambda \geq \kappa$ there is a $(\kappa, |V_\lambda|^+)$-extender E such that $V_\lambda \subset \text{Ult}_E$ and $\lambda < j_E(\kappa)$.* □

Hence "strongness" is expressible in ZFC.

We conclude by introducing a large cardinal property that was isolated by Woodin and that has played a central role in the study of determinacy and inner models:

Definition 20.31. A cardinal δ is a *Woodin cardinal* if for all $A \subset V_\delta$ there are arbitrarily large $\kappa < \delta$ such that for all $\lambda < \delta$ there exists an elementary embedding $j : V \to M$ with critical point κ, such that $j(\kappa) > \lambda$, $V_\lambda \subset M$, and $A \cap V_\lambda = j(A) \cap V_\lambda$.

Being a Woodin cardinal is expressible in ZFC, in terms of extenders. Every supercompact cardinal is Woodin, and below a Woodin cardinal δ, there are δ cardinals that are λ-strong for every $\lambda < \delta$. While Woodin cardinals are inaccessible (and Mahlo), the least Woodin cardinal is not weakly compact, as δ being Woodin is a Π^1_1 property of $(V_\delta, \in)$.

Exercises

20.1. If κ is strongly compact then $\mathcal{L}_{\kappa,\kappa}$ satisfies the Compactness Theorem.

[Verify Łoś's Theorem]

20.2. If κ is strongly compact, $\lambda \geq \kappa$, and $A \subset \lambda$, then λ^+ is an ineffable cardinal in $L[A]$.

[Let U be as in Theorem 20.3, let $M = \text{Ult}_U(L[A])$, $N = \text{Ult}^-_U(L[A])$, let $j : L[A] \to M$, $i : L[A] \to N$, and let $k : N \to M$ be as there. Again, $M = N$, and $i(\lambda^+)$ is the least ordinal moved. By Lemma 17.32, N thinks that $i(\lambda^+)$ is ineffable; hence λ^+ is ineffable in $L[A]$.]

20.3. The following are equivalent, for $\kappa \le \lambda$:

(i) There is a fine measure on $P_\kappa(\lambda)$.
(ii) For any set S, every κ-complete filter on S generated by at most λ sets can be extended to a κ-complete ultrafilter on S.

20.4. If U is a normal measure on $P_\kappa(\lambda)$, then every closed unbounded subset of $P_\kappa(\lambda)$ is in U.

[If $C \subset P_\kappa(\lambda)$ is closed unbounded, then $D = \{j(x) : x \in C\}$ is a directed subset of $j(C)$ and $|D| = \lambda^{<\kappa} < j(\kappa)$. Hence $\bigcup D \in j(C)$, and since $\bigcup D = \{j(\gamma) : \gamma < \lambda\}$, we have $C \in U$.]

20.5. Let $\lambda \ge \kappa$ and let U be a normal measure on $P_\kappa(\lambda)$. The ultraproduct $\text{Ult}_U\{(V_{\lambda_x}, \in) : x \in P_\kappa(\lambda)\}$ is isomorphic to $(V_\lambda, \in)$.

20.6. If κ is inaccessible then $V_\kappa \prec_{\Sigma_1} V$.

20.7. If κ is supercompact then $V_\kappa \prec_{\Sigma_2} V$.

[Let $x \in V_\kappa$ such that $\exists y\, \varphi(x, y)$ where φ is Π_1. Let $j : V \to M$ be such that $y \in M \cap V_{j(\kappa)}$. In M, $V_{j(\kappa)} \vDash \exists y\, \varphi(x, y)$, hence $V_\kappa \vDash \exists y\, \varphi(x, y)$.]

Let κ be supercompact and let $\lambda \ge \kappa$ be a cardinal. A normal measure D on $P_\kappa(\lambda)$ is *strongly normal* if there exists $X \in D$ such that for every function f on X, if for each nonempty $x \in X$, $f(x)$ is in X, $f(x) \subset x$ and $f(x) \ne x$, then f is constant on some $Y \in D$.

