24. The Singular Cardinal Problem

In this chapter we use combinatorial methods to prove theorems (in ZFC) on cardinal arithmetic of singular cardinals. We introduce a powerful theory of Shelah, the *pcf* theory, and apply the theory to present a most remarkable result of Shelah on powers of singular cardinals.

The Galvin-Hajnal Theorem

Following Silver's Theorem 8.12 on singular cardinals of uncountable cofinality, Galvin and Hajnal proved a related result:

Theorem 24.1 (Galvin-Hajnal [1975]). *Let $\aleph_\alpha$ be a strong limit singular cardinal of uncountable cofinality. Then $2^{\aleph_\alpha} < \aleph_\gamma$ where $\gamma = (2^{|\alpha|})^+$.*

Note that the theorem gives a nontrivial information only if $\aleph_\alpha$ is not a fixed point of the aleph function.

In order to simplify the notation, we consider the special case $\alpha = \omega_1$. The following lemma implies the theorem (as in the proof of Silver's Theorem). Two functions f and g on ω_1 are almost disjoint if $\{\alpha : f(\alpha) = g(\alpha)\}$ is at most countable.

Lemma 24.2. *Assume that $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ for all $\alpha < \omega_1$. Let F be an almost disjoint family of functions*

$$F \subset \prod_{\alpha<\omega_1} A_\alpha$$

such that $|A_\alpha| < \aleph_{\omega_1}$ for all $\alpha < \omega_1$. Then $|F| < \aleph_\gamma$ where $\gamma = (2^{\aleph_1})^+$.

Proof. We first introduce the following relation among functions $\varphi : \omega_1 \to \omega_1$

$$\varphi < \psi \quad \text{if and only if} \quad \{\alpha < \omega_1 : \varphi(\alpha) \geq \psi(\alpha)\} \text{ is nonstationary.} \tag{24.1}$$

Since the closed unbounded filter is σ-complete, it follows that there is no infinite descending sequence

$$\varphi_0 > \varphi_1 > \varphi_2 > \dots .$$

Otherwise, the set $\{\alpha < \omega_1 : \varphi_n(\alpha) \le \varphi_{n+1}(\alpha) \text{ for some } n\}$ is nonstationary and so there is an α such that

$$\varphi_0(\alpha) > \varphi_1(\alpha) > \varphi_2(\alpha) > \ldots,$$

a contradiction.

Hence the relation $\varphi < \psi$ is well-founded and we can define the rank $\|\varphi\|$ of φ in this relation (called the *norm* of φ) such that

$$\|\varphi\| = \sup\{\|\psi\| + 1 : \psi < \varphi\}.$$

Note that $\|\varphi\| = 0$ if and only if $\varphi(\alpha) = 0$ for a stationary set of α's.

Lemma 24.2 follows from

Lemma 24.3. *Assume that $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ for all $\alpha < \omega_1$. Let $\varphi : \omega_1 \to \omega_1$ and let F be an almost disjoint family of functions*

$$F \subset \prod_{\alpha<\omega_1} A_\alpha$$

such that

$$|A_\alpha| \le \aleph_{\alpha+\varphi(\alpha)}$$

for every $\alpha < \omega_1$. Then $|F| \le \aleph_{\omega_1+\|\varphi\|}$.

To prove Lemma 24.2 from Lemma 24.3, we let φ be such that $|A_\alpha| \le \aleph_{\alpha+\varphi(\alpha)}$. If ϑ is the length of the well-founded relation $\varphi < \psi$, then certainly $|\vartheta| \le 2^{\aleph_1}$ and so $\vartheta < (2^{\aleph_1})^+$. Hence $\omega_1 + \|\varphi\| < (2^{\aleph_1})^+$ for every φ and Lemma 24.2 follows. □

Proof of Lemma 24.3. By induction on $\|\varphi\|$. If $\|\varphi\| = 0$, then $\varphi(\alpha) = 0$ on a stationary set and the statement is precisely Lemma 8.16.

To handle the case $\|\varphi\| > 0$, we first generalize the definition of $\varphi < \psi$. Let $S \subset \omega_1$ be a stationary set. We define

(24.2) $\quad \varphi <_S \psi \quad$ if and only if $\quad \{\alpha \in S : \varphi(\alpha) \ge \psi(\alpha)\}$ is nonstationary.

The same argument as before shows that $\varphi <_S \psi$ is a well-founded relation and so we define the norm $\|\varphi\|_S$ accordingly. Note that if $S \subset T$, then $\|\varphi\|_T \le \|\varphi\|_S$. In particular, $\|\varphi\| \le \|\varphi\|_S$, for any stationary S. Moreover,

$$\|\varphi\|_{S\cup T} = \min\{\|\varphi\|_S, \|\varphi\|_T\} \tag{24.3}$$

as can easily be verified.

For every $\varphi : \omega_1 \to \omega_1$, we let I_φ be the collection of all nonstationary sets along with those stationary S such that $\|\varphi\| < \|\varphi\|_S$. If S is stationary and X is nonstationary, then $\|\varphi\|_{S\cup X} = \|\varphi\|_S$. This and (24.3) imply that I_φ is a proper ideal on ω_1.

If $\|\varphi\|$ is a limit ordinal, then

$$S = \{\alpha < \omega_1 : \varphi(\alpha) \text{ is a successor ordinal}\} \in I_\varphi$$

because if $S \notin I_\varphi$, then $\|\varphi\| = \|\varphi\|_S = \|\psi\|_S + 1$, where $\psi(\alpha) = \varphi(\alpha) - 1$ for all $\alpha \in S$. Hence

$$\{\alpha < \omega_1 : \varphi(\alpha) \text{ is a limit ordinal}\} \notin I_\varphi.$$

Similarly, if $\|\varphi\|$ is a successor ordinal, then

$$\{\alpha < \omega_1 : \varphi(\alpha) \text{ is a successor ordinal}\} \notin I_\varphi.$$

Now we are ready to proceed with the induction.

(a) Let $\|\varphi\|$ be a limit ordinal, $\|\varphi\| > 0$. Let

$$S = \{\alpha < \omega_1 : \varphi(\alpha) > 0 \text{ and is a limit ordinal}\}.$$

It follows that $S \notin I_\varphi$.

We may assume that $A_\alpha \subset \aleph_{\alpha+\varphi(\alpha)}$ for every α, and so we have $f(\alpha) < \aleph_{\alpha+\varphi(\alpha)}$ for every $f \in F$. Given $f \in F$, we can find for each $\alpha \in S$ some $\beta < \varphi(\alpha)$ such that $f(\alpha) < \omega_{\alpha+\beta}$; call this $\beta = \psi(\alpha)$. For $\alpha \notin S$, let $\psi(\alpha) = \varphi(\alpha)$. Since $S \notin I_\varphi$, we have $\|\psi\| \le \|\psi\|_S < \|\varphi\|_S = \|\varphi\|$. We also have $f \in F_\psi$, where

$$F_\psi = \{f \in F : f(\alpha) < \omega_{\alpha+\psi(\alpha)} \text{ for all } \alpha\},$$

and so

$$F = \bigcup\{F_\psi : \|\psi\| < \|\varphi\|\}.$$

By the induction hypothesis, $|F_\psi| \le \aleph_{\omega_1+\|\psi\|} < \aleph_{\omega_1+\|\varphi\|}$ for every ψ such that $\|\psi\| < \|\varphi\|$. Since the number of functions $\psi : \omega_1 \to \omega_1$ is $2^{\aleph_1}$, and $2^{\aleph_1} < \aleph_{\omega_1}$, we have $|F| \le \aleph_{\omega_1+\|\varphi\|}$.

(b) Let $\|\varphi\|$ be a successor ordinal, $\|\varphi\| = \gamma + 1$. Let

$$S_0 = \{\alpha < \omega_1 : \varphi(\alpha) \text{ is a successor}\}.$$

It follows that $S_0 \notin I_\varphi$.

Again, we may assume that $A_\alpha \subset \omega_{\alpha+\varphi(\alpha)}$ for each $\alpha < \omega_1$. First we prove that for every $f \in F$, the set

$$F_f = \{g \in F : \exists S \subset S_0,\ S \notin I_\varphi,\ (\forall \alpha \in S)\ g(\alpha) \le f(\alpha)\}$$

has cardinality $\aleph_{\omega_1+\gamma}$. If $S \subset S_0$ and $S \notin I_\varphi$, let

$$F_{f,S} = \{g \in F : (\forall \alpha \in S)\ g(\alpha) \le f(\alpha)\}.$$

Let $\psi : \omega_1 \to \omega_1$ be such that $\psi(\alpha) = \varphi(\alpha) - 1$ for $\alpha \in S$, and $\psi(\alpha) = \varphi(\alpha)$ otherwise. Since $S \notin I_\varphi$, we have $\|\psi\| \le \|\psi\|_S < \|\varphi\|_S = \|\varphi\| = \gamma + 1$ and so $\|\psi\| = \gamma$. Since $F_{f,S} \subset \prod_{\alpha<\omega_1} B_\alpha$, where $|B_\alpha| \le \aleph_{\alpha+\psi(\alpha)}$ for all α, we use the induction hypothesis to conclude that $|F_{f,S}| \le \aleph_{\omega_1+\gamma}$. Then it follows that $|F_f| \le \aleph_{\omega_1+\gamma}$.

To complete the proof, we construct a sequence

$$\langle f_\xi : \xi < \vartheta \rangle \tag{24.4}$$

such that $\vartheta \le \aleph_{\omega_1+\gamma+1}$ and

$$F = \bigcup \{F_{f_\xi} : \xi < \vartheta\}. \tag{24.5}$$

Given f_ν, $\nu < \xi$, we let $f_\xi \in F$ (if it exists) be such that $f_\xi \notin F_{f_\nu}$, for all $\nu < \xi$. Then the set

$$\{\alpha \in S_0 : f_\xi(\alpha) \le f_\nu(\alpha)\}$$

belongs to I_φ, and so $f_\nu \in F_{f_\xi}$, for each $\nu < \xi$.

Since $|F_{f_\xi}| \le \aleph_{\omega_1+\gamma}$ and $F_{f_\xi} \supset \{f_\nu : \nu < \xi\}$, it follows that $\xi < \aleph_{\omega_1+\gamma+1}$ if f_ξ exists. Thus the sequence (24.4) has length $\vartheta \le \aleph_{\omega_1+\gamma+1}$. Then we have

$$F = \bigcup \{F_{f_\xi} : \xi < \vartheta\}$$

and so $|F| \le \aleph_{\omega_1+\gamma+1}$. □

Ordinal Functions and Scales

The proof of the Galvin-Hajnal Theorem suggests that ordinal functions play an important role in arithmetic of singular cardinals. We shall now embark on a systematic study of ordinal functions and introduce Shelah's pcf theory.

