21. Large Cardinals and Forcing

Many forcing techniques have been developed specifically for use with large cardinals. Firstly, when investigating the effect of large cardinals on cardinal arithmetic, it is desirable to establish the relative consistency of statements about the continuum function with the existence of various large cardinals. The main technical question here is whether large cardinal properties are preserved under various forcing extensions. Secondly, it follows from the Covering Theorem that an attempt to violate the Singular Cardinal Hypothesis, or even to change cofinalities, necessarily involves large cardinals. Indeed, forcing techniques have been developed for changing cofinalities and for violating SCH that use large cardinals. And thirdly, as the large cardinal hierarchy serves as a gauge of consistency strength, forcing that uses large cardinals provides an upper bound for the consistency strength of the problems that the forcing proves consistent.

Mild Extensions

We begin with an early discovery that "mild" forcing extensions do not effect large cardinal properties, i.e., whether κ is a large cardinal is not changed by forcing of size less than κ.

Theorem 21.1 (Lévy-Solovay). *Let κ be a measurable cardinal in the ground model. Let $(P, <)$ be a notion of forcing such that $|P| < \kappa$. Then κ is measurable in the generic extension.*

Proof. We give a proof using elementary embeddings since similar arguments will be used in subsequent constructions; for a direct proof, see Exercise 21.1. Let $B = B(P)$; since $|B| < \kappa$, we may assume $B \in V_\kappa$. We can also assume that P is a dense subset of B. Let G be a generic ultrafilter on B; let us work in $V[G]$.

Since κ is measurable in V, there is an elementary embedding $j : V \to M$ with critical point κ, and M transitive. We shall extend j to an elementary embedding (denoted also j) of $V[G]$ into $M[G]$, thus showing that κ is measurable in $V[G]$.

Since $B \in V_\kappa$, we have $j(B) = B$, and B is a complete Boolean algebra in M. Since G is generic over V, G is also generic over M. Note that the interpretation of Boolean names in M^B by G is the same whether computed in V or in M. We define $j(x)$ for $x \in V[G]$ as follows: Let $\dot{x} \in V^B$ be a name for x, $x = \dot{x}^G$. Let

$$j(x) = (j(\dot{x}))^G. \tag{21.1}$$

Since $\dot{x} \in V^B$, we have $j(\dot{x}) \in M^B$ and so $(j(\dot{x}))^G \in M[G]$. However, we have to show that the definition (21.1) does not depend on which name for x we choose.

Let $\dot{y}$ be another B-valued name and let $p \in G$ be such that

$$p \Vdash \dot{x} = \dot{y}. \tag{21.2}$$

When we apply j to (21.2), we have (in M)

$$j(p) \Vdash j(\dot{x}) = j(\dot{y}).$$

But $j(p) = p \in G$ and therefore

$$(j(\dot{x}))^G = (j(\dot{y}))^G.$$

Finally we show that $j : V[G] \to M[G]$ is elementary. Let φ be a formula such that

$$V[G] \vDash \varphi(x, \ldots).$$

Let $\dot{x}, \ldots$ be such that $(\dot{x})^G = x, \ldots$. There is some $p \in G$ such that

$$p \Vdash \varphi(\dot{x}, \ldots). \tag{21.3}$$

Applying j to (21.3), we get (in M)

$$p \vDash \varphi(j(\dot{x}), \ldots)$$

(because $j(p) = p$). Hence

$$M[G] \vDash \varphi(j(\dot{x}), \ldots)$$

and since φ was arbitrary, j is elementary. □

It turns out that practically every large cardinal property is unchanged by mild extension:

Theorem 21.2. *Let κ be an infinite cardinal, and let $(P, <)$ be a notion of forcing such that $|P| < \kappa$. Let G be a V-generic filter on P. Then κ is inaccessible (Mahlo, weakly compact, Ramsey, measurable, strongly compact, supercompact, huge, strong, Woodin) in V if and only if it is inaccessible (Mahlo, weakly compact, Ramsey, measurable, strongly compact, supercompact, huge, strong, Woodin) in $V[G]$.*

Proof. If κ is inaccessible in V, then firstly κ is regular in $V[G]$ because all cardinals and cofinalities above $|P|$ are preserved. Secondly, if $\alpha < \kappa$, then $(2^\alpha)^{V[G]} \le |B(P)|^\alpha < \kappa$ and hence κ is inaccessible in $V[G]$. Conversely, if κ is inaccessible in $V[G]$, then κ is inaccessible in V.

If κ is Mahlo in V, then firstly κ is inaccessible in $V[G]$. Secondly, every $\alpha > |P|$ is a regular cardinal in $V[G]$ if and only if it is a regular cardinal in V and so the set $S = \{\alpha < \kappa : |P| < \alpha$ and α is regular in $V[G]\}$ is stationary in V. It is easy to see that every closed unbounded set $C \subset \kappa$ in $V[G]$ has a closed unbounded subset D in V (Exercise 21.2). Thus, S is also stationary in $V[G]$ and hence κ is Mahlo in $V[G]$. Conversely, if κ is Mahlo in $V[G]$, then κ is Mahlo in V because $V \subset V[G]$.

If κ is weakly compact in V, let $F : [\kappa]^2 \to \{0,1\}$ be a partition of $[\kappa]^2$ in $V[G]$. Let $\dot{F} \in V^B$ be its name such that $\|\dot{F} : [\kappa]^2 \to \{0,1\}\| = 1$. For all $\alpha \ne \beta \in \kappa$, let $G(\alpha,\beta) = \|\dot{F}(\alpha,\beta) = 0\|$; G is (in V) a partition of $[\kappa]^2$ into $|B(P)|$ pieces. Since $|B(P)| < \kappa$, there is an $H \subset \kappa$ of size κ homogeneous for G, and it is easy to see that H is homogeneous for F.

Conversely, if κ is weakly compact in $V[G]$, let $F : [\kappa]^2 \to \{0,1\}$ be a partition of $[\kappa]^2$ in V. There is, in $V[G]$, a set $K \subset \kappa$ of size κ, homogeneous for F. As in Exercise 21.2, K has an unbounded subset $H \in V$; hence F has in V a homogeneous set of size κ.

The argument for Ramsey cardinals is exactly the same as for weakly compact cardinals.

If κ is measurable in V, then it is measurable in $V[G]$ by Theorem 21.1. Conversely, if κ is measurable in $V[G]$, let $U \in V[G]$ be a κ-complete nonprincipal ultrafilter on κ, let J be the dual prime ideal, and let $\dot{J} \in V^B$ be its name. (Without loss of generality we assume that $\|\dot{J}$ is a κ-complete nonprincipal prime ideal$\| = 1$.) Let $I = \{X \subset \kappa : \|X \in \dot{J}\| = 1\}$. It is easy to verify that I is a κ-complete ideal containing all singletons. We claim that I is $|P|^+$-saturated: If $p \Vdash \check{X} \notin \dot{J}$ and $p \Vdash \check{Y} \notin \dot{J}$, then $p \Vdash \check{X} \cap \check{Y} \notin \dot{J}$ (because $\dot{J}$ is prime). Hence if X and Y are such that $X \notin I$, $Y \notin I$, and $X \cap Y \in I$, then $\|\check{X} \notin \dot{J}\| \cdot \|\check{Y} \notin \dot{J}\| = 0$, and it follows that I is $|P|^+$-saturated. However, since I is ν-saturated for some $\nu < \kappa$ and κ is inaccessible, κ is measurable, by Exercise 21.3.

If κ is strongly compact, let $\lambda \ge \kappa$ and let us show that in $V[G]$, there is a fine measure on $P_\kappa(\lambda)$. Let U be a fine measure on $P_\kappa(\lambda)$ in V, and let $j = j_U$ be the canonical elementary embedding $j_U : V \to \text{Ult}_U(V)$. The standard argument shows that $X \in U$ if and only if $H \in j(X)$, where H is the set in $\text{Ult}_U(V)$ represented by the function $d(Z) = Z$ on $P_\kappa(\alpha)$; also, $H \supset j``\lambda$ (and is equal to it if U is normal). Similarly, as in the proof of Theorem 21.1 we extend j to $V[G]$ as follows:

$$j(x) = (j(\dot{x}))^G$$

where $\dot{x}$ is a name for x; the definition is legitimate because we assume, without loss of generality, that $P \in V_\kappa$ and hence $j(p) = p$ for all $p \in P$, and

$j(P) = P$. Now we define, in $V[G]$, an ultrafilter W on $P_\kappa(\lambda)$ as follows:

$$X \in W \quad \text{if and only if} \quad H \in j(X).$$

It is routine to verify that W is a fine measure on $P_\kappa(\lambda)$; for instance, if $Z_0 \in P_\kappa(\lambda)$, then $\{Z \in P_\kappa(\lambda) : Z \supset Z_0\} \in W$ because $j(Z_0) = \{j(\alpha) : \alpha \in Z_0\} \subset H$.

