25. Descriptive Set Theory

Descriptive set theory is the study of definable sets of real numbers, in particular projective sets, and is mostly interested in how well behaved these sets are. A prototype of such results is Theorem 11.18 stating that $\mathbf{\Sigma}^1_1$ sets are Lebesgue measurable, have the Baire property, and have the perfect set property. This chapter continues the investigations started in Chapter 11. Throughout, we shall work in set theory ZF + DC (the Principle of Dependent Choice).

The Hierarchy of Projective Sets

Modern descriptive set theory builds on both the classical descriptive set theory and on recursion theory. It has become clear in the 1950's that the topological approach of classical descriptive set theory and the recursion theoretic techniques of logical definability describe the same phenomena. Modern descriptive set theory unified both approaches, as well as the notation. An additional ingredient is in the use of infinite games and determinacy; we shall return to that subject in Part III.

We first reformulate the hierarchy of projective sets in terms of the *lightface* hierarchy Σ^1_n, Π^1_n and Δ^1_n and its *relativization* for real parameters. While we introduce these concepts explicitly for subsets of the Baire space $\mathcal{N} = \omega^\omega$, analogous definitions and results apply to product spaces $\mathcal{N} \times \mathcal{N}$, $\mathcal{N}^r$ as well as the spaces ω, ω^k, $\omega^k \times \mathcal{N}^r$.

Definition 25.1.

(i) A set $A \subset \mathcal{N}$ is Σ^1_1 if there exists a recursive set $R \subset \bigcup_{n=0}^\infty (\omega^n \times \omega^n)$ such that for all $x \in \mathcal{N}$,

$$x \in A \quad \text{if and only if} \quad \exists y \in \omega^\omega \, \forall n \in \omega \, R(x{\restriction}n, y{\restriction}n). \tag{25.1}$$

(ii) Let $a \in \mathcal{N}$; a set $A \subset \mathcal{N}$ is $\Sigma^1_1(a)$ (Σ^1_1 in a) if there exists a set R recursive in a such that for all $x \in \mathcal{N}$,

$$x \in A \quad \text{if and only if} \quad \exists y \in \omega^\omega \, \forall n \in \omega \, R(x{\restriction}n, y{\restriction}n, a{\restriction}n).$$

(iii) $A \subset \mathcal{N}$ is Π^1_n (in a) if the complement of A is Σ^1_n (in a).
(iv) $A \subset \mathcal{N}$ is Σ^1_{n+1} (in a) if it is the projection of a Π^1_n (in a) subset of $\mathcal{N} \times \mathcal{N}$.
(v) $A \subset \mathcal{N}$ is Δ^1_n (in a) if it is both Σ^1_n and Π^1_n (in a).

A similar lightface hierarchy exists for Borel sets: A set $A \subset \mathcal{N}$ is Σ^0_1 (*recursive open* or *recursively enumerable*) if

$$A = \{x : \exists n\ R(x{\restriction}n)\} \tag{25.2}$$

for some recursive R, and Π^0_1 (*recursive closed*) if it is the complement of a Σ^0_1 set. Thus Σ^1_1 sets are projections of Π^0_1 sets, and as every open set is Σ^0_1 in some $a \in \mathcal{N}$ (namely an a than codes the corresponding union of basic open intervals), we have

$$\boldsymbol{\Sigma}^1_1 = \bigcup_{a \in \mathcal{N}} \Sigma^1_1(a),$$

and more generally, every $\boldsymbol{\Sigma}^1_n$ ($\boldsymbol{\Pi}^1_n$) set is Σ^1_n (Π^1_n) in some parameter $a \in \mathcal{N}$.

For $n \in \omega$, the lightface hierarchy of Σ^0_n and Π^0_n sets describes the *arithmetical* sets: For instance, a set A is Σ^0_3 if

$$A = \{x \in \mathcal{N} : \exists m_1\, \forall m_2\, \exists m_3\, R(m_1, m_2, x{\restriction}m_3)\}$$

for some recursive R, etc. Arithmetical sets are exactly those $A \subset \mathcal{N}$ that are definable (without parameters) in the model $(\mathit{HF}, \in)$ of hereditary finite sets.