Let A be an infinite set and let I be an ideal on A.

Definition 24.4. For ordinal functions f, g on A, let

$$\begin{array}{lll} f =_I g & \text{if and only if} & \{a \in A : f(a) \ne g(a)\} \in I, \\ f \le_I g & \text{if and only if} & \{a \in A : f(a) > g(a)\} \in I, \\ f <_I g & \text{if and only if} & \{a \in A : f(a) \ge g(a)\} \in I. \end{array}$$

If F is a filter on A, then $f <_F g$ means $f <_I g$ where I is the dual ideal, and similarly for $f \le_F g$ and $f =_F g$.

The relation $\le_I$ is a partial ordering (of equivalence classes). If S is a set of ordinal functions on A then g is an *upper bound* of S if $f \le_I g$ for all $f \in S$, and g is a *least upper bound* of S if it is an upper bound and if $g \le_I h$ for every upper bound h.

The relation $<_I$ is also a partial ordering (different from $\le_I$ unless I is a prime ideal), and if I is σ-complete then $<_I$ is well-founded. If I is the nonstationary ideal on a regular uncountable cardinal κ, then the rank of an ordinal function f on κ is the (Galvin-Hajnal) *norm* $\|f\|$.

The following lemma shows that for every $\eta < \kappa^+$ there is a *canonical* function f_η on κ of norm η:

Lemma 24.5. *Let κ be a regular uncountable cardinal. There exist ordinal functions f_η, $\eta < \kappa^+$, on κ such that*

(i) $f_0(\alpha) = 0$ *and* $f_{\eta+1}(\alpha) = f_\eta(\alpha) + 1$, *for all* $\alpha < \kappa$,
(ii) *if η is a limit ordinal then f_η is a least upper bound of $\{f_\xi : \xi < \eta\}$ in $\le_{I_{\mathrm{NS}}}$.*

The functions are unique up to $=_{I_{\mathrm{NS}}}$, and for every stationary set $S \subset \kappa$, $\|f_\eta\|_S = \eta$.

Proof. Let $\langle \xi_\nu : \nu < \operatorname{cf}\eta\rangle$ be some sequence with limit η. If $\operatorname{cf}\eta < \kappa$, let $f_\eta(\alpha) = \sup\{f_{\xi_\nu}(\alpha) : \nu < \operatorname{cf}\eta\}$, and if $\operatorname{cf}\eta = \kappa$, let $f_\eta(\alpha) = \sup\{f_{\xi_\nu}(\alpha) : \nu < \alpha\}$ (for every limit ordinal α), the *diagonal limit* of f_ξ, $\xi < \eta$. $\square$

For $\eta \ge \kappa^+$, canonical functions may or may not exist. The existence of f_η for all ordinals η is equiconsistent with a measurable cardinal. For the relation between canonical functions and canonical stationary sets, see Exercise 24.10.

A subset A of a partially ordered set $(P, <)$ is *cofinal* if for every $p \in P$ there exists some $a \in A$ such that $p \le a$. The *cofinality* of $(P, <)$ is the smallest size of a cofinal set (it need not be a regular cardinal—see Exercise 24.11). The *true cofinality* of $(P, <)$ is the least cardinality of a cofinal chain (if it exists—see Exercise 24.12). The true cofinality is a regular cardinal (or 1 if P has a greatest element).

Consider again an infinite set A, an ideal I on A, and an indexed set $\{\gamma_a : a \in A\}$ of limit ordinals.

Definition 24.6. A *scale* in $\prod_{a\in A}\gamma_a$ is a $<_I$-increasing transfinite sequence $\langle f_\alpha : \alpha < \lambda\rangle$ of functions in $\prod_{a\in A}\gamma_a$ that is cofinal in $\prod_{a\in A}\gamma_a$ in the partial ordering $<_I$.

If $\prod_{a\in A}\gamma_a$ has a λ-*scale* (i.e., a scale of length λ) and λ is a regular cardinal then it has true cofinality λ, and is λ-*directed*, i.e., every set $B \subset \prod_{a\in A}\gamma_a$ of size $< \gamma$ has an upper bound. The ordinal function $\langle \gamma_a : a \in A\rangle$ is the least upper bound of $\prod_{a\in A}\gamma_a$; moreover, it is an *exact upper bound*:

Definition 24.7. In a partially ordered set $(P, <)$, g is an *exact upper bound* of a set S if S is cofinal in the set $\{f \in P : f < g\}$.

The following theorem is a precursor of the pcf theory. We note that the pcf theory shows, among others, that different sequences $\langle \lambda_n : n < \omega\rangle$ with the same limit will generally result in different cofinalities of $\prod_{n<\omega}\lambda_n$.

Theorem 24.8 (Shelah). *Let κ be a strong limit cardinal of cofinality ω. There exists an increasing sequence $\langle \lambda_n : n < \omega\rangle$ of regular cardinals with limit κ such that the true cofinality of $\prod_{n<\omega}\lambda_n$ modulo the ideal of finite sets is equal to κ^+.*

Proof. Let I be the ideal of finite subsets of ω. We shall find the λ_n's and a κ^+-scale in $\prod_n \lambda_n$ in the partial ordering $<_I$.

First we choose any increasing sequence κ_n, $n < \omega$, of regular cardinals with limit κ. As every subset of $\prod_{n<\omega} \kappa_n$ of size κ has an upper bound in $(\prod_{n<\omega} \kappa_n, <_I)$, we can construct inductively a $<_I$-increasing κ^+-sequence $F = \langle f_\xi : \xi < \kappa^+ \rangle$ of functions in $\prod_n \kappa_n$.

Lemma 24.9. *There exists a function $g : \omega \to \kappa$ that is an upper bound of F in $<_I$, and is $\le_I$-minimal among such upper bounds.*

Proof. Let $g_0 = \langle \kappa_n : n < \omega \rangle$; we shall construct a maximal transfinite $\le_I$-decreasing sequence $\langle g_\nu \rangle_\nu$ of upper bounds of F. It suffices to show that the length of the sequence $\langle g_\nu \rangle_\nu$ is not a limit ordinal: Then the last function is $\le_I$-minimal.

Thus let ϑ be a limit ordinal, and let $\langle g_\nu : \nu < \vartheta \rangle$ be a $\le_I$-decreasing sequence of upper bounds for F. We shall find a function g such that $g >_I f_\xi$ for all $\xi < \kappa^+$, and $g \le_I g_\nu$ for all $\nu < \vartheta$.

First we claim that $|\vartheta| \le 2^{\aleph_0}$. Thus assume that $|\vartheta| \ge (2^{\aleph_0})^+$ and consider the partition $G : [\vartheta]^2 \to \omega$ defined as follows (for $\alpha < \beta$):

$$G(\alpha, \beta) = \text{the least } n \text{ such that } g_\alpha(n) > g_\beta(n).$$

By the Erdős-Rado Partition Theorem 9.6 there exists an infinite set of ordinals $\alpha_0 < \alpha_1 < \alpha_2 < \ldots$ such that for some n, $g_{\alpha_0}(n) > g_{\alpha_1}(n) > g_{\alpha_2}(n) > \ldots$, a contradiction.

Let $A = \bigcup_{\nu<\vartheta} \operatorname{ran}(g_\nu)$ and let $S = A^\omega$. Since $|\vartheta| \le 2^{\aleph_0}$, we have $|S| \le 2^{\aleph_0}$. For every $g \in S$, if g is not an upper bound for F, let ξ_g be such that $f_{\xi_g} \not<_I g$. Since $|S| \le 2^{\aleph_0}$, there is some $\eta < \kappa^+$ greater than all the ξ_g's. Now let

$$g(n) = \text{the least } \gamma \in A \text{ such that } \gamma > f_\eta(n).$$

The function g is an upper bound for F: If not then $f_{\xi_g} \not<_I g$ but $f_{\xi_g} <_I f_\eta <_I g$. We complete the proof of the lemma by showing that $g \le_I g_\nu$ for all $\nu < \vartheta$. If $\nu < \vartheta$ then $g_\nu(n) > f_\eta(n)$ for all but finitely many n and, since $g_\nu(n) \in A$, we have $g_\nu \ge g$. □

Let g be the function given by Lemma 24.9. We claim that g is an exact upper bound of F. If not, let $f <_I g$ be such that $f \not<_I f_\xi$ for all ξ. For each $\xi < \kappa^+$, let A_ξ be the infinite set of all n such that $f(n) > f_\xi(n)$. Since $2^{\aleph_0} < \kappa$, there exists an infinite set A, such that for κ^+ many ξ's, $f(n) > f_\xi(n)$ for all $a \in A$. It follows that $f{\restriction}A >_I f_\xi{\restriction}A$ for every $\xi < \kappa^+$, and therefore the function $g' = f{\restriction}A \cup g{\restriction}(\omega - A) \le_I g$ is an upper bound of F but $g' \ne_I g$, a contradiction.

Now, if g is increasing with limit κ and if every $g(n)$ is a regular cardinal, then we let $\lambda_n = g(n)$ and are done. In general, all but finitely many $g(n)$ are limit ordinals; without loss of generality, all are. For each n, let Y_n be a closed unbounded subset of $g(n)$ whose order-type is a regular cardinal γ_a. Note that

$\sup_n \gamma_n = \kappa$; otherwise, $|\prod_n Y_n| < \kappa$ and hence bounded by some f_ξ. So let $\langle \lambda_n : n < \omega \rangle = \langle \gamma_{k_n} : n < \omega \rangle$ be an increasing subsequence of $\langle \gamma_n \rangle_n$.

For each $f \in F$, let h_f be the function

$$h_f(n) = \text{the least } \alpha \in Y_{k_n} \text{ such that } \alpha \geq f(k_n).$$

and let $H = \{h_f : f \in F\}$. For every $f \in \prod_n Y_n$ there exists some $h \in H$ such that $f <_I h$. Also, $|H| = \kappa^+$ since every smaller set of functions is bounded by some f_ξ. Thus we can find in H a $<_I$-increasing transfinite sequence $\langle h_\xi : \xi < \kappa^+ \rangle$ such that for every $f \in \prod_n Y_n$, there is a ξ with $f <_I h_\xi$. By copying $\prod_n Y_n$ onto $\prod_n \lambda_n$, we get a sequence $\langle h_\xi : \xi < \kappa^+ \rangle$ with the required properties. □

As an application of Theorem 24.8 we give a short proof of Kunen's Theorem 17.7, due to Zapletal [1996].