Conversely, if κ is strongly compact in $V[G]$, let S be a set in V and let F be a κ-complete filter on S (in V); let us show that there is a κ-complete ultrafilter in V extending F. Every set $X \subset F$ of size $< \kappa$ in $V[G]$ is included in some $Y \subset F$ of size $< \kappa$ such that $Y \in V$ (this is because $|P| < \kappa$). Hence F generates a κ-complete filter in $V[G]$ and that in turn is included in a κ-complete ultrafilter on U. Let J be the dual prime ideal. As in the proof for measurable cardinals above, the ideal $I = \{X \subset S : \|X \in \dot{J}\| = 1\}$ is κ-complete and $|P|^+$-saturated, and clearly $X \in F$ implies $S - X \in I$. Since κ is inaccessible and I is ν-saturated for some $\nu < \kappa$, I has an atom A. If $X \in F$, then $X \cap A \notin I$ and so $\{X \subset S : X \cap A \notin I\}$ is a κ-complete ultrafilter extending F.

The proofs for the remaining large cardinal properties are similar. □

There are numerous examples when κ ceases to be large when we use a notion of forcing of size $\geq \kappa$ (a good example is the Lévy collapse). The example in Exercise 21.4 is quite interesting since inaccessibility of κ is preserved by any notion of forcing that is α-distributive for all $\alpha < \kappa$.

Kunen-Paris Forcing

It is an immediate consequence of the Lévy-Solovay Theorem that if κ is a measurable cardinal, $\lambda < \kappa$, and F is a function on regular cardinals below λ such that (i) $F(\alpha) \leq F(\beta)$ if $\alpha \leq \beta$, (ii) $\mathrm{cf}(F(\alpha)) > \alpha$, and (iii) $F(\alpha) < \kappa$ for all α in its domain, then there is a model in which κ is measurable and $2^\alpha = F(\alpha)$ for all $\alpha \in \mathrm{dom}(F)$.

One can also prescribe the values of the continuum function on regular cardinals greater than the measurable cardinal; this can be done by a κ-closed forcing; see Exercise 21.5.

The proof of the next theorem uses a method that preserves measurability of κ while adding subsets to an unbounded set of cardinals below κ. It is vital however that the set $A \subset \kappa$ has a normal measure 0.

Theorem 21.3 (Kunen-Paris). *Assume* GCH *and let κ be a measurable cardinal. Let D be a normal measure on κ and let A be a set of regular cardinals below κ such that $A \notin D$; let F be a function on A such that $F(\alpha) < \kappa$ for all $\alpha \in A$, and:*

(i) $\mathrm{cf}\, F(\alpha) > \alpha$;

(ii) $F(\alpha_1) \le F(\alpha_2)$ *whenever* $\alpha_1 \le \alpha_2$.

Then there is a generic extension $V[G]$ *of* V *with the same cardinals and cofinalities, such that* κ *is measurable in* $V[G]$, *and for every* $\alpha \in A$,

$$V[G] \vDash 2^\alpha = F(\alpha). \tag{21.4}$$

Moreover, given any regular cardinal $\lambda > \kappa^+$, *we can find* $V[G]$ *so that there are* λ *normal measures on* κ *in* $V[G]$.

Proof. Let $j : V \to M$ be the elementary embedding given by the ultrapower Ult_D. As we assume that $A \notin D$, we have $\kappa \notin j(A)$.

Let $(P, <)$ be the Easton product of P_α, $\alpha \in A$, where each P_α is the notion of forcing that adjoins $F(\alpha)$ subsets of α. Thus conditions are 0–1 functions whose domain consists of triples (α, ξ, η) where $\alpha \in A$, $\xi < \alpha$, and $\eta < F(\alpha)$, and such that for every regular cardinal γ,

$$|\{(\alpha, \xi, \eta) \in \mathrm{dom}(p) : \alpha \le \gamma\}| < \gamma.$$

In particular, $|p| < \kappa$ for all $p \in P$, hence $P \subset V_\kappa$ and so $j(p) = p$ for each $p \in P$.

We shall however use not P but $j(P)$ as our notion of forcing. Thus $j(P)$ is, in M, the Easton product of P_α for $\alpha \in j(A)$. Note that $P \subset j(P)$ and that $j(P)$ is isomorphic to $P \times Q$ where $P = (jP)^{<\kappa}$ and $Q = (jP)^{\ge\kappa}$.

Let G be a V-generic filter on $j(P)$. We claim that $V[G]$ has the same cardinals and cofinalities as V and satisfies (21.4) and that κ is a measurable cardinal in $V[G]$. Let $G_1 = G \cap P$; since $j(P)$ is isomorphic to $P \times Q$, there is a $V[G_1]$-generic filter G_2 on Q such that $V[G] = V[G_1 \times G_2]$.

As we have noted before, $\kappa \notin j(A)$, and so $Q = (jP)^{\ge\kappa}$ is in fact $= (jP)^{>\kappa}$, and hence is, in M, κ-closed. But since $M^\kappa \subset M$, Q is κ-closed in V. Moreover, we have $|P| = \kappa$ and $|j(P)| = |j(\kappa)| = \kappa^+$. Thus for each regular $\lambda \le \kappa$, we can break $j(P)$ into a product of two notions of forcing, one that satisfies the λ^+-chain condition and one that is λ-closed, and hence all cardinals $\le \kappa^+$ are preserved. Since $|j(P)| = \kappa^+$, all cardinals $> \kappa^+$ are also preserved.

We prove (21.4) similarly to Easton's Theorem. For $\alpha \in A$, we regard $j(P)$ as a product $(jP)^{>\alpha} \times (jP)^{\le\alpha}$; and since $(jP)^{\le\alpha} = P^{\le\alpha}$ and $(jP)^{>\alpha}$ is α-closed, we conclude that $(2^\alpha)^{V[G]} = F(\alpha)$.

The crucial step is to show that κ is a measurable cardinal in $V[G]$. We shall first extend $j : V \to M$ to an elementary embedding

$$j : V[G_1] \to M[G]. \tag{21.5}$$

We define $j(x)$ for $x \in V[G_1]$ as follows: If $\dot{x} \in V^P$ is a name for x, we let

$$j(x) = (j(\dot{x}))^G \tag{21.6}$$

where $j(\dot{x})$ is, in M, a $j(P)$-name. As in the proof of Theorem 21.1, we show that (21.6) does not depend on the choice of $\dot{x}$. Since $j(p) = p$ for all $p \in P$

and because $G_1 \subset G$, it follows that if some $p \in G_1$ forces $\dot{x} = \dot{y}$, then (in M) $p \Vdash j(\dot{x}) = j(\dot{y})$ and therefore $(j(\dot{x}))^G = (j(\dot{y}))^G$. The same reasoning shows that $j : V[G_1] \to M[G]$ is elementary.

Using (21.5), we define an $V[G_1]$-ultrafilter U on κ as follows:

$$X \in U \quad \text{if and only if} \quad \kappa \in j(X), \tag{21.7}$$

for all $X \subset \kappa$ in $V[G_1]$. A standard argument shows that U is κ-complete; and since j extends the original $j = j_D$, U is nonprincipal.

Now we use again the fact that $j(P)$ is isomorphic to $P \times Q$, where $|P| = \kappa$ and Q is κ-closed. Thus every subset of κ is in $V[G_1]$ and therefore U is in $V[G]$ a κ-complete nonprincipal ultrafilter on κ.

To get a model with λ normal measures on κ we modify the construction above as follows: Let R be the κ^+-product of λ copies of Q, i.e., the set of all functions $f \in Q^\lambda$ such that $|\{i < \lambda : f(i) \neq \emptyset\}| \le \kappa$. We consider the notion of forcing $P \times R$.

Let $G \times H$ be a generic filter on $P \times R$. We claim that the model $V[G \times H]$ has the required properties.

Since R is κ-closed, all subsets of κ are contributed by G; hence $2^\kappa = \kappa^+$ holds in $V[G \times H]$. Standard arguments show that cardinals are preserved and $2^{\kappa^+} = \lambda$ in $V[G \times H]$.