20.8. The following are equivalent:

(i) D is strongly normal.
(ii) There is $X \in D$ such that if $\{Z_x : x \in X\} \subset D$, then $\triangle_{x \in X} Z_x \in D$ where $\triangle_{x \in X} Z_x = \{y : y \in Z_x \text{ for each } x \subset y \text{ such that } x \ne y \text{ and } x \in X\}$.
(iii) D has this partition property: If $F : [P_\kappa(\lambda)]^2 \to \{0, 1\}$ is a partition, then there is $X \in D$ such that F is constant on $\{\{x, y\} \in [X]^2 : x \subsetneqq y \text{ or } y \subsetneqq x\}$.
(iv) There is $X \in D$ such that if $x, y \in X$, $x \ne y$ and $x \subset y$, then $\lambda_x < \kappa_y$.

[(i) $\to$ (ii): Let $X \in D$ be a witness to strong normality. Prove by contradiction that D is closed under $\triangle_{x \in X} Z_x$.

(ii) $\to$ (iii): Let $F : [P_\kappa(\lambda)]^2 \to \{0, 1\}$; for each x, let $F_x : \hat{x} \to \{0, 1\}$ be $F_x(y) = F(x, y)$. For each x there is $Z_x \subset \hat{x}$, $Z_x \in D$, such that F_x is constant on Z_x. Let $X \in D$ be as in (ii) and such that the constant value of F_x is the same for all $x \in X$. Then $X \cap \triangle_{x \in X} Z_x$ is homogeneous for F in the sense of (iii).

(iii) $\to$ (iv): Note that if $X \in D$, then there exist $x, y \in X$ such that $x \subsetneqq y$ and $\lambda_x < \kappa_y$.

(iv) $\to$ (i): Let $X \in D$ be as in (iv) and let $f : X \to X$ be such that $f(x) \subset x$ and $f(x) \ne x$ for all x. In Ult_D, if $x \in j_D(X)$ and $x \subset j``\lambda$, then $|x| < j``\lambda \cap \kappa = \kappa$ and hence $x = j(y)$ for some $y \in P_\kappa(\lambda)$. Hence $(jf)(j``\lambda) = j(y)$ for some y and so $f(x) = y$ for almost all x.]

It has been proved that if κ is supercompact, then every $P_\kappa(\lambda)$ has a strongly normal measure; however, not every normal measure is necessarily strongly normal:

20.9. If $\lambda > \kappa$ is measurable, then there is a normal measure U on $P_\kappa(\lambda)$ that is not strongly normal.

[Let $j : V \to M$ be elementary, κ least moved, $j(\kappa) > \lambda$, and $M^\lambda \subset M$. Let D be a normal measure on λ. Let us define a normal measure U on $P_\kappa(\lambda)$ as follows: $X \in U$ if and only if $\{\alpha < \lambda : j``\alpha \in j(X)\} \in D$. If $X \in U$, then there exist $\alpha < \beta$ such that $j``\alpha$ and $j``\beta$ are in $j(X)$; hence $M \vDash \exists x, y \in j(X)$ such that x is an initial segment of y. Thus $\exists x, y \in X$ such that x is an initial segment of y, and so $\lambda_x \ge \kappa_y$.]

20.10. If κ is extendible then $V_\kappa \prec_{\Sigma_3} V$.

[Use Exercise 20.7, and show that there are arbitrarily large inaccessible $\lambda > \kappa$ such that $V_\kappa \prec V_\lambda$.]

20.11. A cardinal κ is huge, with $j : V \to M$ and $j(\kappa) = \lambda$ if and only if there is a normal κ-complete ultrafilter U on $\{X \subset \lambda : \text{order-type}(X) = \kappa\}$.

[$X \in U$ if and only if $j``\lambda \in j(X)$.]

20.12. Let κ be the least huge cardinal and let μ be the least supercompact cardinal. Then $\kappa < \mu$.

[If $\kappa = \mu$ then by 20.27, 20.25, 20.24, and 20.23 we get: $V_\mu \vDash \text{VP}$, $V_\mu \vDash (\exists$ supercompact $\alpha)$, there is a supercompact $\alpha < \mu$, a contradiction. If $\kappa < \mu$, let $j : V \to M$ with $\lambda = j(\kappa)$ and $M^\lambda \subset M$. Since μ is supercompact, let $i : V \to N$ be such that $i(\mu) > \lambda$ and $V_{\lambda+2} \subset N$. If U is a normal measure witnessing the hugeness of κ, then $U \in N$, and hence $N \vDash (\exists$ huge cardinal below $i(\mu))$. Thus there exists a huge cardinal below μ, a contradiction.]