Assume that $j : V \to M$ is elementary, with critical point κ, and let $\lambda = \lim_n j^n(\kappa)$. As λ is a strong limit cardinal of cofinality ω, let $\langle \lambda_n : n < \omega \rangle$ be an increasing sequence of regular cardinals with limit λ such that $\kappa < \lambda_0$ and that $\prod_n \lambda_n$ has a λ^+-scale $F = \langle f_\xi : \xi < \lambda^+ \rangle$ (modulo finite). Since $j(\lambda) = \lambda$, we have $j(\lambda^+) = \lambda^+$, and $j(F)$ is a λ^+-scale in $\prod_n j(\lambda_n)$.

Since $j``\lambda^+$ is cofinal in $j(\lambda^+) = \lambda^+$, $j``F$ is cofinal in $j(F)$ and thus in $\prod_n j(\lambda_n)$. However, let $g \in \prod_n j(\lambda_n)$ be the function $g(n) = \sup j``\lambda_n$; we have $g(n) < j(\lambda_n)$ because $j(\lambda_n)$ is regular. If $f \in \prod_n \lambda_n$ then $g > j(f)$ pointwise because $j(f) = j``f$. Hence g is an upper bound for $j``F$, a contradiction. □

Toward the pcf theory, we shall now prove several results on ordinal functions and scales. Let I be an ideal on A.

Lemma 24.10. *If $\lambda > 2^{|A|}$ is a regular cardinal then every $<_I$-increasing λ-sequence of ordinal functions on A has an exact upper bound.*

Proof. Let $F = \langle f_\alpha : \alpha < \lambda \rangle$ be $<_I$-increasing. Let M be an elementary submodel of H_ϑ for a sufficiently large ϑ such that $I \in M$, $F \in M$, $|M| = 2^{|A|}$ and $M^{|A|} \subset M$. For every α, let

$$g_\alpha(a) = \text{the least } \beta \in M \text{ such that } \beta \geq f_\alpha(a) \qquad (a \in A).$$

Since $M^{|A|} \subset M$, we have $g_\alpha \in M$, and since $|M| < \lambda$, there exists some $f \in M$ such that $f = g_\alpha$ for λ many α's. Since $\langle f_\alpha \rangle_\alpha$ is increasing and $f \geq_I f_\alpha$ for λ many α's, f is an upper bound of F.

To show that whenever $h <_I f$ then $h <_I f_\alpha$ for some α, it is enough to show this for every $h \in M$. Thus let $h \in M$ be such that $h <_I f$.

Let α be any α such that $f = g_\alpha$. For every $a \in A$ such that $h(a) < g_\alpha(a)$ we necessarily have $h(a) < f_\alpha(a)$ because $h(a) \in M$ and $g_\alpha(a)$ is the least $\beta \in M$ such that $\beta \geq f_\alpha(a)$. Hence $h <_I f_\alpha$. □

If F is a set of ordinal functions on A and g is an upper bound of F, then we say that F is *bounded below* g if it has an upper bound $h <_I g$; F is *cofinal* in g if it is cofinal in $\prod_{a\in A} g(a)$. If $X \in I^+$ then $f <_I g$ on X, etc., means $f <_{I\restriction X} g$ where $I\restriction X$ is the ideal generated by $I \cup \{A - X\}$.

Corollary 24.11. *If $\lambda > 2^{|A|}$ is regular, $F = \langle f_\alpha : \alpha < \lambda\rangle$ is $<_I$-increasing and g is an upper bound of F, then either F is bounded below g, or F is cofinal in g, or $A = X \cup Y$ with $X, Y \in I^+$ such that F is bounded below g on X and is cofinal in g on Y.*

Proof. Let f be an exact upper bound of F and let $X = \{a \in A : f(a) < g(a)\}$. $\square$

Corollary 24.12. *Let $\lambda > 2^{|A|}$ be a regular cardinal, let γ_a, $a \in A$, be limit ordinals, and assume that $\prod_{a\in A} \gamma_a$ is λ-directed in $<_I$. Then either $\prod_{a\in A} \gamma_a$ is λ^+-directed, or has a λ-scale, or $A = X \cup Y$ with $X, Y \in I^+$ such that $\prod_{a\in A} \gamma_a$ has a λ-scale on X and is λ^+-directed on Y.*

Proof. Assume that $\prod_{a\in A} \gamma_a$ is λ-directed but not λ^+-directed, and let $S \subset \prod_{a\in A} \gamma_a$ be such that $|S| = \lambda$ and S is not bounded. Using the λ-directness, we construct an increasing sequence $F = \langle f_\alpha : \alpha < \lambda\rangle$ such that for every $f \in S$, there exists an $\alpha < \lambda$ such that $f <_I f_\alpha$. As F is not bounded, there exists some $Z \in I^+$ such that F is a scale on Z.

Now let $\mathcal{Z}$ be the collection of all $Z \in I^+$ that have a λ-scale, and for each $Z \in \mathcal{Z}$ let $\langle f_\alpha^Z : \alpha < \lambda\rangle$ be a λ-scale on Z. Let $S = \{f_\alpha^Z : \alpha < \lambda, Z \in \mathcal{Z}\}$; since $2^{|A|} = \lambda$, we have $|S| = \lambda$, and we can construct an increasing λ-sequence $F = \langle f_\alpha : \alpha < \lambda\rangle$ such that for every $f \in S$ there is an $\alpha < \lambda$ with $f \le_I f_\alpha$.

Either F is a scale, or $A = X \cup Y$ such that F is bounded on X and cofinal on Y. To complete the proof, we show that $\prod_{a\in A} \gamma_a$ is λ^+-directed; i.e., that for every set of size λ is bounded on X. If not, we repeat the argument above and find a $Z \subset X$ that has a scale. This contradicts the fact that S is bounded on X. $\square$

Definition 24.13. Let $F = \langle f_\alpha : \alpha < \lambda\rangle$, λ regular, be a $<_I$-increasing sequence of ordinal functions on A and let $\gamma < \lambda$ be a regular uncountable cardinal. F is *γ-rapid* if for every $\beta < \lambda$ of cofinality γ there exists a closed unbounded set $C \subset \beta$ such that for every limit ordinal $\alpha < \beta$, $f_\alpha >_I s_{C\cap\alpha}$, where $s_{C\cap\alpha}$ is the pointwise supremum of $\{f_\xi(a) : \xi \in C \cap \alpha\}$:

$$s_{C\cap\alpha}(a) = \sup\{f_\xi(a) : \xi \in C \cap \alpha\} \qquad (a \in A).$$

Lemma 24.14. *Let $F = \langle f_\alpha : \alpha < \lambda\rangle$ be γ-rapid, with $\gamma > |A|$. For each $a \in A$, let $S_a \subset \lambda$ be such that $|S_a| < \gamma$. Then there exists an $\alpha < \lambda$ with the property that for every $h \in \prod_{a\in A} S_a$, if $h >_I f_\alpha$, then h is an upper bound of F.*

Proof. Assume by contradiction that for every $\alpha < \lambda$ there exists an $h \in \prod_{a\in A} S_a$ such that $h >_I f_\alpha$ but h is not an upper bound of F. By induction, we construct a continuous increasing sequence α_ξ, $\xi < \gamma$, and functions $h_\xi \in \prod_{a\in A} S_a$ such that for every ξ, $f_{\alpha_\xi} <_I h_\xi$ and $f_{\alpha_{\xi+1}} \not\leq_I h_\xi$. Let $\beta = \lim_{\xi\to\gamma} \alpha_\xi$.

As F is γ-rapid, there exists a closed unbounded $C \subset \beta$ such that $f_\alpha >_I s_{C\cap\alpha}$ for every $\alpha \in C$. We may assume that $\alpha_\xi \in C$ for every $\xi < \gamma$ (otherwise replace $\{\alpha_\xi\}_{\xi<\gamma}$ by its intersection with C).

For each $\xi < \gamma$ we have $s_{C\cap\alpha_\xi} <_I f_{\alpha_\xi} <_I h_\xi \not\geq_I f_{\alpha_{\xi+1}}$ and so there exists some $a_\xi \in A$ such that

$$s_{C\cap\alpha_\xi}(a_\xi) < f_{\alpha_\xi}(a_\xi) < h_\xi(a_\xi) < f_{\alpha_{\xi+1}}(a_\xi).$$

As $\gamma > |A|$, there exist a set $Z \subset \gamma$ of size γ and some $a \in A$ such that $a_\xi = a$ for all $\xi \in Z$. Now if ξ and η are in Z, such that $\xi+1 < \eta$, then $\alpha_{\xi+1} \in C\cap\alpha_\eta$ and we have

$$h_\xi(a) < f_{\alpha_{\xi+1}}(a) \leq s_{C\cap\alpha_\eta}(a) < h_\eta(a).$$

This is a contradiction because $|S_a| < \gamma$ while $|Z| = \gamma$. □

Corollary 24.15. *If $F = \langle f_\alpha : \alpha < \lambda\rangle$ is γ-rapid, with $|A| < \gamma < \lambda$, and if f is the least upper bound of F, then* $\operatorname{cf} f(a) \geq \gamma$ *for I-almost all $a \in A$.*

Proof. Let f be an upper bound of F, and assume that $B = \{a \in A : \operatorname{cf} f(a) < \gamma\} \in I^+$. We shall find an upper bound h of F such that $h <_I f$ on B.

For $a \in B$, let S_a be a cofinal subset of $f(a)$ of size $< \gamma$. By Lemma 24.14 there is an $\alpha < \lambda$ such that for every $h \in \prod_{a\in B} S_a$, $h >_I f_\alpha$ on B implies that h is an upper bound of F on B. Given this α, we consider a function $h \in \prod_{a\in B} S_a$ as follows: If $f_\alpha(a) < f(a)$, let $h(a) \in S_a$ be such that $f_\alpha(a) < h(a) < f(a)$. The function h is an upper bound of F on B, and $h <_I f$ on B. □

Theorem 24.16 (Shelah). *Let κ be a regular uncountable cardinal, and let $I = I_{\mathrm{NS}}$ be the nonstationary ideal on κ. Let $\langle \eta_\xi : \xi < \kappa\rangle$ be a continuous increasing sequence with limit η. Then $\prod_{\xi<\kappa} \aleph_{\eta_\xi+1}$ has true cofinality $\aleph_{\eta+1}$ (in $<_I$).*

We shall prove this theorem only under the assumption $2^\kappa < \aleph_\eta$ (we only need the weaker version for the proof of Theorem 24.33). For the general proof, see Burke and Magidor [1990].