To find λ distinct normal measures, let us look more closely at the definition (21.7) of U: U has a name $\dot{U} \in V^{P \times Q}$ such that for all $p \in P$ and $q \in Q$, and any name $\dot{X} \in V^P$ for a subset of κ,

$$p \cup q \Vdash (\dot{X} \in \dot{U} \leftrightarrow \kappa \in j(\dot{X})). \tag{21.8}$$

If q is represented in Ult_D by $\langle q_\alpha : \alpha < \kappa \rangle$ (with $q_\alpha \in P^\alpha$ for each $\alpha < \kappa$), we have

$$p \cup q \Vdash \dot{X} \in \dot{U} \quad \text{if and only if} \quad \{\alpha : p \cup q_\alpha \Vdash \alpha \in \dot{X}\} \in D. \tag{21.9}$$

For each $i < \lambda$, let Q_i denote the ith copy of Q and let $\dot{U}_i$ be the canonical name for a normal measure using Q_i instead of Q in (21.8). It suffices to show that for any $i \neq k < \lambda$, $\dot{U}_i$ and $\dot{U}_k$ denote different measures in $V[G \times H]$.

The last assertion follows by a standard argument using genericity, and we leave its proof to the reader: Given $i \neq k$ and a condition (p, r) in $P \times R$, use (21.9) to find a stronger condition (p', r') and some P-valued name $\dot{X}$ such that (p', r') forces $\dot{X} \in \dot{U}_i$ but $\dot{X} \notin \dot{U}_k$. □

Silver's Forcing

We shall now construct a model that has a measurable cardinal κ for which $2^\kappa > \kappa^+$. By Corollary 19.25, the consistency strength of this is more than measurability. It has been established that the failure of GCH at a measurable cardinal is equiconsistent with the existence of a measurable cardinal κ of Mitchell's order κ^{++}. The lower bound is obtained by Mitchell's method of iterated ultrapowers from Chapter 19, while the upper bound follows from improvements (due to Woodin and Gitik) on Silver's forcing construction presented below.

Theorem 21.4 (Silver). *If there exists a supercompact cardinal κ, then there is a generic extension in which κ is a measurable cardinal and $2^\kappa > \kappa^+$.*

Silver's construction uses iterated forcing. As $2^\kappa > \kappa^+$ for a measurable cardinal implies that $2^\alpha > \alpha^+$ for many α below κ, the iteration adjoins not only subsets of κ, but, iteratively, subsets of regular cardinals below κ. The iteration combines direct and inverse limits, in a manner similar to Easton's forcing.

Definition 21.5. Let $\alpha \geq 1$, and let P_α be an iterated forcing of length α (see Definition 16.29). P_α is an iteration with *Easton support* if for every $p \in P_\alpha$ and every regular cardinal $\gamma \leq \alpha$, $|s(p) \cap \gamma| < \gamma$. Equivalently, for every limit ordinal $\gamma \leq \alpha$, P_γ is a direct limit if γ is regular, and an inverse limit otherwise.

When using iterated forcing that combines direct and inverse limits, we can apply Theorem 16.30 to calculate the chain condition, and Exercise 16.19 to calculate the degree of closedness. We shall need the following variant:

Definition 21.6. A notion of forcing $(P, <)$ is λ-*directed closed* if whenever $D \subset P$ is such that $|D| \leq \lambda$ and for any $d_1, d_2 \in D$ there is some $e \in D$ with $e \leq d_1$ and $e \leq d_2$, then there exists a $p \in P$ such that $p \leq d$ for all $d \in D$.

Lemma 21.7.

(i) *If P is λ-directed closed, and if $\Vdash_P \dot{Q}$ is λ-directed closed, then $P * \dot{Q}$ is λ-directed closed.*

(ii) *If $\operatorname{cf} \alpha > \lambda$, if P_α is a direct limit and if for each $\beta < \alpha$, P_β is λ-directed closed, then P_α is λ-directed closed.*

(iii) *Let P_α be a forcing iteration of $\langle \dot{Q}_\beta : \beta < \alpha \rangle$ such that for each limit ordinal $\beta \leq \alpha$, P_β is either a direct limit or an inverse limit. Assume that for each $\beta < \alpha$, $\dot{Q}_\beta$ is a λ-directed closed forcing in V^{P_β}. If for every limit ordinal $\beta \leq \alpha$ such that $\operatorname{cf} \beta \leq \lambda$, P_β is an inverse limit, then P_α is λ-directed closed.*

Proof. (i) Let $D = \{(p_\alpha, \dot{q}_\alpha) : \alpha < \lambda\}$ be a directed subset of $P * \dot{Q}$. Clearly, $D_1 = \{p_\alpha : \alpha < \lambda\}$ is a directed subset of P and hence there is $p \in P$ stronger than all p_α, $\alpha < \lambda$. Since for any $\alpha, \beta < \lambda$ there is $\gamma < \lambda$ such that $(p_\gamma, \dot{q}_\gamma) \leq (p_\alpha, \dot{q}_\alpha)$ and $(p_\gamma, \dot{q}_\gamma) \leq (p_\beta, \dot{q}_\beta)$, it is clear that $p \Vdash (\dot{q}_\gamma \leq \dot{q}_\alpha$ and $\dot{q}_\gamma \leq \dot{q}_\beta)$ and thus p forces that $\{\dot{q}_\alpha : \alpha < \lambda\}$ is a directed subset of $\dot{Q}$. Hence

$$p \Vdash \exists q \in \dot{Q} \text{ stronger than all the } \dot{q}_\alpha$$

and therefore there is a $\dot{q}$ such that $\|\dot{q} \in \dot{Q}\| = 1$ and

$$p \Vdash \dot{q} \leq \dot{q}_\alpha \text{ for all } \alpha < \lambda.$$

It follows that $(p, \dot{q}) \leq (p_\alpha, \dot{q}_\alpha)$ for all $\alpha < \lambda$.

(ii) Let D be a directed subset of P, $|D| \leq \lambda$. For each $d \in D$ there is $\gamma_d < \alpha$ such that if $d = \langle p_\beta : \beta < \alpha \rangle$, then $p_i = 1$ for all $i \geq \gamma_d$. Since $\lambda < \operatorname{cf} \alpha$, there is $\gamma < \alpha$ such that each $d \in D$ is as follows:

$$d = (d{\restriction}\gamma)^\frown 1^\frown 1^\frown 1^\frown \ldots.$$

Now $D_\gamma = \{(d{\restriction}\gamma) : d \in D\}$ is a directed subset of P_γ; and since P_γ is λ-directed closed, D_γ has a lower bound $p \in P_\gamma$. Then $p^\frown 1^\frown 1^\frown 1^\frown \ldots$ is a lower bound for D in P.

(iii) By induction on α. It follows from (i) and (ii) that the assertion is true if α is a successor or if P_α is the direct limit. Thus assume that P_α is an inverse limit.

Let $D = \langle p^\nu : \nu < \lambda \rangle$ be a directed subset of P_α; for each ν let $p^\nu = \langle p^\nu_\beta : \beta < \alpha \rangle$. We shall construct, by induction on $\beta < \alpha$, a function $p = \langle p_\beta : \beta < \alpha \rangle \in P_\alpha$ stronger than all p^ν, $\nu < \lambda$.

We construct p such that for each $\beta < \alpha$, $p{\restriction}\beta$ is in P_β and is stronger than all $p^\nu{\restriction}\beta$, $\nu < \lambda$. Having constructed $p{\restriction}\beta$, we let p_β be such that

$$p{\restriction}\beta \Vdash p_\beta \leq p^\nu_\beta \text{ for all } \nu < \lambda.$$

Moreover, if $p^\nu_\beta = 1$ for all $\nu < \lambda$, we let $p_\beta = 1$ too.