The following lemma gives a list of closure properties of projective relations on $\mathcal{N}$. We use the logical (rather than set-theoretic) notation for Boolean operations; compare with Lemma 13.10.

Lemma 25.2. *Let $n \geq 1$.*

(i) *If A, B are $\Sigma^1_n(a)$ relations, then so are $\exists x\, A$, $A \wedge B$, $A \vee B$, $\exists m\, A$, $\forall m\, A$.*
(ii) *If A, B are $\Pi^1_n(a)$ relations, then so are $\forall x\, A$, $A \wedge B$, $A \vee B$, $\exists m\, A$, $\forall m\, A$.*
(iii) *If A is $\Sigma^1_n(a)$, then $\neg A$ is Π^1_n; if A is $\Pi^1_n(a)$, then $\neg A$ is Σ^1_n.*
(iv) *If A is $\Pi^1_n(a)$ and B is $\Sigma^1_n(a)$, then $A \to B$ is $\Sigma^1_n(a)$; if A is $\Sigma^1_n(a)$ and B is $\Pi^1_n(a)$, then $A \to B$ is $\Pi^1_n(a)$.*
(v) *If A and B are $\Delta^1_n(a)$, then so are $\neg A$, $A \wedge B$, $A \vee B$, $A \to B$, $A \leftrightarrow B$, $\exists m\, A$, $\forall m\, A$.*

Proof. We prove the lemma for $n = 1$; the general case follows by induction. Moreover, clauses (ii)–(v) follow from (i).

First, let $A \in \Sigma^1_1(a)$ and let us show that $\exists x\, A$ is $\Sigma^1_1(a)$. We have

$$(x, y) \in A \leftrightarrow \exists z\, \forall n\, (x{\restriction}n, y{\restriction}n, z{\restriction}n, n) \in R,$$

where R is recursive in a. Thus

$$y \in \exists x\, A \leftrightarrow \exists x\, \exists z\, \forall n\, (x{\restriction}n, y{\restriction}n, z{\restriction}n, n) \in R.$$

We want to contract the two quantifiers $\exists x\, \exists z$ into one. Let us consider some recursive homeomorphism between $\mathcal{N}$ and $\mathcal{N}^2$, e.g., for $u \in \mathcal{N}$ let u^+ and u^- be

$$u^+(n) = u(2n), \qquad u^-(n) = u(2n+1), \qquad (n \in \boldsymbol{N}).$$

There exists a relation R' recursive in R, such that for all $u, y \in \mathcal{N}$,

$$\text{(25.3)} \quad \forall n\, (u{\restriction}n, y{\restriction}n, n) \in R' \quad \text{if and only if} \quad \forall k\, (u^+{\restriction}k, y{\restriction}k, u^-{\restriction}k, k) \in R.$$

Namely, if $n = 2k$ (or $n = 2k+1$), we let $(s, t, n) \in R'$ just in case $\text{length}(s) = \text{length}(t) = n$ and

$$(\langle s(0), \ldots, s(2k-2)\rangle, \langle t(0), \ldots, t(k-1)\rangle, \langle s(1), \ldots, s(2k-1)\rangle, k) \in R.$$

Now (25.3) implies that

$$y \in \exists x\, A \leftrightarrow \exists u\, \forall n\, (u{\restriction}n, y{\restriction}n, n) \in R',$$

and hence $\exists x\, A$ is $\Sigma^1_1(a)$.

Now let A and B be $\Sigma^1_1(a)$:

$$\begin{aligned} x \in A &\leftrightarrow \exists z\, \forall n\, (x{\restriction}n, z{\restriction}n, n) \in R_1, \\ x \in A &\leftrightarrow \exists z\, \forall n\, (x{\restriction}n, z{\restriction}n, n) \in R_2 \end{aligned}$$

where both R_1 and R_2 are recursive in a. Note that

$$x \in A \wedge B \leftrightarrow \exists z_1\, \exists z_2\, \forall n\, [(x{\restriction}n, z_1{\restriction}n, n) \in R_1 \wedge (x{\restriction}n, z_2{\restriction}n, n) \in R_2]$$

and hence, by contraction of $\exists z_1\, \exists z_2$, there is some R, recursive in R_1 and R_2 such that

$$x \in A \wedge B \leftrightarrow \exists z\, \forall n\, (x{\restriction}n, z{\restriction}n, n) \in R.$$

Thus $A \wedge B$ is $\Sigma^1_1(a)$.