Proof. Let $\lambda = \aleph_{\eta+1}$. We wish to find a λ-scale. It is not difficult to see that $\prod_{\xi<\kappa} \aleph_{\eta_\xi+1}$ is λ-directed. By Corollary 24.12 (as we assume $2^\kappa < \lambda$), if there is no λ-scale then there is a stationary set $S \subset \kappa$ such that $\prod_{\xi\in S} \aleph_{\eta_\xi+1}$ is λ^+-directed.

We shall construct a $<_I$-increasing λ-sequence in $\prod_{\xi\in S} \aleph_{\eta_\xi+1}$ that is γ-rapid for all regular $\gamma < \aleph_\eta$. For every limit ordinal $\beta < \lambda$, let $C_\beta \subset \beta$ be closed unbounded, of size $\operatorname{cf}\beta$. We construct $F = \langle f_\alpha : \alpha < \lambda\rangle$ by induction. Let α be a limit ordinal. For each limit $\beta > \alpha$, let s_β be the pointwise supremum of $\{f_\nu : \nu \in C_\beta \cap \alpha\}$. For eventually all $\xi < \kappa$, $s_\nu(\xi) < \aleph_{\eta_{\xi+1}}$, so $s_\nu \in \prod_{\xi\in S} \aleph_{\eta_\xi+1}$. Since $\prod_{\xi\in S} \aleph_{\eta_\xi+1}$ is λ^+-directed, we can find f_α so that $f_\alpha >_I s_\beta$ on S for all limit $\beta < \lambda$. This guarantees that F is γ-rapid for every regular uncountable $\gamma < \lambda$.

By Lemma 24.10, F has an exact upper bound g, and without loss of generality, $g(\xi) \le \aleph_{\eta_\xi+1}$ for all $\xi \in S$. We claim that $g(\xi) \ge \aleph_{\eta_\xi+1}$ for almost all $\xi \in S$, and hence F is a scale on S, contrary to the assumption on S. If $g(\xi) < \aleph_{\eta_\xi+1}$ for stationary many ξ, then $\operatorname{cf} g(\xi) < \aleph_{\eta_\xi}$, and hence for some $\gamma < \aleph_{\eta+1}$, $\operatorname{cf} g(\xi) < \gamma$ for stationary many ξ. This contradicts Corollary 24.15, as F is γ-rapid for all $\gamma < \lambda$. □

The pcf Theory

Shelah's pcf theory is the theory of *possible cofinalities* of ultraproducts of sets of regular cardinals. Let A be a set of regular cardinals, and let D be an ultrafilter on A. $\prod A = \prod_{a\in A}\{a : a \in A\}$ denotes the product $\{f : \operatorname{dom}(f) = A$ and $f(a) \in a\}$; the ultraproduct $\prod A/D$ is linearly ordered, and $\operatorname{cof} D = \operatorname{cof} \prod A/D$ is its cofinality.

Definition 24.17. If A is a set of regular cardinals, then

$$\operatorname{pcf} A = \{\operatorname{cof} D : D \text{ is an ultrafilter on } A\}.$$

The set $\operatorname{pcf} A$ is a set of regular cardinals, includes A (for every $a \in A$ consider the principal ultrafilter given by a), has cardinality at most $2^{2^{|A|}}$ and satisfies $\operatorname{pcf}(A_1 \cup A_2) = \operatorname{pcf} A_1 \cup \operatorname{pcf} A_2$.

We shall investigate the structure of pcf in the next section. In this section we explore the relation between pcf and cardinal arithmetic. Instead of the general theory we concentrate on the special case when $A = \{\aleph_n\}_{n=0}^\infty$. We prove the following theorem:

Theorem 24.18 (Shelah). *If $\aleph_\omega$ is a strong limit cardinal then*

$$\max(\operatorname{pcf}\{\aleph_n\}_{n=0}^\infty) = 2^{\aleph_\omega}.$$

A stronger theorem is true: If $2^{\aleph_0} < \aleph_\omega$ then $\max(\operatorname{pcf}\{\aleph_n\}_{n=0}^\infty) = \aleph_\omega^{\aleph_0}$; again, we refer the reader to Burke and Magidor [1990].

We say that a set of regular cardinals A is an *interval* if it contains every regular λ such that $\min A \le \lambda < \sup A$.

Lemma 24.19. *Let A be an interval of regular cardinals such that* $\min A = (2^{|A|})^+$. *Then* $\operatorname{pcf} A$ *is an interval.*

Proof. Let D be an ultrafilter on A and let λ be a regular cardinal such that $\min A \le \lambda < \operatorname{cof} D$. We shall find an ultrafilter E on A such that $\operatorname{cof} E = \lambda$.

Let $\{f_\alpha : \alpha < \operatorname{cof} D\}$ be a D-increasing sequence in $\prod A$. Since $\lambda > 2^{|A|}$, the sequence has a least upper bound g in $\le_D$ (by Lemma 24.10). For each $a \in A$ let $h(a) = \operatorname{cf} g(a)$ and let S_a be a cofinal subset of $g(a)$ of order-type $h(a)$. It is easy to see that $\prod_{a\in A} S_a/D$ has an increasing λ-sequence cofinal in g, and hence $\prod_{a\in A} h(a)/D$ has a cofinal sequence $\{h_\alpha : \alpha < \lambda\}$.

For D-almost all a, $h(a) > 2^{|A|}$: This is because the number of functions from A into $2^{|A|}$ is less than λ. Thus we may assume that $h(a) \in A$ for all $a \in A$. Let E be the ultrafilter on A defined by

$$E = \{X \subset A : h^{-1}(X) \in D\}.$$

We now construct, by induction on α, functions g_α, $\alpha < \lambda$, such that the sequence $\{g_\alpha \circ h : \alpha < \lambda\}$ is D-increasing and cofinal in h. Then $\{g_\alpha : \alpha < \lambda\}$ is E-increasing and cofinal in $\prod A/E$. □

Corollary 24.20. *If $\aleph_\omega$ is a strong limit cardinal, then $\operatorname{pcf}\{\aleph_n\}_{n=0}^\infty$ is an interval and $\sup \operatorname{pcf}\{\aleph_n\}_{n=0}^\infty < \aleph_{\aleph_\omega}$.*

Proof. Apply Lemma 24.19 to the interval $A = [(2^{\aleph_0})^+, \aleph_\omega)$, and use $|\operatorname{pcf} A| \le 2^{2^{\aleph_0}} < \aleph_\omega$. □

Toward the proof of Theorem 24.18, we assume that $\aleph_\omega$ is strong limit and let

$$\lambda = \sup \operatorname{pcf}\{\aleph_n\}_{n=0}^\infty.$$

We shall show that $2^{\aleph_\omega} = \lambda$. Since $\operatorname{cf} 2^{\aleph_\omega} > \aleph_\omega$ (by König's Theorem) and $\lambda < \aleph_{\aleph_\omega}$, it follows that $2^{\aleph_\omega}$ is a successor cardinal, and therefore $2^{\aleph_\omega} = \max(\operatorname{pcf}\{\aleph_n\}_{n=0}^\infty)$.

Lemma 24.21. *There exists a family F of functions in $\prod_{n=0}^\infty \aleph_n$, $|F| = \lambda$, such that for every $g \in \prod_{n=0}^\infty \aleph_n$ there is some $f \in F$ with $g(n) \le f(n)$ for all n.*

Proof. For every ultrafilter D on ω choose a sequence $\langle f_\alpha^D : \alpha < \operatorname{cof} D\rangle$ that is cofinal in $\prod_{n=0}^\infty \aleph_n/D$, and let F be the set of all $f = \max\{f_{\alpha_1}^{D_1}, \ldots, f_{\alpha_m}^{D_m}\}$ where $\{D_1, \ldots, D_m\}$ is a finite set of ultrafilters and $\{\alpha_1, \ldots, \alpha_m\}$ a finite set of ordinals. Since $\lambda > \aleph_\omega > 2^{2^{\aleph_0}}$, we have $|F| = \lambda$.

Assume, by contradiction, that there is a $g \in \prod_{n=0}^\infty \aleph_n$ that is not majorized by any $f \in F$. Thus if we let, for every D and every α, $X_\alpha^D = \{n : g(n) > f_\alpha^D(n)\}$, then the family $\{X_\alpha^D\}_{\alpha,D}$ has the finite intersection property, and so extends to an ultrafilter U. Then $g <_U f_\alpha^U$ for some α, a contradiction. □

Let us fix such a family F of size λ, and let $k < \omega$ be such that $2^{\aleph_0} \le \aleph_k$ and $\lambda < \aleph_{\aleph_k}$. Let ϑ be sufficiently large, and consider elementary submodels of

$(H_\vartheta, \in, <)$ where $<$ is some well-ordering of H_ϑ. For every countable subset a of $\aleph_\omega$ we shall construct an elementary chain of models M^a_α, of length ω_k. Each M^a_α will have size $\aleph_k$ and will be such that $M^a_\alpha \supset a \cup \omega_k$.

We choose M^a_0 of size $\aleph_k$ so that $M^a_0 \supset a \cup \omega_k$. If $\alpha < \omega_k$ is a limit ordinal, we let $M^a_\alpha = \bigcup_{\beta<\alpha} M^a_\beta$. Given M^a_α, we find $M^a_{\alpha+1}$ as follows. Let

$$\chi^a_\alpha(n) = \sup(M^a_\alpha \cap \omega_n) \qquad (\text{all } n > k), \tag{24.6}$$

the *characteristic function* of M^a_α. There exists a function $f^a_\alpha \in F$ such that $f^a_\alpha(n) \geq \chi^a_\alpha(n)$ for all $n > k$; let $M^a_{\alpha+1}$ be such that $f^a_\alpha \in M^a_{\alpha+1}$.