If $\gamma \leq \alpha$ is a limit ordinal, we have to show that $\langle p_\beta : \beta < \gamma \rangle \in P_\gamma$. If P_γ is the inverse limit, then there is nothing to prove, so let us assume that P_γ is the direct limit. By the assumption, we have $\operatorname{cf} \gamma > \lambda$ and therefore there is a $\delta < \gamma$ such that for all $\nu < \lambda$, $p^\nu_\beta = 1$ for all β such that $\delta \leq \beta < \gamma$. Hence we have $p_\beta = 1$ for all β such that $\delta \leq \beta < \gamma$, and so $\langle p_\beta : \beta < \gamma \rangle \in P_\gamma$. Thus we have $p = \langle p_\beta : \beta < \alpha \rangle \in P_\alpha$, and it is clear from the construction that $p \leq p^\nu$ for all $\nu < \lambda$. □

An important feature of iterated forcing is that often, under reasonable assumptions, an iteration $P_{\alpha+\beta}$ is equivalent to $P_\alpha * \dot{P}^{(\alpha)}_\beta$ where $P^{(\alpha)}_\beta$ is an iteration of length β inside V^{P_α}. The following lemma is used in applications of iteration with Easton support:

Lemma 21.8 (The Factor Lemma). *Let $P_{\alpha+\beta}$ be a forcing iteration of $\langle \dot{Q}_\xi : \xi < \alpha + \beta \rangle$, where each P_ξ, $\xi \leq \alpha + \beta$ is either a direct limit or inverse limit. In V^{P_α}, let $\dot{P}^{(\alpha)}_\beta$ be the forcing iteration of $\langle \dot{Q}_{\alpha+\xi} : \xi < \beta \rangle$ such that for every limit ordinal $\xi < \beta$, $\dot{P}^{(\alpha)}_\xi$ is either a direct or inverse limit, according to whether $P_{\alpha+\xi}$ is a direct limit or inverse limit. If $P_{\alpha+\xi}$ is an inverse limit for every limit ordinal $\xi \leq \beta$ such that $\operatorname{cf} \xi \leq |P_\alpha|$, then $P_{\alpha+\beta}$ is isomorphic to $P_\alpha * \dot{P}^{(\alpha)}_\beta$.*

This formulation is not quite accurate. The name $\dot{Q}_{\alpha+\xi}$ is in $V^{P_{\alpha+\xi}}$ while $P_\xi^{(\alpha)}$ is an iteration in $V^{P_{\alpha+\beta}}$ that at stage ξ should use a V^{P_α}-name for a name $\dot{Q}_\xi^{(\alpha)} \in V^{\dot{P}_\xi^{(\alpha)}}$. However, the Factor Lemma yields, for each ξ, an isomorphism between $V^{P_{\alpha+\xi}}$ and the Boolean-valued model $V^{\dot{P}_\xi^{(\alpha)}}$ inside V^{P_α}, and so $\dot{Q}_{\alpha+\xi}$ is identified with a V^{P_α}-name for some $\dot{Q}_\xi^{(\alpha)} \in V^{\dot{P}_\xi^{(\alpha)}}$.

Proof. By induction on β. Let β be an ordinal number; we shall construct an isomorphism π between $P_\alpha * \dot{P}_\beta^{(\alpha)}$ and $P_{\alpha+\beta}$.

If $\beta = 0$, then $P_\alpha * \dot{P}_\beta^{(\alpha)} = \{(p, 1) : p \in P_\alpha\}$ and we let $\pi(p, 1) = p$. Thus let $\beta > 0$. A typical element of $P_\alpha * \dot{P}_\beta^{(\alpha)}$ is a pair $(p, \dot{q})$ where $p \in P_\alpha$ and $\dot{q}$ is an element of V^{P_α} such that in V^{P_α}, $\dot{q}$ is a β-sequence and satisfies the conditions on iterated forcing; in particular, for each $\xi < \beta$, $\dot{q}\restriction\xi$ is in $\dot{P}_\xi^{(\alpha)}$ and the ξth term of $\dot{q}$ is in $\dot{Q}_{\alpha+\xi}$.

We shall define a β-sequence $\langle p_{\alpha+\xi} : \xi < \beta\rangle$ and let $\pi(p, \dot{q}) = p^\frown\langle p_\alpha, p_{\alpha+1}, \ldots, p_{\alpha+\xi}, \ldots\rangle$. This mapping π will be an isomorphism between $P_\alpha * \dot{P}_\beta^{(\alpha)}$ and $P_{\alpha+\beta}$. For $\xi < \beta$, let $\dot{q}_\xi \in V^{P_\alpha}$ be such that $\dot{q}_\xi$ is the ξth term of $\dot{q}$. Hence

$$\Vdash_{P_\alpha} (\Vdash_{\dot{P}_\xi^{(\alpha)}} \dot{q}_\xi \in \dot{Q}_{\alpha+\xi}).$$

By the induction hypothesis, $P_{\alpha+\xi}$ is isomorphic to $P_\alpha * \dot{P}_\xi^{(\alpha)}$. Let $p_{\alpha+\xi} \in V^{P_{\alpha+\xi}}$ be the element corresponding to $\dot{q}_\xi$ under the isomorphism between $(V^{P\alpha})^{\dot{P}_\xi^{(\alpha)}}$ and $V^{P_{\alpha+\xi}}$.

Let $\pi(p, \dot{q}) = p^\frown\langle p_\alpha, p_{\alpha+1}, \ldots, p_{\alpha+i}, \ldots\rangle$. All we have to do now is to show that π is an isomorphism between $P_\alpha * \dot{P}_\beta^{(\alpha)}$ and $P_{\alpha+\beta}$. We shall show that for each $(p, \dot{q}) \in P_\alpha * \dot{P}_\beta^{(\alpha)}$, $\pi(p, \dot{q})$ is in $P_{\alpha+\beta}$ and leave the rest to the reader, namely to show that

$$(p, \dot{q}) \le (p', \dot{q}') \quad \text{if and only if} \quad \pi(p, \dot{q}) \le \pi(p', \dot{q}').$$

We want to show that for each $\gamma \le \beta$, $p^\frown\langle p_{\alpha+\xi} : \xi < \gamma\rangle$ is an element of $P_{\alpha+\gamma}$. We need to show that if γ is a limit ordinal and $P_{\alpha+\gamma}$ is the direct limit, then there exists $i_0 < \gamma$ such that $p_{\alpha+i} = 1$ for all i, $i_0 \le i < \gamma$.

Thus let $\gamma \le \beta$ be a limit ordinal such that $P_{\alpha+\gamma}$ is the direct limit of $P_{\alpha+\xi}$, $\xi < \gamma$. Hence in V^{P_α}, $\dot{P}_\gamma^{(\alpha)}$ is the direct limit, and therefore

$$\text{(21.10)} \qquad \Vdash_{P_\alpha} (\exists\xi_0 < \gamma)(\forall\xi \ge \xi_0) \text{ the } \xi\text{th term of } \dot{q} \text{ is } 1.$$

Now we have made an assumption that if $P_{\alpha+\gamma}$ is the direct limit, then $\operatorname{cf}\gamma > |P_\alpha|$. It is easy to see that because $|P_\alpha| < \operatorname{cf}\gamma$, (21.10) implies that there exists $\xi_0 < \gamma$ such that for all $\xi \ge \xi_0$, $\Vdash_{P_\alpha} \dot{q}_i = 1$. Thus for all $\xi \ge \xi_0$, $\Vdash_{P_{\alpha+\xi}} p_{\alpha+\xi} = 1$ and hence $p_{\alpha+\xi} = 1$ for all $\xi \ge \xi_0$. □

Proof of Theorem 21.4. Let κ be a supercompact cardinal and assume $2^\kappa = \kappa^+$. We shall construct a generic extension in which κ is measurable and $2^\kappa = \kappa^{++}$.

We use iterated forcing with Easton support, successively adjoining to each inaccessible cardinal $\alpha \leq \kappa$, α^{++} subsets of α. At limit stages of the iteration we use direct limits when the ordinal is a regular cardinal and inverse limits otherwise.

Let us define, by induction on α, the αth iterate P_α (and the corresponding forcing relation $\Vdash_\alpha$ and the algebra $B_\alpha = B(P_\alpha)$) and the B_α-valued notion of forcing $\dot{Q}_\alpha$:

(21.11) (i) If α is an inaccessible cardinal, let $\dot{Q}_\alpha$ be the notion of forcing in V^{P_α} that adjoins α^{++} subsets of α; that is, we let $\dot{Q}_\alpha$ be in V^{P_α}, the set of all 0–1 functions p whose domain is a subset of size $< \alpha$ of $\alpha \times \alpha^{++}$ (and $\dot{Q}_\alpha$ is ordered by $\supset$). If α is not an inaccessible cardinal, let $\dot{Q}_\alpha = \{1\}$ (as usual, 1 denotes the greatest element of each notion of forcing).