The following argument shows that the case $A \vee B$ can be reduced to the case $\exists m\, C$. Let us define R as follows ($s, t \in \mathit{Seq}$, $m, n \in \boldsymbol{N}$):

$$\begin{aligned} (s, m, t, n) \in R \leftrightarrow \text{either } & m = 1 \text{ and } (s, t, n) \in R_1 \\ \text{or } & m = 2 \text{ and } (s, t, n) \in R_2. \end{aligned}$$

R is recursive in R_1 and R_2, and

$$\begin{aligned} x \in A \vee B &\leftrightarrow \exists z\, \forall n\, (x{\restriction}n, z{\restriction}n, n) \in R_1 \vee \exists z\, \forall n\, (x{\restriction}n, z{\restriction}n, n) \in R_2 \\ &\leftrightarrow \exists m\, \exists z\, \forall n\, (x{\restriction}n, m, z{\restriction}n, n) \in R \\ &\leftrightarrow x \in \exists m\, C \end{aligned}$$

where C is $\Sigma^1_1(a)$.

The contraction of quantifiers $\exists m\,\exists z$ is easier than the contraction $\exists x\,\exists z$ above. We employ the following recursive homomorphism between $\mathcal{N}$ and $\omega \times \mathcal{N}$: $h(u) = (u(0), u')$, where

$$u'(n) = u(n+1) \qquad (n \in \boldsymbol{N}).$$

If

$$(x, m) \in A \leftrightarrow \exists z\, \forall n\, (x{\restriction}n, m, z{\restriction}n, n) \in R,$$

then we leave it to the reader to find a relation R', recursive in R, such that for all $u, x \in \mathcal{N}$,

$$\forall n\, (x{\restriction}n, u{\restriction}n, n) \in R' \leftrightarrow \forall k\, (x{\restriction}k, u(0), u'{\restriction}k, k) \in R.$$

Then

$$x \in \exists m\, A \leftrightarrow \exists u\, \forall n\, (x{\restriction}n, u{\restriction}n, n) \in R'.$$

It remains to show that if A is $\Sigma^1_1(a)$, then $\forall m\, A$ is $\Sigma^1_1(a)$. Let

$$(x, m) \in A \leftrightarrow \exists z\, \forall n\, (x{\restriction}n, m, z{\restriction}n, n) \in R$$

where R is recursive in a. Thus

$$x \in \forall m\, A \leftrightarrow \forall m\, \exists z\, \forall n\, (x{\restriction}n, m, z{\restriction}n, n) \in R. \tag{25.4}$$

We want to replace the quantifiers $\forall m\, \exists z$ by $\exists u\, \forall m$ and then contract the two quantifiers $\forall m\, \forall n$ into one. Let us consider the pairing function $\Gamma : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{N}$ and the following homeomorphism between $\mathcal{N}$ and $\mathcal{N}^\omega$: For each $u \in \mathcal{N}$, let u_m, $m \in \boldsymbol{N}$, be

$$u_m(n) = u(\Gamma(m, n)) \qquad (m, n \in \boldsymbol{N}).$$

Now we can replace $\forall m\, \exists z$ in (25.4) by $\exists u\, \forall m$ (note that in the forward implication we use the Countable Axiom of Choice):

$$\forall m\, \exists z\, \forall n\, (x{\restriction}n, m, z{\restriction}n, n) \in R \leftrightarrow \exists u\, \forall m\, \forall n\, (x{\restriction}n, m, u_m{\restriction}n, n) \in R. \tag{25.5}$$