Then we let $M^a = \bigcup_{\alpha<\omega_k} M^a_\alpha$, and

$$\chi^a(n) = \sup(M^a \cap \omega_n) \qquad (\text{all } n > k).$$

Lemma 24.22. *If a and b are countable subsets of $\aleph_\omega$ and if $\chi^a = \chi^b$, then $M^a \cap \aleph_\omega = M^b \cap \aleph_\omega$.*

Proof. By induction on n we show that $M^a \cap \aleph_n = M^b \cap \aleph_n$, for all $n \geq k$. This is true for $n = k$; thus assume that this is true for n and prove it for $n+1$. Both $M^a \cap \aleph_{n+1}$ and $M^b \cap \aleph_{n+1}$ contain a closed unbounded subset of the ordinal $\chi^a(n+1) = \chi^b(n+1)$ (of cofinality $\aleph_k$), and so there is a cofinal subset C of this ordinal such that $C \subset M^a$ and $C \subset M^b$. For every $\gamma \geq \omega_n$ in C there is a one-to-one function π that maps ω_n onto γ. If we let π be the $\prec$-least such function in H_ϑ, then π is both in M^a and in M^b. It follows that $\gamma \cap M^a = \gamma \cap M^b$. Consequently, $\omega_{n+1} \cap M^a = \omega_{n+1} \cap M^b$ and the lemma follows. $\square$

We shall complete the proof of Theorem 24.18 by showing that the set $\{\chi^a : a \subset \aleph_\omega \text{ countable}\}$ has size at most λ. Since each M^a has $\aleph_k$ countable subsets it will follow that there are at most λ countable subsets of $\aleph_\omega$, and therefore $2^{\aleph_\omega} = \lambda$.

For each a and each n we have

$$\chi^a(n) = \sup\nolimits_{\alpha<\omega_k} \chi^a_\alpha(n) = \sup\nolimits_{\alpha<\omega_k} f^a_\alpha(n).$$

If S is any subset of ω_k of size $\aleph_k$, then $\chi^a(n) = \sup\{f^a_\alpha(n) : \alpha \in S\}$ and so the set $\{f^a_\alpha : \alpha \in S\}$ determines χ^a.

Lemma 24.23. *There exists a family F_λ of λ subsets of λ, each of size $\aleph_k$, such that for every subset $Z \subset \lambda$ of size $\aleph_k$ there exists an $X \in F_\lambda$ such that $X \subset Z$.*

Proof. We prove (by induction on α) that for every ordinal α such that $2^{\aleph_k} \leq \alpha \leq \lambda$ there is a family $F_\alpha \subset [\alpha]^{\aleph_k}$, $|F_\alpha| \leq |\alpha|$ such that for every $Z \in [\alpha]^{\aleph_k}$ there is an $X \in F_\alpha$ such that $X \subset Z$. This is true for $\alpha = 2^{\aleph_k}$. If α is not a cardinal, then F_α can be obtained by a one-to-one transformation from $F_{|\alpha|}$. If α is a cardinal then since $\alpha \leq \lambda < \aleph_{\aleph_k}$, we have $\operatorname{cf}\alpha \neq \aleph_k$, and it follows that $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$ has the required property. $\square$

Now we complete the proof of Theorem 24.18. For each countable subset a of $\aleph_\omega$ let $Z_a = \{f^a_\alpha : \alpha < \omega_k\}$; each Z_a is a subset of F, and $|Z| = \aleph_k$. Apply Lemma 24.23 to the set F (instead of λ) and obtain a family $F_\lambda \subset [F]^{\aleph_k}$ such that for each a there exists some $X \in F_\lambda$ such that $X \subset Z$. Since $|X| = \aleph_k$, X determines χ^a. It follows that $|\{\chi^a : a \subset \aleph_\omega \text{ countable}\}| \le \lambda$. $\square$

The Structure of pcf

Let A be a set of regular cardinals and let $\operatorname{pcf} A$ denote the set of all possible cofinalities of $\prod A$. First we mention some facts about pcf:

(24.7)
- (i) $A \subset \operatorname{pcf} A$.
- (ii) If $A_1 \subset A_2$ then $\operatorname{pcf} A_1 \subset \operatorname{pcf} A_2$.
- (iii) $\operatorname{pcf}(A_1 \cup A_2) = \operatorname{pcf} A_1 \cup \operatorname{pcf} A_2$.
- (iv) $|\operatorname{pcf} A| \le 2^{2^{|A|}}$.
- (v) $\sup \operatorname{pcf} A \le |\prod A|$.

In Lemma 24.19 we showed:

- (vi) If A is an interval and $2^{|A|} < \min A$ then $\operatorname{pcf} A$ is an interval.

This is true in general, under the assumption $|A| < \min A$ (see Shelah [1994]).

In the following Lemma 24.24 we prove

- (vii) If $|\operatorname{pcf} A| < \min A$ then $\operatorname{pcf}(\operatorname{pcf} A) = \operatorname{pcf} A$.

Finally, Theorem 24.18 is true in general, and under weaker assumptions; we state this without a proof.

- (viii) If A is an interval without a greatest element and $(\min A)^{|A|} < \sup A$, then $(\sup A)^{|A|} = \max \operatorname{pcf} A$.

For proof, see e.g. Burke and Magidor [1990].

Lemma 24.24. *If* $|\operatorname{pcf} A| < \min A$ *then* $\operatorname{pcf}(\operatorname{pcf} A) = \operatorname{pcf} A$.

Proof. Let $B = \operatorname{pcf} A$. For each $\lambda \in B$ choose D_λ on A such that $\operatorname{cof} D_\lambda = \lambda$, and let $\langle f^\lambda_\alpha : \alpha < \lambda\rangle$ be cofinal in $\prod A/D_\lambda$. Let $\mu \in \operatorname{pcf} B$; choose D on B with $\operatorname{cof} D = \mu$, and let $\langle g_\alpha : \alpha < \mu\rangle$ be cofinal in $\prod B/D$. Let

$$E = \{X \subset A : \{\lambda \in B : X \in D_\lambda\} \in D\}.$$

E is an ultrafilter on A and we shall show that $\operatorname{cof} E = \mu$, thus proving $\mu \in \operatorname{pcf} A$, and hence $\operatorname{pcf} B = B$.

For every $\alpha < \mu$, let

$$h_\alpha(a) = \sup_{\lambda\in B} f^\lambda_{g_\alpha(\lambda)}(a) \qquad (\text{all } a \in A).$$

Since $\min A > |B|$, we have $h_\alpha(a) < a$ for all $a \in A$. We will show that for each $h \in \prod A$, eventually all h_α are $\geq_E h$. The we can find a subsequence of $\langle h_\alpha : \alpha < \mu\rangle$ that is cofinal in $\prod A/E$.

Let $h \in \prod A$. For each $\lambda \in B$ there exists a $g(\lambda) < \lambda$ such that $h <_{D_\lambda} f^\lambda_{g(\lambda)}$. For eventually all $\alpha < \mu$ we have $g <_D g_\alpha$, and we claim that whenever $g <_D g_\alpha$ then $h <_E h_\alpha$.

Let α be such that $g <_D g_\alpha$. Let $X = \{a \in A : h(a) < h_\alpha(a)\}$. If λ is such that $g(\lambda) < g_\alpha(\lambda)$ then for D_λ-almost all a, $h(a) < f^\lambda_{g(\lambda)}(a) < f^\lambda_{g_\alpha(\lambda)}(a) \leq h_\alpha(a)$ and hence $a \in X$. Thus $X \in D_\lambda$ for D-almost all λ, and so $X \in E$. $\square$

The fundamental theorem of the pcf theory is the following.

Theorem 24.25 (Shelah). *If A is a set of regular cardinals such that $2^{|A|} < \min A$, then there exist sets $B_\lambda \subset A$, $\lambda \in \operatorname{pcf} A$, such that for every $\lambda \in \operatorname{pcf} A$*

(a) $\lambda = \max \operatorname{pcf} B_\lambda$.
(b) $\lambda \notin \operatorname{pcf}(A - B_\lambda)$.
(c) $\prod\{a : a \in B_\lambda\}$ *has a λ-scale* mod J_λ *where J_λ is the ideal generated by the sets B_ν, $\nu < \lambda$.*

(To see that J_λ is an ideal, we observe that if $X \in J_\lambda$ then $X \subset B_{\nu_1} \cup \ldots \cup B_{\nu_k}$, hence $\operatorname{pcf} X \subset \operatorname{pcf} B_{\nu_1} \cup \ldots \cup \operatorname{pcf} B_{\nu_k}$ and so by (a), $\lambda \notin \operatorname{pcf} X$. Hence $X \neq A$.)

The theorem is true under the weaker assumption $|A| < \min A$; see Shelah [1994] or Burke and Magidor [1990].

Note that (a) and (b) can be formulated as follows:

(a) For every ultrafilter D on B_λ, $\operatorname{cof} D \leq \lambda$; and there exists some D on B_λ such that $\operatorname{cof} D = \lambda$.
(b) For every ultrafilter D on A, if $\operatorname{cof} D = \lambda$ then $B_\lambda \in D$.

The sets B_λ, $\lambda \in \operatorname{pcf} A$, are called the *generators* of $\operatorname{pcf} A$. It follows from (a) and (b) that the cofinality of an ultrafilter on A is determined by which generators it contains:

$$\operatorname{cof} D = \text{the least } \lambda \text{ such that } B_\lambda \in D. \tag{24.8}$$

Corollary 24.26. *If $2^{|A|} < \min A$ then $|\operatorname{pcf} A| \leq 2^{|A|}$.*

Proof. The number of generators is at most $2^{|A|}$. $\square$

Corollary 24.27. *If $\aleph_\omega$ is strong limit then $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$.*

Proof. Corollary 24.26, Corollary 24.20 and Theorem 24.18. $\square$

Corollary 24.28. *If* $2^{|A|} < \min A$ *then* $\operatorname{pcf} A$ *has a greatest element.*

Proof. Assume that $\operatorname{pcf} A$ does not have a greatest element. Then the set $\{A - B_\lambda : \lambda \in \operatorname{pcf} A\}$ has the finite intersection property, and so extends to an ultrafilter D. By (b), $B_{\operatorname{cof} D} \in D$, a contradiction. □

Proof of Theorem 24.25. We shall apply the results on ordinal functions proved earlier in this chapter. If I is an ideal on a set A of regular cardinals then we say that I has a λ-scale if $\prod A$ has a λ-scale in $<_I$; similarly, we say that I is λ-*directed* if $\prod A$ is λ-directed in $\leq_I$.