(ii) P_α is the set of all α-sequences $\langle p_\xi : \xi < \alpha \rangle$ satisfying the following:

(a) For every $\gamma < \alpha$, $p{\restriction}\gamma \in P_\gamma$ and $\Vdash_\gamma p_\gamma \in \dot{Q}_\gamma$.

(b) If α is a regular cardinal, then $\exists \xi_0 \, \forall \xi \geq \xi_0 \, p_\xi = 1$.

(iii) If $p, q \in P_\alpha$, then $p \leq_\alpha q$ if and only if

$$(\forall \gamma < \alpha)(p{\restriction}\gamma \leq_\gamma q{\restriction}\gamma \text{ and } p{\restriction}\gamma \Vdash_\gamma p_\gamma \text{ is stronger than } q_\gamma).$$

Finally, let $P = P_{\kappa+1}$, and let $B = B(P)$.

Let G be a generic filter on P and let $V[G]$ be the generic extension of V by G. We shall prove that κ is a measurable cardinal in $V[G]$ and that $V[G] \Vdash 2^\kappa = \kappa^{++}$. Since P is isomorphic to the two-step iteration $P_\kappa * \dot{Q}_\kappa$, we have $V[G] = V[G_\kappa][H_\kappa]$, where G_κ is V-generic on P_κ and H_κ is $V[G_\kappa]$-generic on $Q_\kappa = (\dot{Q}_\kappa)^{G_\kappa}$. Now P_κ is the direct limit of P_α, $\alpha < \kappa$; and since κ is a Mahlo cardinal, there is a stationary set of $\alpha < \kappa$ such that P_α is also a direct limit. Since $|P_\alpha| < \kappa$ for all $\alpha < \kappa$, it follows by Theorem 16.30 that P_κ satisfies the κ-chain condition and hence κ is a regular cardinal in $V[G_\kappa]$. Also, $|P_\kappa| = \kappa$, and hence $V[G_\kappa]$ satisfies $(\forall \alpha < \kappa)\ 2^\alpha \leq \kappa$. In $V[G_\kappa]$, Q_κ is a notion of forcing that adjoins κ^{++} subsets of κ and preserves all cardinals. Thus $V[G] \Vdash (\kappa$ is a regular cardinal and $2^\kappa = \kappa^{++})$.

It remains to prove that κ is a measurable cardinal in $V[G]$. This will be done by first constructing an elementary embedding of $V[G]$ and then showing that the induced measure is in $V[G]$.

Let $\lambda = \kappa^{++}$. Since κ is supercompact, there is an elementary embedding $j : V \to M$ with critical point κ such that $M^\lambda \subset M$ and $j(\kappa) > \lambda$. It follows that $|P| = \lambda$, $P \in M$, and moreover, P is defined in M by the same definition (21.11).

Since $P \in M$, G is also an M-generic filter on P, and we can consider the model $M[G]$. We need the following lemma:

Lemma 21.9. $(M[G])^\lambda \cap V[G] \subset M[G]$.

Proof. It suffices to show that if $f \in V[G]$ is a function from λ into ordinals, then $f \in M[G]$. Let $\dot{f}$ be a name for f and let $p_0 \in G$ be a condition that forces that $\dot{f}$ is a function from λ into the ordinals. For each $\alpha < \lambda$, let

$$A_\alpha = \{p \le p_0 : \exists \beta\, p \Vdash \dot{f}(\alpha) = \beta\}.$$

Each A_α is dense below p_0 (and hence $A_\alpha \cap G \ne \emptyset$). For each $\alpha < \lambda$ and each $p \in A_\alpha$, let $g(\alpha, p)$ be the unique β such that $p \Vdash \dot{f}(\alpha) = \beta$. Since $|P| = \lambda$, we have $|g| = \lambda$ and hence $g \in M$. Now it is easy to see that $f \in M[G]$ because it is defined in $M[G]$ as follows: $f(\alpha) =$ the unique β such that for some $p \in G$, $g(\alpha, p) = \beta$. □

Let us now consider $j(P)$. In M, $j(P)$ is a notion of forcing obtained by iteration up to $j(\kappa) + 1$. We claim that in M we can apply the Factor Lemma to $j(P)$ at $\alpha = \kappa + 1$. First we note that $(jP)_\alpha = P_\alpha$ for all $\alpha < \kappa$, and since $(jP)_\kappa$ is the direct limit, we have $(jP)_\kappa = P_\kappa$. Since $\dot{Q}_\kappa$ is the same in V and M, it follows that $(jP)_{\kappa+1} = P_{\kappa+1}$. The first nontrivial step above α in the iteration occurs at the least inaccessible cardinal (in M) above κ, thus the first nontrivial direct limit is taken far above λ and then only at regular cardinals. Since $|P_{\kappa+1}| = \lambda$, the assumption of the Factor Lemma is satisfied.

Hence $j(P)$ is isomorphic to a two-step iteration (in M)

$$(jP)_{\kappa+1} * (jP)^{(\kappa+1)}_{j(\kappa)+1}. \tag{21.12}$$

Now the first factor of (21.12) is equal to $P_{\kappa+1} = P$. Let us denote $\dot{Q}$ the second factor. By the Factor Lemma, $\dot{Q}$ is, in M^P, a notion of forcing obtained by iteration, with Easton support, from $\kappa + 1$ to $j(\kappa) + 1$. At each $\xi > \kappa$, the iteration uses a notion of forcing in M^{P_ξ} that is either trivial or adjoins ξ^{++} subsets of ξ (if ξ is inaccessible in M); in either case, the notion of forcing is λ-directed closed in M^{P_ξ}. By Lemma 21.7, $\dot{Q}$ is λ-directed closed in M^P. Thus we can write

$$j(P) = P * \dot{Q} \tag{21.13}$$

where $\dot{Q} \in M^P$ is a λ-directed closed notion of forcing. Thus $Q = \dot{Q}^G$ is a λ-directed closed notion of forcing in $M[G]$.

Let $p \in P$. Then by (21.13), $j(p)$ is (represented by) a pair $(s, \dot{q})$ where $s \in P$ and $\dot{q} \in M^P$ is in $\dot{Q}$. By the definition of P, $p = \langle p_\xi : \xi < \kappa + 1\rangle$ and there is $\xi_0 < \kappa$ such that $p_\xi = 1$ for all ξ, $\xi_0 \le \xi < \kappa$. Thus $j(p) = \langle p'_\xi : \xi < j(\kappa) + 1\rangle$ and $p'_\xi = 1$ for all ξ, $\xi_0 \le \xi < j(\kappa)$. In particular, $p'_\kappa = 1$; and since $p'_\xi = p_\xi$ for all $\xi < \kappa$, and $s = j(p){\restriction}(\kappa + 1)$, we have $s = (p{\restriction}\kappa)^\frown 1$. This implies that if $p \in G$ and $j(p) = (s, \dot{q})$, then $s \in G$.

Now let

$$D = \{q \in Q : \text{for some } p \in G,\ q = (\dot{q})^G \text{ where } j(p) = (s, \dot{q})\}.$$

Since P has size λ, we have $j{\restriction}P \in M$ and therefore $D \in M[G]$. It is easy to see that D is directed, i.e., if $q_1, q_2 \in D$, then there is $q \in D$ such that

$q \leq q_1$ and $q \leq q_2$ (this is because G is directed). We have (in $M[G]$), $|D| \leq |G| \leq |P| = \lambda$; and because Q is λ-directed closed, there exists some $a \in Q$ (a *master condition*) such that $a \leq q$ for all $q \in D$.

We shall now consider a generic extension of $V[G]$. Let H be a $V[G]$-generic filter on Q such that H contains the master condition a. Since H is also $M[G]$-generic, and $j(P) = P * \dot{Q}$, there is an M-generic filter K on $j(P)$ such that $M[K] = M[G][H]$; in fact

$$K = \{(s, \dot{q}) : s \in G \text{ and } (\dot{q})^G \in H\}.$$

Now we extend the elementary embedding $j : V \to M$ to an embedding of $V[G]$ into $M[K]$. We work in $V[G][H]$ and define, for all $x \in V[G]$,

$$j(x) = (j(\dot{x}))^K \tag{21.14}$$

where $\dot{x}$ is some P-name for x.