Let $\alpha : \boldsymbol{N} \to \boldsymbol{N}$ and $\beta : \boldsymbol{N} \to \boldsymbol{N}$ be the inverses of the function Γ: If $\Gamma(m, n) = k$, then $m = \alpha(k)$ and $n = \beta(k)$. From (25.4) and (25.5) we get

$$x \in \forall m\, A \leftrightarrow \exists u\, \forall k\, (x{\restriction}\beta(k), \alpha(k), u_{\alpha(k)}{\restriction}\beta(k), \beta(k)) \in R. \tag{25.6}$$

Now it suffices to show that there exists a relation $R' \subset Seq^2 \times \boldsymbol{N}$, recursive in R, such that for all $u, x \in \mathcal{N}$,

$$\forall k\, (x{\restriction}k, u{\restriction}k, k) \in R' \leftrightarrow \forall k\, (x{\restriction}\beta(k), \alpha(k), u_{\alpha(k)}{\restriction}\beta(k), \beta(k)) \in R. \tag{25.7}$$

The relation R' is found in a way similar to the relation R' in (25.3), and we leave the details as an exercise.

Hence $\forall m\, A$ is $\Sigma^1_1(a)$ because by (25.6) and (25.7),

$$x \in \forall m\, A \leftrightarrow \exists u\, \forall k\, (x{\restriction}k, u{\restriction}k, k) \in R'.$$

In Lemma 11.8 we proved the existence of a universal $\boldsymbol{\Sigma}^1_n$ set. An analysis of the proof (and of Lemma 11.2) yields a somewhat finer result: There exists a Σ^1_n set $A \subset \mathcal{N}^2$ (lightface) that is a universal $\boldsymbol{\Sigma}^1_n$ set.

$\mathbf{\Pi}^1_1$ Sets

We formulate a normal form for Π^1_1 sets in terms of trees. This is based on the idea that analytic sets are projections of closed sets, and that closed sets in $\mathcal{N}$ are represented by sets $[T]$ where T is a sequential tree; cf. (4.6). Let us consider the product space $\mathcal{N}^r$, for an arbitrary integer $r \geq 1$. As in the case $r = 1$, the closed subsets of $\mathcal{N}^r$ can be represented by trees: Let Seq_r denote the set of all r-tuples $(s_1, \ldots, s_r) \in Seq^r$ such that $\text{length}(s_1) = \ldots = \text{length}(s_r)$. A set $T \subset Seq_r$ is an *(r-dimensional sequential) tree* if for every $(s_1, \ldots, s_r) \in T$ and each $n \leq \text{length}(s_1)$, $(s_1{\restriction}n, \ldots, s_r{\restriction}n)$ is also in T. Let

$$(25.8) \qquad [T] = \{(a_1, \ldots, a_r) \in \mathcal{N}^r : \forall n\, (a_1{\restriction}n, \ldots, a_r{\restriction}n) \in T\}.$$

The set $[T]$ is closed, and every closed set in $\mathcal{N}^r$ has the form (25.8), for some tree T.

We call a sequential tree $T \subset Seq_r$ *well-founded* if $[T] = \emptyset$, i.e., if the reverse inclusion on T is a well-founded relation. T is *ill-founded* if it is not well-founded.

For $T \subset Seq_{r+1}$ and for each $x \in \mathcal{N}$, let

$$(25.9) \quad T(x) = \{(s_1, \ldots, s_r) \in Seq_r : (x{\restriction}n, s_1, \ldots, s_r) \in T \quad \text{where } n = \text{length}\, s_i\}.$$

Now if $A \subset \mathcal{N}$ is analytic, there exists a tree $T \subset Seq_2$ such that A is the projection of $[T]$; consequently, for all $x \in \mathcal{N}$ we have

$$(25.10) \qquad x \in A \quad \text{if and only if} \quad T(x) \text{ is ill-founded.}$$

More generally, if A is Σ^1_1, let R be recursive such that

$$x \in A \leftrightarrow \exists y \in \mathcal{N}\, \forall n\, R(x{\restriction}n, y{\restriction}n)$$

and define $T = \{(t, s) \in Seq_2 : \forall n \leq \text{length}(s)\, R(t{\restriction}n, s{\restriction}n)\}$. For all $x \in \mathcal{N}$, we have $T(x) = \{s \in Seq : \forall n \leq \text{length}(s)\, R(x{\restriction}n, s{\restriction}n)\}$ and $x \in A$ if and only if $T(x)$ is ill-founded.