We construct the generators B_λ by induction, so that for each cardinal $\kappa \leq \sup \operatorname{pcf} A$ the following conditions are satisfied:

(24.9)
- (i) the ideal J_κ generated by $\{B_\lambda : \lambda < \kappa$ and $\lambda \in \operatorname{pcf} A\}$ is κ-directed;
- (ii) if $\kappa \notin \operatorname{pcf} A$ then J_κ is κ^+-directed;
- (iii) if $\kappa \in \operatorname{pcf} A$ and κ is not a maximal element of $\operatorname{pcf} A$ then there exists a $B_\kappa \in J_\kappa^+$ such that J_κ has a κ-scale on B_κ and $J_\kappa[B_\kappa]$, the ideal generated by $J_\kappa \cup \{B_\kappa\}$, is a κ^+-directed ideal;
- (iv) if $\kappa = \max(\operatorname{pcf} A)$ then J_κ has a κ-scale on A (and we let $B_\kappa = A$).

If the conditions (24.9) are satisfied, then the sets B_λ satisfy Theorem 24.25:

To prove (a), let $\lambda \in \operatorname{pcf} A$. Choose an ultrafilter D on B_λ that extends the dual filter of J_λ. J_λ has a λ-scale on B_λ, and this scale is also a scale for $<_D$; therefore $\operatorname{cof} D = \lambda$, and so $\lambda \in \operatorname{pcf} B_\lambda$. Also, if D is any ultrafilter on B_λ, then either $D \cap J_\lambda = \emptyset$ in which case $\operatorname{cof} D = \lambda$, or else there is some $\nu < \lambda$ such that $B_\nu \in D$. If ν is the least such ν then D is an ultrafilter on B_ν and $D \cap J_\nu = \emptyset$. Since J_ν has a ν-scale on B_ν, we have $\operatorname{cof} D = \nu$. In either case, $\operatorname{cof} D \leq \lambda$.

To prove (b), let D be an ultrafilter on A such that $B_\lambda \notin D$; we claim that $\operatorname{cof} D \neq \lambda$. Either $D \ni B_\lambda$ for some $\nu < \lambda$ in which case $\operatorname{cof} D < \lambda$, or else $D \cap J_\lambda[B_\lambda] \neq \emptyset$, and since $J_\lambda[B_\lambda]$ is λ^+-directed, D is λ^+-directed, and we have $\operatorname{cof} D > \lambda$.

Finally, (c) follows from (24.9)(iii) and (iv). We prove (24.9) by induction on $\kappa \leq \sup \operatorname{pcf} A$:

(i) If $\kappa \leq \min A$ then $J_\kappa = \{\emptyset\}$ is κ-directed. If κ is a limit cardinal, then $J_\kappa = \bigcup_{\lambda<\kappa} J_\lambda$ and the claim follows easily. If $\kappa = \lambda^+$ then either $\lambda \notin \operatorname{pcf} A$ and $J_\kappa = J_\lambda$ is λ^+-directed by (ii), or $\lambda \in \operatorname{pcf} A$ and $J_\kappa = J_\lambda[B_\lambda]$ is λ^+-directed by (iii).

(ii) Let $\kappa \notin \operatorname{pcf} A$ and $\kappa \geq \min A$; hence $\kappa > 2^{|A|}$. If κ is singular, then it is easy to see that since J_κ is κ-directed, it is κ^+-directed. If κ is regular, assume by contradiction that J_κ is κ-directed but not κ^+-directed. By Corollary 24.12, J_κ has a κ-scale on some $X \in J_\kappa^+$. Let D be any ultrafilter on X such that $D \cap J_\kappa = \emptyset$. Then $\operatorname{cof} D = \kappa$ and so $\kappa \in \operatorname{pcf} A$, a contradiction.

(iii) Let $\kappa \in \operatorname{pcf} A$ be such that $\kappa < \sup \operatorname{pcf} A$. We claim that J_κ is not κ^+-directed and that J_κ does not have a κ-scale on A. Then a B_κ exists by Corollary 24.12. Assume by contradiction that J_κ is κ^+-directed, and let D be any ultrafilter on A. If $D \ni B_\lambda$ for some $\lambda < \kappa$, then $\operatorname{cof} D < \kappa$. Otherwise, $D \cap J_\kappa = \emptyset$ and since J_κ is κ^+-directed, D is κ^+-directed and so $\operatorname{cof} D > \kappa$. In either case $\operatorname{cof} D \neq \kappa$, hence $\kappa \notin \operatorname{pcf} A$, a contradiction.

Now assume that J_κ does have a κ-scale on A. Then for every ultrafilter D on A, either $D \ni B_\lambda$ for some $\lambda < \kappa$, and then $\operatorname{cof} D < \kappa$, or $D \cap J_\kappa = \emptyset$, so D has a κ-scale and $\operatorname{cof} D = \kappa$. Hence $\kappa = \max(\operatorname{pcf} A)$, a contradiction.

(iv) Let $\kappa = \max(\operatorname{pcf} A)$ and again assume, by contradiction, that J_κ does not have a scale on A. Then by Corollary 24.12 there exists a $Y \in J_\kappa^+$ such that $J_\kappa[Y]$ is κ^+-directed. If D is any ultrafilter on A such that $D \cap J_\kappa[Y] = \emptyset$ then $<_D$ is κ^+-directed and so $\operatorname{cof} D > \kappa$. Hence κ is not the maximal element of $\operatorname{pcf} A$, a contradiction. □

The same argument that shows that $\operatorname{pcf} A$ has a greatest element yields the following property of pcf, called *compactness*:

Corollary 24.29. *Let B_λ, $\lambda \in \operatorname{pcf} A$, be generators of $\operatorname{pcf} A$. For every $X \subset A$ there exists a finite set $\{\nu_1, \dots, \nu_k\} \subset \operatorname{pcf} X$ such that $X \subset B_{\nu_1} \cup \ldots \cup B_{\nu_k}$.*

Proof. Assume the contrary. Then $\{X - B_\nu : \nu \in \operatorname{pcf} X\}$ has the finite intersection property and there exists an ultrafilter D on X such that $B_\nu \notin D$ for all $\nu \in \operatorname{pcf} X$. If $\lambda = \operatorname{cof} D$ then $B_\lambda \in D$ by Theorem 24.25(b), a contradiction. □

We conclude this section with the following improvement of Theorem 24.16:

Corollary 24.30. *Let κ be a regular uncountable cardinal, and let $\aleph_\eta$ be a singular cardinal of cofinality κ such that $2^\kappa < \aleph_\eta$. Then there is a closed unbounded set $C \subset \eta$ such that $\max(\operatorname{pcf}\{\aleph_{\alpha+1} : \alpha \in C\}) = \aleph_{\eta+1}$; $\prod_{\alpha \in C} \aleph_{\alpha+1}$ has true cofinality $\aleph_{\eta+1}$ mod I where I is the ideal of all bounded subsets of C.*

Proof. Let C_0 be any closed unbounded subset of η of order-type κ such that $2^\kappa < \aleph_{\alpha_0}$ where $\alpha_0 = \min C_0$. Let $A_0 = \{\aleph_{\alpha+1} : \alpha \in C_0\}$, let $\lambda = \aleph_{\eta+1}$, and let B_λ be a generator for $\operatorname{pcf} A_0$, for this λ (by Theorem 24.16, $\lambda \in \operatorname{pcf} A_0$). Let $X = \{\alpha \in C_0 : \aleph_{\alpha+1} \in B_\lambda\}$. If D is any ultrafilter on C_0 that extends the closed unbounded filter, then by Theorem 24.16, $\operatorname{cof} \prod_{\alpha \in C_0} \aleph_{\alpha+1}/D = \lambda$, and by Theorem 24.25(b), $X \in D$. Thus X contains a closed unbounded set C. Let $A = \{\aleph_{\alpha+1} : \alpha \in C\}$. By Theorem 24.25(a), $\max(\operatorname{pcf} A) \leq \lambda$, and therefore $= \lambda$.

Now let B_ν, $\nu \leq \lambda$, denote the generators of $\operatorname{pcf} A$. Every B_ν for $\nu < \lambda$ is a bounded subset of A and so the ideal of all bounded subsets of A extends J_λ, the ideal generated by the B_ν, $\nu < \lambda$. Thus $\prod_{\alpha \in C} \aleph_{\alpha+1}/I$ has a λ-scale. □

Transitive Generators and Localization

Let A be a set of regular cardinals with $2^{|A|} < \min A$, let B_λ, $\lambda \in \operatorname{pcf} A$, be generators for $\operatorname{pcf} A$, and let J_κ be, for each $\kappa \le \max(\operatorname{pcf} A)$, the ideal generated by $\{B_\lambda : \lambda < \kappa\}$. The following shows that the ideals J_κ are independent of the choice of generators for $\operatorname{pcf} A$:

(24.10) For every $X \subset A$, $X \in J_\kappa$ if and only if $\operatorname{cof} D < \kappa$ for every ultrafilter D on X.

To see this, note first that if $X \in J_\kappa$ then $X \subset B_{\nu_1} \cup \ldots \cup B_{\nu_k}$ for some $\nu_1, \ldots, \nu_k < \kappa$, and so $\max(\operatorname{pcf} X) < \kappa$. Conversely, if $X \notin J_\kappa$ then the set $\{X - B_\lambda : \lambda < \kappa\}$ has the finite intersection property, and so there exists an ultrafilter D on X such that $B_\lambda \notin D$ for all $\lambda < \kappa$. By Theorem 24.25(b), $\operatorname{cof} D \ge \kappa$. Each generator B_λ is uniquely determined up to equivalence mod J_λ; if B is any set such that $B \triangle B_\lambda \in J_\lambda$, then B also satisfies (a) and (b) of Theorem 24.25. To see this, note that by (24.10), if $X \triangle Y \in J_\lambda$ then $\operatorname{pcf} X - \lambda = \operatorname{pcf} Y - \lambda$; thus $\max \operatorname{pcf} B = \lambda$ and $\lambda \notin \operatorname{pcf}(A - B)$.

We shall now produce generators for pcf that are *transitive*:

Lemma 24.31 (Transitive Generators). *Let A be a set of regular cardinals such that $A = \operatorname{pcf} A$ and $(2^{|A|})^+ < \min A$. There exist generators B_λ, $\lambda \in A$, for* pcf A *with the property*

$$\text{(24.11)} \qquad \textit{if } \mu \in B_\lambda \textit{ then } B_\mu \subset B_\lambda.$$

In other words, the relation "$\mu \in B_\lambda$" of μ and λ is transitive. The lemma holds under weaker assumptions on A; see Shelah [1994].

Proof. Let B_λ, $\lambda \in A$, be generators for $\operatorname{pcf} A$. We shall replace each B_λ by an equivalent generator $\overline{B_\lambda}$ so that (24.11) is satisfied.