We have to show that the definition (21.14) does not depend on the choice of the name $\dot{x}$; the verification of elementarity of j is then straightforward. Here we use the master condition a. If $p \in G$, then $j(p) = (s, \dot{q})$ where $s \in P$ and $\Vdash \dot{q} \in \dot{Q}$. We have shown that $s \in G$, and if $q = (\dot{q})^G$, then, because $p \in G$, we have $q \geq a$ and therefore $q \in H$. Thus $(s, \dot{q}) \in K$, and it follows that

$$j``G \subset K.$$

Now if $p \in G$ forces $\dot{x} = \dot{y}$, then $j(p) \in K$ forces $j(\dot{x}) = j(\dot{y})$ and hence $(j(\dot{x}))^K = (j(\dot{y}))^K$.

Thus we have (in $V[G][H]$) an elementary embedding

$$j : V[G] \to M[K]$$

and we can define, in the usual way, a $V[G]$-ultrafilter on κ:

$$U = \{X \subset \kappa : \kappa \in j(X)\}.$$

U is nonprincipal and κ-complete. It suffices to show that $U \in V[G]$; then $V[G]$ satisfies that κ is a measurable cardinal.

By Lemma 21.9, Q is λ-closed not only in $M[G]$, but also in $V[G]$. Thus the generic extension $V[G][H]$ of $V[G]$ does not add any new λ-sequences in $V[G]$, and because $|U| = \lambda$ we have $U \in V[G]$. $\square$

Prikry Forcing

Let us address the following problem: Can one construct a generic extension in which all cardinals are preserved but the cofinality of some cardinals is different from their cofinality in the ground model? Obviously, in order to do this we have to change some (weakly) inaccessible cardinal into a singular cardinal. Corollary 18.31 of Jensen's Covering Theorem tells us that for this, it is necessary to assume at least $0^\sharp$ in the ground model. Thus we formulate the problem as follows: Let κ be some large cardinal and let $\lambda < \kappa$ be a regular cardinal. Find a cardinal preserving generic extension, in which $\operatorname{cf}\kappa = \lambda$.

The forcing presented below was devised by Karel Prikry and has become a standard tool of the large cardinal theory.

Theorem 21.10 (Prikry). *Let κ be a measurable cardinal. There is a generic extension in which $\operatorname{cf}\kappa = \omega$ and no cardinals are collapsed. Moreover, every bounded subset of κ in $V[G]$ is in the ground model.*

Proof. Let κ be a measurable cardinal and let D be a normal measure on κ. Let $(P, <)$ be the following notion of forcing. A forcing condition is a pair $p = (s, A)$ where $s \in [\kappa]^{<\omega}$, i.e., s is a finite subset of κ, and $A \in D$. A condition (s, A) is stronger than a condition (t, B) if:

(21.15) (i) t is an initial segment of s, i.e., $t = s \cap \alpha$ for some α;
(ii) $A \subset B$;
(iii) $s - t \subset B$.

We immediately note that any two conditions (s, A), (s, B) with the same first coordinate are compatible, and hence any antichain $W \subset P$ has size at most κ. We also note that if (s, A) and (t, B) are compatible, then either s is an initial segment of t, or t is an initial segment of s.

Let G be a generic filter on P. We shall show that in $V[G]$, κ has cofinality ω, that every bounded subset of κ is in V, and that all cardinals and cofinalities above κ are preserved.

The last statement is immediate since P satisfies the κ^+-chain condition. It is also easy to show that $\operatorname{cf}\kappa = \omega$ in $V[G]$: If (s, A) and (t, B) are in G, then either s is an initial segment of t or vice versa; hence $S = \bigcup\{s : (s, A) \in G$ for some $A\}$ is a subset of κ of order type ω. By the genericity of G, S is clearly an unbounded subset of κ, and hence $\operatorname{cf}\kappa = \omega$.

It remains to show that if $X \in V[G]$ is such that $X \subset \lambda$ for some $\lambda < \kappa$, then $X \in V$. For this, we need the property of P stated in Lemma 21.12 below. The proof uses Theorem 10.22 which states that every partition of $[\kappa]^{<\omega}$ into less than κ pieces has a homogeneous set $H \in D$.

Lemma 21.11. *Let σ be a sentence of the forcing language. There exists a set $A \in D$ such that the condition $(\emptyset, A)$ decides σ, i.e., either $(\emptyset, A) \Vdash \sigma$, or $(\emptyset, A) \Vdash \neg\sigma$.*

Proof. Let S^+ be the set of all $s \in [\kappa]^{<\omega}$ such that $(s, X) \Vdash \sigma$ for some X and let $S^- = \{s : \exists X\, (s, X) \Vdash \neg\sigma\}$. Let $T = [\kappa]^{<\omega} - (S^+ \cup S^-)$. By Theorem 10.22, there is a set $A \in D$ such that for every n, either $[A]^n \subset S^+$ or $[A]^n \subset S^-$ or $[A]^n \subset T$. We shall prove that $(\emptyset, A)$ decides σ.

If not, then there are conditions (s, X) and (t, Y), both stronger than $(\emptyset, A)$ such that one forces σ and the other forces $\neg\sigma$. We may assume that $|s| = |t|$, say $|s| = |t| = n$. Since $(s, X) \le (\emptyset, A)$, we have $s \in [A]^n$; and similarly, $t \in [A]^n$. But $s \in S^+$ and $t \in S^-$, which is a contradiction since S^+ and S^- are disjoint and therefore cannot both have a nonempty intersection with $[A]^n$. □

Lemma 21.12. *Let σ be a sentence of the forcing language and let (s_0, A_0) be a condition. Then there exists a set $A \subset A_0$ in D such that the condition (s_0, A) decides σ.*

Proof. A slight modification of the preceding proof; we may assume that $\min(A_0) > \max(s_0)$. Let S^+ be the set of all $s \in [A_0]^{<\omega}$ such that $(s_0 \cup s, X) \Vdash \sigma$ for some $X \subset A_0$ and let S^- be defined similarly. As before, there exists some $A \subset A_0$ in D such that for no n, $[A]^n$ intersects both S^+ and S^-. It follows that (s_0, A) decides σ. □

Now let $\lambda < \kappa$ and let $X \subset \lambda$; we will show that $X \in V$. Let $\dot{X}$ be a name for X, and let $p_0 \in G$ be a condition such that $p \Vdash \dot{X} \subset \lambda$. It suffices to show that for each $p \le p_0$ there is a $q \le p$ and a set $Z \subset \lambda$ such that $q \Vdash \dot{X} = Z$.

Let $p \le p_0$, $p = (s, A)$. For each $\alpha < \lambda$, there is, by Lemma 21.12, a set $A_\alpha \subset A$ in D such that (s, A_α) decides the sentence $\alpha \in \dot{X}$. Let $B = \bigcap_{\alpha<\lambda} A_\alpha$; we have $B \in D$ and $q = (s, B)$ decides $\alpha \in \dot{X}$ for each $\alpha < \lambda$. Thus if $Z = \{\alpha < \lambda : q \Vdash \alpha \in \dot{X}\}$, we have $q \Vdash \dot{X} = Z$.

This completeness the proof of Theorem 21.10. □

An immediate consequence of Theorems 21.4 and 21.10 is the independence of SCH:

Corollary 21.13. *It is consistent* (*relative to the existence of a supercompact cardinal*) *that there is a strong limit singular cardinal κ such that $2^\kappa > \kappa^+$.*

Proof. Let κ be a supercompact cardinal. First we construct a generic extension in which κ is measurable and $2^\kappa > \kappa^+$. Then we extend the model further to make κ a singular cardinal. The new model still satisfies $2^\kappa > \kappa^+$, and κ is a strong limit cardinal. □

We now prove a characterization of Prikry generic sequences due to Mathias. Let us first generalize the *diagonal intersection* as follows: If $\{A_s : s \in [\kappa]^{<\omega}\}$ is a collection of subsets of κ, let

$$\triangle_s A_s = \{\alpha < \kappa : \alpha \in \bigcap\{A_s : \max(s) < \alpha\}\}. \tag{21.16}$$

It is routine to show that every normal ultrafilter on κ is closed under diagonal intersections (21.16).

Theorem 21.14 (Mathias). *Let M be a transitive model of* ZFC, *let U be, in M, a normal measure on κ, and let P be the Prikry forcing defined from U. For every set $S \subset \kappa$ of order-type ω, S is P-generic over M if and only if for every $X \in U$, $S - X$ is finite.*

Proof. In the easy direction, let G be a generic filter on P and let $S = \bigcup\{s : (s, A) \in G\}$. For every $X \in U$, $S - X$ is finite because for every condition (s, A), the stronger condition $(s, A \cap X)$ forces that every $\alpha \in S$ above s is in X.