20.13. The least huge cardinal is greater than the least measurable cardinal.

[Show that $M \vDash \kappa$ is measurable; hence there exists a measurable cardinal less than κ.]

A cardinal κ is *n-huge* if there exists an elementary $j : V \to M$ with critical point κ such that $M^{j^n(\kappa)} \subset M$.

20.14. If κ is $(n+1)$-huge then there is a normal measure D on κ such that $\{\alpha < \kappa : \alpha \text{ is } n\text{-huge}\} \in D$.

20.15. If there exists an elementary $j : V_\lambda \to V_\lambda$ with critical point κ then κ is n-huge for every n.

20.16. If there is a strong cardinal, then $V \neq L[A]$ for any set A.

20.17. If κ is strong then $o(\kappa) = (2^\kappa)^+$.

[As in Corollary 20.20(iii).]

20.18. If κ is strong then $V_\kappa \prec_{\Sigma_2} V$.

[As in Exercise 20.7.]

20.19. If κ is strong, then there exists a function $g : \kappa \to V_\kappa$ such that for every x and every $\lambda \geq \kappa$ such that $\lambda \geq |\,\text{TC}(x)|$ there exists a (κ, λ)-extender E such that $j_E(g)(\kappa) = x$.

20.20. A (κ, λ)-extender $\{E_a : a \in [\lambda]^{<\omega}\}$ has well-founded limit ultrapower if and only if for every $\langle a_m : m \in \omega\rangle$ and every sequence $\langle X_m : m \in \omega\rangle$ such that $X_m \in E_{a_m}$, there exists a function $h : \bigcup_{m\in\omega} a_m \to \kappa$ such that $h``a_m \in X_m$ for all m.

Historical Notes

Strongly compact cardinals were introduced by Keisler and Tarski in [1963/64]; supercompact cardinals were defined by Reinhardt and Solovay. Theorem 20.3 is due to Vopěnka and Hrbáček [1966]; Theorem 20.4 is due to Kunen [1971b].

Solovay discovered that the Singular Cardinal Hypothesis holds above a compact cardinal (Theorem 20.8); see [1974].

Menas and Magidor obtained several results on the relative strength of compact and supercompact cardinals. Menas in [1974/75] showed that it is consistent (relative to existence of compact cardinals) that there is a compact cardinal that is not supercompact. Magidor in [1976] improved Menas' result by showing that it is possible that the least measurable cardinal is strongly compact (while by Lemma 20.16 it is not supercompact) and also showed that it is consistent (relative to supercompact cardinals) that there exists just one compact cardinal and is supercompact.

Kunen's proof of Theorem 20.4 uses a lemma of Ketonen (Lemma 20.5); Lemmas 20.9 and 20.10 which Solovay used in his proof of Theorem 20.8, are also due to Ketonen; see [1972/73].

Most results on supercompact cardinals (e.g., Lemmas 20.16 and 20.19) are due to Solovay; see Solovay, Reinhardt, and Kanamori [1978]; Magidor's paper [1971a] gives a number of normal measures on $P_\kappa(\lambda)$ (Corollary 20.20(ii)). The example of a strongly compact nonsupercompact cardinal (Lemma 20.17 and Corollary 20.18) is due to Menas [1974/75]. Theorem 20.21 is due to Laver [1978].

Extendible cardinals were introduced by Reinhardt; he proved that extendible cardinals are supercompact; see [1974]. The present proof of Theorem 20.24, as well as Lemmas 20.23 and 20.25 are due to Magidor [1971b].

Lemma 20.27: Powell [1972].

A good source for further results on very large cardinals is the paper [1978] of Kanamori and Magidor, as well as Kanamori's book [1994].

Strong cardinals were used by Mitchell [1979a] to develop a theory of inner models for weak versions of supercompactness, and further studied by Baldwin [1986]. Extenders were introduced by Jensen and Dodd; see Dodd [1982].

Woodin cardinals were introduced by Woodin in 1984. They were used, among others, in the proof of projective determinacy by Martin and Steel [1989].

Exercises 20.7, 20.10, 20.11, 20.14, and 20.15: Solovay, Reinhardt and Kanamori [1978].

Exercise 20.9: Solovay.

Exercise 20.12: Morgenstern [1977].

Exercise 20.19: Gitik and Shelah [1989].