For each $\lambda \in A$ there exists a sequence $\langle f^\lambda_\alpha : \alpha < \lambda\rangle$ of functions in $\prod A$ that is $<_{J_\lambda}$-increasing and is cofinal on B_λ. Moreover, by Lemma 24.10 we may assume that for each λ and each α of cofinality greater than $2^{|A|}$, f^λ_α is an exact upper bound of $\{f^\lambda_\beta : \beta < \alpha\}$.

Let $\kappa = (2^{|A|})^+$. Let ϑ be sufficiently large, and consider elementary submodels of $(H_\vartheta, \in, <)$ where $<$ is some well-ordering of H_ϑ. Consider a continuous elementary chain

$$M_0 \prec M_1 \prec \ldots \prec M_\eta \prec \ldots \prec M_\kappa = M \prec H_\vartheta$$

of models, each of size κ, such that M_0 contains A, each $\lambda \in A$, all subsets of A, each $\langle f^\lambda_\alpha : \alpha < \lambda\rangle$, every function from a subset of A into $A^{<\omega}$, and such that

$$\text{(24.12)} \qquad \langle M_\xi : \xi \le \eta\rangle \in M_{\eta+1} \qquad (\text{all } \eta < \kappa).$$

Let χ_η, $\eta \le \kappa$, be the *characteristic functions* of M_η:

(24.13) $$\chi_\eta(\lambda) = \sup(M_\eta \cap \lambda) \qquad (\text{for all } \lambda \in A),$$

and let $\chi = \chi_\kappa$, the characteristic function of M. Each χ_η ($\eta < \kappa$) belongs to $M_{\eta+1}$ and therefore to M. If $\xi < \eta$ then $\chi_\xi(\lambda) < \chi_\eta(\lambda)$ for all $\lambda \in A$, and $\langle \chi_\eta(\lambda) : \eta < \kappa \rangle$ is an increasing continuous sequence with limit $\chi(\lambda) < \lambda$.

We claim that for each $\lambda \in A$, χ is the $<_{J_\lambda}$-exact upper bound of $\langle f^\lambda_\alpha : \alpha \in M \cap \lambda \rangle$ on B_λ and consequently,

(24.14) $$f^\lambda_{\chi(\lambda)}(\mu) = \chi(\mu) \qquad \text{for } J_\lambda\text{-almost all } \mu \in B_\lambda.$$

If $\alpha \in M \cap \lambda$ then $f^\lambda_\alpha \in M$ and so $f^\lambda_\alpha(\mu) < \chi(\mu)$ for all $\mu \in A$. Hence χ is an upper bound of $\langle f^\lambda_\alpha : \alpha \in M \cap \lambda \rangle$. To show that χ is the $<_{J_\lambda}$-exact upper bound on B_λ, it suffices to show that for each $\eta < \kappa$, $\chi_\eta <_{J_\lambda} f^\lambda_\alpha$ on B_λ for some $\alpha \in M \cap \lambda$, since χ is the pointwise supremum of $\{\chi_\eta : \eta < \kappa\}$, and $|A| < \kappa$. Thus let $\eta < \kappa$; there exists an $\alpha < \lambda$ such that $\chi_\eta <_{J_\lambda} f^\lambda_\alpha$ on B_λ, and since M is an elementary submodel, there exists such an α in M.

Since $\operatorname{cf} \chi(\lambda) = \kappa > 2^{|A|}$, $f^\lambda_{\chi(\lambda)}$ is a $<_{J_\lambda}$-exact upper bound of $\{f^\lambda_\alpha : \alpha \in M \cap \lambda\}$ on B_λ, and (24.14) follows.

Now we let, for each $\lambda \in A$,

(24.15) $$B^*_\lambda = \{\mu \in B_\lambda : f^\lambda_{\chi(\lambda)}(\mu) = \chi(\mu)\};$$

if follows from (24.14) that B^*_λ is J_λ-equivalent to B_λ.

The transitive generators $\overline{B_\lambda}$ are defined as follows:

(24.16) $\nu \in \overline{B_\lambda}$ if and only if there exists a finite increasing sequence (with $k \ge 0$) $\langle \nu_0, \dots, \nu_k \rangle$ such that $\nu_0 = \nu$, $\nu_k = \lambda$ and $\nu_i \in B^*_{\nu_{i+1}}$ for every $i = 0, \dots, k-1$.

It is clear that $\overline{B_\lambda}$ is transitive, $B^*_\lambda \subset \overline{B_\lambda}$, and $\lambda = \max \overline{B_\lambda}$. It remains to prove that $\overline{B_\lambda}$ is J_λ-equivalent to B_λ; for that it suffices to show that $\overline{B_\lambda} \in J_{\lambda^+} = J_\lambda[B_\lambda]$.

For each $\nu \in \overline{B_\lambda}$, fix a finite sequence $\varphi(\nu) = \langle \nu_0, \dots, \nu_k \rangle$ to satisfy (24.16). Note that the function φ on $\overline{B_\lambda}$ belongs to M. Let $\langle g_\alpha : \alpha < \lambda \rangle$ be the λ-sequence of functions in $\prod A$ defined as follows:

If $\nu \notin \overline{B_\lambda}$, we let $g_\alpha(\nu) = 0$. If $\nu \in \overline{B_\lambda}$ then $\varphi(\nu) = \langle \nu_0, \dots, \nu_k \rangle$ with $\nu_0 = \nu$ and $\nu_k = \lambda$, and we consider the sequence $\langle \beta_0, \dots, \beta_k \rangle$, where $\beta_i < \nu_i$ for each i, obtained as follows (by descending induction):

(24.17) $$\begin{aligned} \beta_k &= \alpha, \\ \beta_i &= f^{\nu_{i+1}}_{\beta_{i+1}}(\nu_i) \qquad (i = k-1, \dots, 0). \end{aligned}$$

and let $g_\alpha(\nu) = \beta_0$.

As M is an elementary submodel and $\varphi \in M$, the sequence $\langle g_\alpha : \alpha < \lambda \rangle$ is defined in M. Since J_{λ^+} is λ^+-directed, there exists a function $g \in \prod A$

such that $g_\alpha < g \bmod J_{\lambda^+}$ for every $\alpha < \lambda$. Since $M \prec H_\vartheta$, such a function g exists in M. Since $g \in M$, we have $g(\nu) < \chi(\nu)$ for all ν and therefore $g_\alpha < \chi \bmod J_{\lambda^+}$ for every $\alpha < \lambda$.

Now let $\alpha = \chi(\lambda)$. We shall finish the proof by showing that $g_\alpha(\nu) = \chi(\nu)$ for every $\nu \in \overline{B_\lambda}$. This implies that $\overline{B_\lambda} \in J_{\lambda^+}$.

So let $\nu \in \overline{B_\lambda}$. Let $\langle \nu_0, \ldots, \nu_k \rangle = \varphi(\nu)$, and let $\langle \beta_0, \ldots, \beta_k \rangle$ be the sequence obtained in (24.17) for $\alpha = \chi(\lambda)$. We claim that for each i, $\beta_i = \chi(\nu_i)$, and therefore $g_\alpha(\nu) = \beta_0 = \chi(\nu_0) = \chi(\nu)$.

For each i we have $\nu_i \in B^*_{\nu_{i+1}}$, and so by (24.15), $f^{\nu_{i+1}}_{\chi(\nu_{i+1})}(\nu_i) = \chi(\nu_i)$. For $i = k$, we have $\beta_k = \alpha = \chi(\lambda) = \chi(\nu_k)$, and then for each $i = k-1, \ldots, 0$, we have by (24.17)

$$\beta_i = f^{\nu_{i+1}}_{\beta_{i+1}}(\nu_i) = f^{\nu_{i+1}}_{\chi(\nu_{i+1})}(\nu_i) = \chi(\nu_i). \qquad \square$$

Using transitive generators we now prove the *Localization Lemma*:

Lemma 24.32 (Localization). *Let A be a set of regular cardinals such that $2^{|\operatorname{pcf} A|} < \min A$, let $X \subset \operatorname{pcf} A$ and let $\lambda \in \operatorname{pcf} X$. There exists a set $W \subset X$ such that $|W| \le |A|$ and such that $\lambda \in \operatorname{pcf} W$.*

Again, the Localization Lemma holds under the weaker assumption $|\operatorname{pcf} A| < \min A$.

Proof. First, since $2^{|X|} < \min X$, there exist generators for $\operatorname{pcf} X$, and in particular there exists a set $Y \subset X$ with $\max(\operatorname{pcf} Y) = \lambda$. Let $\overline{A} = \operatorname{pcf} A$. By (24.7)(vii) we have $\operatorname{pcf} \overline{A} = \overline{A}$, and since $2^{|\overline{A}|} < \min A$, we can find transitive generators B_ν, $\nu \in \overline{A}$, for $\operatorname{pcf} \overline{A}$.

For every $\nu \in Y$, let $B^A_\nu = B_\nu \cap A$. Since $Y \subset \operatorname{pcf} A$, there exists an ultrafilter D on A with $\operatorname{cof} D = \nu$, and by Theorem 24.25, $B_\nu \in D$. Hence $\nu \in \operatorname{pcf} B^A_\nu$. Let

$$E = \bigcup\{B^A_\nu : \nu \in Y\}.$$

Since $\nu \in \operatorname{pcf} E$ for every $\nu \in Y$, we have $Y \subset \operatorname{pcf} E$, hence $\operatorname{pcf} Y \subset \operatorname{pcf}\operatorname{pcf} E$, and since (by (24.7)(vii)) $\operatorname{pcf}\operatorname{pcf} E = \operatorname{pcf} E$, we have $\operatorname{pcf} Y \subset \operatorname{pcf} E$. In particular, $\lambda \in \operatorname{pcf} E$.

Since $E \subset A$, there exists a set $W \subset Y$ of size $\le |A|$ such that $E \subset \bigcup\{B^A_\nu : \nu \in W\}$. We shall prove that $\lambda \in \operatorname{pcf} W$.