For the other direction, let $S \subset \kappa$, of order-type ω, be such that $S - X$ is finite for all $X \in U$. We want to show that the filter

$$G = \{(s, A) \in P : s \text{ is an initial segment of } S \text{ and } S - s \subset A\}$$

is M-generic; let $D \in M$ be an open dense subset of P and let us show that $G \cap D \neq \emptyset$.

For each $s \in [\kappa]^{<\omega}$, let $F : [\kappa]^{<\omega} \to \{0, 1\}$ be a partition such that $F(t) = 1$ if and only if $\max(s) < \min(t)$ and $\exists X\ (s \cup t, X) \in D$. Let $A_s \in U$ be a homogeneous set for F; if there is an X such that $(s, X) \in D$, let $B_s = A_s \cap X$, and otherwise, let $B_s = A_s$. Let $A = \triangle_s B_s$ be the diagonal intersection. Since D is open dense, we have for all $s \in [\kappa]^{<\omega}$:

$$\text{If } \exists X\ (s, X) \in D \text{ then } (s, A \smallsetminus s) \in D \tag{21.17}$$

where $A \smallsetminus s = A - (\max(s) + 1)$.

By the assumption on S, S has an initial segment s such that $S - s \subset A$. By density of D there exist a $t \in [B \smallsetminus s]^{<\omega}$ and X such that $(s \cup t, X) \in D$. Let $u \subset S - s$ be such that $|u| = |t|$; the homogeneity of $A \smallsetminus s \subset A_s$ for F_s implies that for some Y, $(s \cup u, Y) \in D$. By (21.17) we have $(s \cup u, A \smallsetminus u) \in D$ and since $(s \cup u, A \smallsetminus u) \in G$, $D \cap G \neq \emptyset$. □

The following theorem shows the relationship between Prikry forcing and iterated ultrapowers. Let U be a normal measure on κ, and consider the iterated ultrapowers $M_\alpha = \mathrm{Ult}^{(\alpha)}$, and the embeddings $i_{\alpha,\beta} : M_\alpha \to M_\beta$. Let $\kappa_\alpha = \kappa^{(\alpha)} = i_{0,\alpha}(\kappa)$, and let $U^{(\alpha)} = i_{0,\alpha}(U)$ be the measure on κ_α in M_α.

Theorem 21.15. *Let $M = M_\omega$, $N = \bigcap_{n<\omega} M_n$ and let $P \in M$ be the Prikry forcing for the measure $U^{(\omega)}$ on κ_ω in M. The set $S = \{\kappa_n : n < \omega\}$ is P-generic over M, and $M[S] = N$.*

Proof. The genericity follows from Lemma 19.10 and Theorem 21.14. N is easily seen to be a model of ZF, and since $M[S] \subset M_n$ for all n, we have $M[S] \subset N$. In order to prove $N \subset M[S]$, it suffices, by Theorem 13.28, to prove that every set of ordinals in N is in $M[S]$.

First we claim that for every ordinal ξ

$$\text{for eventually all } n < \omega,\ i_{n,\omega}(\xi) = i_{\omega,\omega+\omega}(\xi). \tag{21.18}$$

As $\xi \in M_\omega$, let n be such that $\xi = i_{n,\omega}(\xi_n)$ for some ξ_n. Thus

$$M_n \vDash \xi \text{ is the image of } \xi_n \text{ under the embedding from } M_n \text{ into } M_\omega.$$

Applying $i_{n,\omega}$, we have

$$i_{n,\omega}(M_n) \vDash i_{n,\omega}(\xi) \text{ is the image of } i_{n,\omega}(\xi_n) \text{ under the embedding from } i_{n,\omega}(M_n) \text{ into } i_{n,\omega}(M_\omega).$$

Since $i_{n,\omega}(M_n) = M_\omega$ and $i_{n,\omega}(M_\omega) = M_{\omega+\omega}$, we get

$$i_{n,\omega}(\xi) \text{ is the image of } \xi \text{ under } i_{\omega,\omega+\omega},$$

establishing (21.18).

Now let x be a set of ordinals in N. Hence $x \in M_n$ for each n. By the representation of iterated ultrapowers, there is for each $n < \omega$ a function f_n on $[\kappa]^n$ such that $x = i_{0,n}(f_n)(\kappa_0, \ldots, \kappa_{n-1})$. Since $i_{n,\omega}(\kappa_i) = \kappa_i$ for $i < n$, we have $i_{n,\omega}(x) = i_{0,\omega}(f)(\kappa_0, \ldots, \kappa_{n-1})$. Now the sequence $\langle i_{0,\omega}(f_n) : n < \omega\rangle = i_{0,\omega}(\langle f_n : n < \omega\rangle)$ is in M_ω and therefore the sequence $\langle i_{n,\omega}(x) : n < \omega\rangle$ is in $M_\omega[\langle\kappa_n : n < \omega\rangle] = M[S]$.

If ξ is an ordinal then $\xi \in x$ if and only if for any n, $i_{n,\omega}(\xi) \in i_{n,\omega}(x)$, and by (21.18), if for eventually all $n < \omega$, $i_{\omega,\omega+\omega}(\xi) \in i_{n,\omega}(x)$. Since $i_{\omega,\omega+\omega}$ is definable in M, x is definable from the sequence $\langle i_{n,\omega}(x) : n < \omega\rangle$ in $M[S]$. Hence $x \in M[S]$, and $N \subset M[S]$. □

Measurability of $\aleph_1$ in ZF

In ZF (without the Axiom of Choice), one can still define measurability in the usual way: An uncountable cardinal κ is measurable if there exists a κ-complete nonprincipal ultrafilter on κ. In the absence of the Axiom of Choice, a measurable cardinal is still regular, but not necessarily a limit cardinal. The absence of AC has no effect on the consistency strength: If U is a nonprincipal κ-complete ultrafilter on $\kappa > \omega$, then in $L[U]$ (a model of ZFC), κ is a measurable cardinal. The following theorem shows that in ZF, $\aleph_1$ can be measurable:

Theorem 21.16. *Let M be a transitive model of* ZFC + "*there is a measurable cardinal.*" *There is a symmetric model $N \supset M$ of* ZF *such that $N \vDash \aleph_1$ is measurable.*

We shall construct a symmetric extension of M. Recall the theory of symmetric models from Chapter 15. We consider a complete Boolean algebra B, a group $\mathcal{G}$ of automorphisms of B, and a normal filter $\mathcal{F}$ on $\mathcal{G}$ (see (15.34)). For every $\dot{x} \in M^B$ we let $\operatorname{sym}(\dot{x})$ be the symmetry group of $\dot{x}$, $\operatorname{sym}(\dot{x}) = \{\pi \in \mathcal{G} : \pi(\dot{x}) = \dot{x}\}$, and call $\dot{x}$ symmetric if $\operatorname{sym}(\dot{x}) \in \mathcal{F}$. We denote *HS* the class of all hereditarily symmetric names. If G is an M-generic ultrafilter on B, we let N be the G-interpretation of the class *HS*; N is a model of ZF and $N \supset M$.

Let us call a subset $A \subset B$ *symmetric* if

$$\{\pi \in \mathcal{G} : \pi(a) = a \text{ for all } a \in A\} \in \mathcal{F}.$$

Lemma 21.17. *Let κ be measurable in M, and let N be a symmetric extension of M (via B, G, $\mathcal{F}$, $\mathcal{G}$). If every symmetric subset of B has size $< \kappa$, then κ is measurable in N.*

Proof. Let U be, in M, a κ-complete nonprincipal ultrafilter on κ. We show that U generates a κ-complete nonprincipal ultrafilter on κ in N. It suffices to show that if $\gamma < \kappa$ and $\{X_\alpha : \alpha < \gamma\}$ is a partition of κ in N, then for some $\alpha < \gamma$, X_α includes some $Y \in U$.

We give the proof for $\gamma = 2$ since the general case is analogous. Let $X \in N$ be a subset of κ, and let $\dot{X} \in HS$ be a symmetric name for X. Let $A = \{\|\alpha \in \dot{X}\| : \alpha < \kappa\}$. If $\pi \in \mathcal{G}$ is such that $\pi(\dot{X}) = \dot{X}$, then (because $\pi(\check{\alpha}) = \check{\alpha}$ for all α) $\pi(a) = a$ for all $a \in A$; thus A is a symmetric subset of B.