Assume, by contradiction, that $\lambda \notin \operatorname{pcf} W$. By compactness (Corollary 24.29) there exist $\lambda_1, \ldots, \lambda_n \in \operatorname{pcf} W$ such that $W \subset B_{\lambda_1} \cup \ldots \cup B_{\lambda_n}$, and since $\max \operatorname{pcf} W \le \max \operatorname{pcf} Y = \lambda$, we have $\lambda_i < \lambda$ for all $i = 1, \ldots, n$. Now

$$E \subset \bigcup\{B_\nu : \nu \in W\} \subset \bigcup\{B_\nu : \nu \in B_{\lambda_1}\} \cup \ldots \cup \bigcup\{B_\nu : \nu \in B_{\lambda_n}\},$$

and since, by transitivity (Lemma 24.31), $\bigcup_{\nu \in B_\mu} B_\nu \subset B_\mu$ for every μ, we have

$$E \subset B_{\lambda_1} \cup \ldots \cup B_{\lambda_n}.$$

It follows that $\operatorname{pcf} E \subset \operatorname{pcf}(B_{\lambda_1} \cup \ldots \cup B_{\lambda_n}) = \operatorname{pcf} B_{\lambda_1} \cup \ldots \cup \operatorname{pcf} B_{\lambda_n}$, and so $\max(\operatorname{pcf} E) \le \max\{\lambda_1, \ldots, \lambda_n\} < \lambda$, a contradiction. $\square$

Shelah's Bound on $2^{\aleph_\omega}$

As an application of the pcf theory, we shall now present the following result of Shelah:

Theorem 24.33 (Shelah). *If $\aleph_\omega$ is a strong limit cardinal then $2^{\aleph_\omega} < \aleph_{\omega_4}$.*

Proof. Let us assume that $\aleph_\omega$ is strong limit. We already know, by Corollary 24.27, that $2^{\aleph_\omega} = \max \operatorname{pcf}\{\aleph_n\}_{n=0}^\infty < \aleph_{\aleph_\omega}$. We shall prove that

$$\max \operatorname{pcf}\{\aleph_n\}_{n=0}^\infty < \aleph_{\omega_4}.$$

Let ϑ be the ordinal such that $2^{\aleph_\omega} = \aleph_{\vartheta+1}$; we shall prove that $\vartheta < \omega_4$.

Lemma 24.34. *There exists an ordinal function on $P(\vartheta)$ with the following properties*:

(24.18) (i) *If $X \subset Y$ then $F(X) \le F(Y)$.*
(ii) *For every limit ordinal $\eta < \vartheta$ of uncountable cofinality there is a closed unbounded set $C \subset \eta$ such that $F(C) = \eta$.*
(iii) *If $X \subset \vartheta$ has order-type ω_1 then there exists some $\gamma \in X$ such that $F(X \cap \gamma) \ge \sup X$.*

Proof. Let $X \subset \vartheta$ and consider the set $A = \{\aleph_{\xi+1} : \xi \in X\}$. As $2^{|A|} = \aleph_k$ for some finite k, $\max(\operatorname{pcf} A)$ exists and is equal to some $\aleph_{\gamma+1}$. We define $F(X) = \gamma$.

It is clear that $X \subset Y$ implies $F(X) \le F(Y)$ and that $F(X) \ge \sup X$.

Property (ii) follows from Corollary 24.30. If $\kappa = \operatorname{cf} \eta$ then $\kappa < \aleph_\omega$ and so $2^\kappa < \aleph_\omega < \aleph_\eta$ and the corollary applies.

Property (iii) is a consequence of the Localization Lemma 24.32: If $X \subset \vartheta$ then $\{\aleph_{\xi+1} : \xi \in X\} \subset \operatorname{pcf}\{\aleph_n\}_{n=0}^\infty$ and since $2^{|\operatorname{pcf}\{\aleph_n\}_n|} \le 2^{2^{\aleph_0}} < \aleph_\omega$, Lemma 24.32 applies (with e.g. $\lambda = \aleph_{\eta+1}$ where $\eta = \sup X$) and X has a countable subset W such that $F(W) \ge \sup X$. □

We complete the proof of Shelah's Theorem by showing that $\vartheta < \omega_4$.

Assume, by contradiction, that $\vartheta \ge \omega_4$. Let $\langle C_\alpha : \alpha \in E^{\aleph_3}_{\aleph_1}\rangle$ be a club-guessing sequence (see Theorem 23.3). Each C_α is a closed unbounded subset of α, and for every closed unbounded $C \subset \omega_3$, the set $\{\alpha \in E^{\aleph_3}_{\aleph_1} : C_\alpha \subset C\}$ is stationary.

Let M_α, $\alpha < \omega_3$, be a continuous elementary chain of models of size $\aleph_3$ that contain the family $\{C_\alpha\}_\alpha$, are closed under F, such that $\langle M_\xi : \xi \le \alpha\rangle \in M_{\alpha+1}$ for each α, and that for each α, $\eta_\alpha = M_\alpha \cap \omega_4$ is an ordinal. Let $\eta : \omega_3 \to \omega_4$ be the continuous function $\eta(\alpha) = \eta_\alpha$. By (24.18)(ii) there is a closed unbounded set $C \subset \omega_3$ such that $F(\eta``C) = \sup_\alpha \eta_\alpha$. Let $\alpha \in E^{\aleph_3}_{\aleph_1}$ be such that $C_\alpha \subset C$. By (24.18)(iii) there exists a $\beta < \alpha$ such that $F(\eta``(C_\alpha \cap \beta)) \ge \eta(\alpha)$. Let $X = \eta``(C_\alpha \cap \beta)$.

Since $C_\alpha \in M_\alpha$ and $\eta{\restriction}\beta \in M_\alpha$, we have $X \in M_\alpha$. Since $X \subset \eta``C$ we have $F(X) \le F(\eta``C) < \omega_4$. As M_α is closed under F, we have $F(X) \in M_\alpha$, and since $\omega_4 \cap M_\alpha = \eta(\alpha)$, it follows that $F(X) < \eta(\alpha)$, a contradiction. □

Exercises

24.1. If $\beta < \omega_1$ and if $2^{\aleph_\alpha} \le \aleph_{\alpha+\beta}$ for a stationary set of α's, then $2^{\aleph_{\omega_1}} \le \aleph_{\omega_1+\beta}$.
[By induction on β: If $\varphi(\alpha) \le \beta$ on a stationary set, then $\|\varphi\| \le \beta$.]

24.2. If $\beta < \omega_1$, if $2^{\aleph_1} < \aleph_{\omega_1}$, and if $\aleph_\alpha^{\aleph_0} \le \aleph_{\alpha+\beta}$ for a stationary set of α's, then $\aleph_{\omega_1}^{\aleph_1} \le \aleph_{\omega_1+\beta}$.

24.3. If $2^{\aleph_\alpha} \le \aleph_{\alpha+2}$ holds for all cardinals of cofinality ω, then the same holds for all singular cardinals.

24.4. If $\aleph_1 \le \operatorname{cf}\aleph_\eta < \aleph_\eta$, if $\beta < \operatorname{cf}\aleph_\eta$, and if $2^{\aleph_\alpha} \le \aleph_{\alpha+\beta}$ for all $\alpha < \eta$, then $2^{\aleph_\eta} \le \aleph_{\eta+\beta}$.

24.5. If $2^{\aleph_\alpha} \le \aleph_{\alpha+\alpha+1}$ for a stationary set of $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} \le \aleph_{\omega_1+\omega_1+1}$.
[If $\varphi(\alpha) = \alpha$ for all $\alpha < \omega_1$, then $\|\varphi\| = \omega_1$.]

24.6. If $2^{\aleph_{\omega_1+\alpha}} < \aleph_{\omega_1+\alpha+\alpha}$ for all $\alpha < \omega_1$, then $2^{\aleph_{\omega_1+\omega_1}} < \aleph_{\omega_1+\omega_1+\omega_1}$.
[Use the sets $A_\alpha = \omega_{\omega_1+\alpha}$.]

24.7. If $2^{\aleph_1} < \aleph_{\omega_1}$ and if $\aleph_\alpha^{\aleph_0} \le \aleph_{\alpha+\alpha+1}$ for all $\alpha < \omega_1$, then $\aleph_{\omega_1}^{\aleph_1} \le \aleph_{\omega_1+\omega_1+1}$.

24.8. If κ is a strong limit cardinal, $\kappa = \aleph_\eta$, and $\operatorname{cf}\kappa \ge \aleph_1$, then $2^\kappa < \aleph_\gamma$, where $\gamma = (|\eta|^{\operatorname{cf}\kappa})^+$.

24.9. If $\aleph_1 \le \operatorname{cf}\kappa < \kappa$ and if $\lambda^{\operatorname{cf}\kappa} < \kappa$ for all $\lambda < \kappa$, then $\kappa^{\operatorname{cf}\kappa} < \aleph_\gamma$, where $\gamma = (|\eta|^{\operatorname{cf}\kappa})^+$.

The next exercise uses the notation from Chapter 8. Let κ be a regular uncountable cardinal, let $M_0 = \kappa$, $M_{\eta+1} = \operatorname{Tr}(M_\eta)$, $M_\eta = \bigcap_{\nu<\operatorname{cf}\eta} M_{\xi_\nu}$ or $M_\eta = \triangle_{\nu<\kappa} M_{\xi_\nu}$ (if $\operatorname{cf}\eta = \kappa$) as long as M_η is stationary.

24.10. Let f_η, $\eta < \kappa^+$, be the canonical functions on κ. Let $S_\eta = \{\alpha < \kappa : o(\alpha) = f_\eta(\alpha)\}$. Show that $S_\eta = M_\eta - M_{\eta+1} \bmod I_{\mathrm{NS}}$ and that $o(S) = \eta$ for every stationary $S \subset S_\eta$.

The sets S_η are the *canonical stationary sets* (of order η).

24.11. Find a partially ordered set of cofinality $\aleph_\omega$; of cofinality 1, 2, 3, etc.

24.12. The lexicographical ordering $\omega \times \omega_1$ does not have true cofinality.

24.13. Let $I = I_{\mathrm{NS}}$ be the nonstationary ideal on ω_1, let c_γ, $\gamma < \omega_1$, be the constant functions (with value γ) on ω_1, and let $d(\alpha) = \alpha$ be the diagonal function. The function d is a least upper bound, but not an exact upper bound of the set $\{c_\gamma : \gamma < \omega_1\}$, in $<_I$.

Historical Notes

The Galvin-Hajnal Theorem appeared in [1975]. Shelah's investigation leading to the pcf theory started in [1978], and the book [1982] contains the first proof of a bound on $2^{\aleph_\omega}$. In a sequence of papers starting in 1978, Shelah developed the theory of possible cofinalities. A complete presentation is in his book [1994].

There are several papers that give an exposition and/or simplified proofs of Shelah's results; we mention Burke and Magidor [1990] and Jech [1992].