Hence $|A| < \kappa$. For each $a \in A$, let $Y_a = \{\alpha : \|\alpha \in \dot{X}\| = a\}$. Clearly, $\{Y_a : a \in A\}$ is a partition of κ into fewer than κ pieces, and hence one $Y = Y_\alpha$ is in U. Now if $a \in G$, then we have $Y \subset X$, and if $a \notin G$, then $Y \subset \kappa - X$. Hence either X or $\kappa - X$ has a subset that is in U. □

Proof of Theorem 21.16. Let κ be a measurable cardinal in M. Let P be the set of all one-to-one finite sequences $p = \langle \alpha_0, \alpha_1, \ldots, \alpha_{n-1}\rangle$ of ordinals less than κ; p is stronger than q if p extends q (P collapses κ; cf. Example 15.20). Let $\mathcal{G}$ be the set of all permutations of κ; every $\pi \in \mathcal{G}$ induces an automorphism of $(P, <)$ as follows:

$$\pi(\langle \alpha_0, \ldots, \alpha_{n-1}\rangle) = \langle \pi(\alpha_0), \ldots, \pi(\alpha_{n-1})\rangle$$

and, in turn, an automorphism of $B = B(P)$. Thus we identify $\mathcal{G}$ with the group of automorphisms so induced.

For each $\gamma < \kappa$, let

$$H_\gamma = \{\pi \in \mathcal{G} : \pi(\alpha) = \alpha \text{ for all } \alpha < \gamma\}$$

and let $\mathcal{F}$ be the normal filter on $\mathcal{G}$ generated by $\{H_\gamma : \gamma < \kappa\}$. Thus $\dot{x} \in M^B$ is symmetric if and only if there is some $\gamma < \kappa$ such that $\pi(\dot{x}) = \dot{x}$ whenever $\pi(\alpha) = \alpha$ for all $\alpha < \gamma$.

Let G be an M-generic filter on B and let N be the symmetric model given by B, G, $\mathcal{F}$, $\mathcal{G}$. We shall show that $\kappa = (\aleph_1)^N$ and that κ is measurable in N.

If $\gamma < \kappa$, then γ is countable in N: Let $\dot{f}$ be the name such that

$$\|\dot{f}(n) = \alpha\| = \sum\{p \in P : p(n) = \alpha\}$$

for all $n < \omega$ and $\alpha < \gamma$. Clearly, $\dot{f}$ is symmetric because $\pi(\dot{f}) = \dot{f}$ for every $\pi \in H_\gamma$, and hence $\dot{f} \in HS$. The interpretation of $\dot{f}$ is a one-to-one function of a subset of ω onto γ.

It remains to show that κ is measurable in N. By Lemma 21.17 it suffices to show that every symmetric $A \subset B$ has size $< \kappa$. Let $A \subset B$ be symmetric. There exists a $\gamma < \kappa$ such that $\pi(a) = a$ for all $a \in A$ and all $\pi \in H_\gamma$.

For every $a \in A$, let $S_a = \{p \in P : p \le a\}$. If $\pi \in H_\gamma$ and $p \in S_a$, then $\pi(p) \in S_a$ because $\pi(p) \le \pi(a) = a$. Let $T_a = \{p \in S_a : p(n) < \gamma + \omega$ for all $n \in \operatorname{dom}(p)\}$. If $\pi \in H_\gamma$ and $p \in T_a$, then $\pi(p) \in S_a$. Conversely, if $p \in S_a$, there is a $\pi \in H_\gamma$ that maps all $\alpha \in \operatorname{ran}(p)$ greater than γ into $\gamma + \omega$; since $\pi \in H_\gamma$ and $p \in S_a$, we have $\pi(p) \in S_a$ and hence $\pi(p) \in T_a$. Thus $S_a = \{\pi(p) : p \in T_a \text{ and } \pi \in H_\gamma\}$, and consequently

$$\text{if } a \ne b \in A, \text{ then } T_a \ne T_b. \tag{21.19}$$

However, each T_a is a set of finite sequences in $\gamma + \omega$; and since κ is inaccessible, (21.19) implies that $|A| < \kappa$. □

Exercises

21.1. Let κ be measurable and $|P| < \kappa$; let U be a κ-complete ultrafilter on κ. Then in $V[G]$, the filter $W = \{X \subset \kappa : X \supset Y \text{ for some } Y \in U\}$ is a κ-complete ultrafilter on κ.

[For instance, to show that W is an ultrafilter, consider $\dot{X} \in V^B$ such that $\|\dot{X} \subset \kappa\| \in G$. The function $\alpha \mapsto \|\alpha \in \dot{X}\|$ is a partition of κ into $|B| < \kappa$ pieces, and by the κ-completeness of U there is a $Y \in U$ such that $\|\alpha \in \dot{X}\|$ is the same for all $\alpha \in Y$. Now either $X \in W$ or $\kappa - X \in W$ according to whether this B-value is in G or not.]

21.2. If κ is an inaccessible cardinal and $|P| < \kappa$, then every closed unbounded set $C \subset \kappa$ in $V[G]$ has a closed unbounded subset in V. (See Lemma 22.25 for a stronger result.)

[For each $b \in B(P)$, let $C_b = \{\alpha : \|\alpha \in \dot{C}\| = b\}$. Using $|B| < \kappa$, show that for some $b \in G$, C_b is unbounded. Let D be the closure of C_b.]

21.3. Let κ be a regular uncountable cardinal and let $\nu < \kappa$. If I is a κ-complete ν-saturated ideal on κ then either κ is measurable or $\kappa \le 2^\nu$.

[Use the proof of Lemma 10.9.]

21.4. Let κ be an inaccessible cardinal. There is a notion of forcing $(P, <)$ such that $|P| = \kappa$ and P is α-distributive for all $\alpha < \kappa$, and such that κ is not a Mahlo cardinal in the generic extension.

[Forcing conditions are sets $p \subset \kappa$ such that $|p \cap \gamma| < \gamma$ for every regular $\gamma \le \kappa$; $p \le q$ if and only if p is an end-extension of q, i.e., if $q = p \cap \alpha$ for some α. To show that for any $\alpha < \kappa$, P does not add any new α-sequence, observe that for every p there is a $q \le p$ such that $P_q = \{r \in P : r \le q\}$ is α-closed.]

21.5. If κ is a measurable cardinal and P is a κ-closed notion of forcing (or just κ-distributive), then κ is measurable in the generic extension.

21.6. It is consistent that $2^{\operatorname{cf}\kappa} < \kappa$ and $\kappa^+ < \kappa^{\operatorname{cf}\kappa} < 2^\kappa$.

[Extend the model in Corollary 21.13 by adding a large number of subsets of ω_1.]

The Prikry model $V[G]$ of Theorem 21.10 provides an example of a singular Rowbottom cardinal. The exercise below shows that κ has in $V[G]$ the combinatorial property equivalent to being a Rowbottom cardinal.

21.7. In the Prikry model, for every partition $F : [\kappa]^{<\omega} \to \lambda$ into $\lambda < \kappa$ pieces there exists a set $H \subset \kappa$ of size κ such that F takes at most $\aleph_0$ values on $[H]^{<\omega}$.

[Let $\dot{F}$ be a name for F and let (s_0, A_0) be a condition (such that $\max(s_0) < \min(A_0)$). Let g be a partition of $[\kappa]^{<\omega} \times [\kappa]^{<\omega}$ into λ pieces, defined as follows: If $s \in [A_0]^{<\omega}$ and for some $X \subset A_0$, $(s_0 \cup s, X) \Vdash \dot{F}(t) = \alpha$, then let $g(s,t) = \alpha$; otherwise, let $g(s,t) = 0$. Show that there is $A \subset A_0$ in D and a countable $S \subset \lambda$ such that $g([A]^{<\omega} \times [A^{<\omega}]) \subset S$. Then $(s_0, A) \Vdash \dot{F}([A]^{<\omega}) \subset S$.]

Historical Notes

Theorem 21.1 is due to Lévy and Solovay [1967]. Theorem 21.3 was proved by Kunen and Paris in [1970/71] Theorem 21.4 is an unpublished result of Silver; an account of Silver's forcing appeared in Menas [1976]. Theorem 21.10 is due to Prikry [1970]. The characterization of Prikry sequences (Theorem 21.14) appeared in Mathias [1973]. Theorem 21.15 was proved by Bukovský [1973, 1977] and by Dehornoy [1975, 1978]. Theorem 21.16 is due to Jech [1968] and Takeuti [1970].

Exercise 21.4: Jensen.

Exercise 21.7: Prikry [1970].