Theorem 25.3 (Normal Form for Π^1_1). *A set $A \subset \mathcal{N}$ is Π^1_1 if and only if there exists a recursive mapping $x \mapsto T(x)$ such that each $T(x)$ is a sequential tree, and*

$$(25.11) \qquad x \in A \quad \textit{if and only if} \quad T(x) \textit{ is well-founded.} \qquad \square$$

Similarly, a relation $A \subset \mathcal{N}^r$ is Π^1_1 if and only if $A = \{\vec{x} : T(\vec{x}) \text{ is well-founded}\}$ where $\langle T(\vec{x}) : \vec{x} \in \mathcal{N}^r \rangle$ is a recursive system of r-dimensional trees.

One consequence of normal forms is that Π^1_1 (and Σ^1_1) relations are absolute for transitive models:

Theorem 25.4 (Mostowski's Absoluteness). *If P is a $\mathbf{\Sigma}^1_1$ property then P is absolute for every transitive model that is adequate for P.*

Proof. "Adequate" here means that the model satisfies enough axioms to know that well-founded trees have a rank function, and contains the parameter in which P is Σ^1_1. The proof is similar to Lemma 13.11.

Let M be a transitive model and let $T \in M$ be a tree such that $P = \{x : T(x) \text{ is ill-founded}\}$. Let $x \in M$. If $M \vDash (T(x)$ is ill-founded) then $T(x)$ is ill-founded. Conversely, if $M \vDash (T(x)$ is well-founded) then $M \vDash (\exists f : T(x) \to \mathit{Ord}$ such that $f(s) < f(t)$ whenever $s \supset t)$ and therefore $T(x)$ is well-founded. □

Trees, Well-Founded Relations and κ-Suslin Sets

Much of modern descriptive set theory depends on a generalization of the Normal Form for Π^1_1 sets. A tree $T \subset \mathit{Seq}_r$ consists of r-tuples of finite sequences. We can also identify T with finite sequences of r-tuples, which enables us to consider a more general concept:

Definition 25.5.

(i) A tree T (on a set X) is a set of finite sequences (in X) closed under initial segments.
(ii) If $s, t \in T$ then $s \leq t$ means $s \supset t$, i.e., t is an initial segment of s.
(iii) If $s \in T$ then $T/s = \{t : s^\frown t \in T\}$.
(iv) If $(T, \leq)$ is well-founded then $\|T\|$ is the height of $\leq$, and for $t \in T$, $\rho_T(t)$ is the rank of t in $\leq$.
(v) $[T] = \{f \in X^\omega : \forall n\, f{\restriction}n \in T\}$.

If S and T are well-founded trees and if $f : S \to T$ is order-preserving then $\|S\| \leq \|T\|$; this is easily verified by induction on rank. But the converse is also true:

Lemma 25.6. *If S and T are well-founded trees and $\|S\| \leq \|T\|$ then there exists an order-preserving map $f : S \to T$.*

Proof. By induction on $\|T\|$. For each $\langle a \rangle \in S$, $\|S/\langle a \rangle\| < \|S\| \leq \|T\|$ and there exists a $t_a \neq \emptyset$ such that $\|S/\langle a \rangle\| \leq \|T/\langle t_a \rangle\|$. Let $f_a : S/\langle a \rangle \to S/t_a$ be order-preserving. Now define $f : S \to T$ as follows: $f(\emptyset) = \emptyset$, and $f(a^\frown s) = t_a^\frown f_a(s)$ whenever $a^\frown s \in S$. □

We remark that the above proof (as well as the existence of rank), uses the Principle of Dependent Choices. If T is ill-founded, note that for *any* S there exists an order-preserving $f : S \to T$ (into an infinite branch of T). Thus we have

Corollary 25.7. *There exists an order-preserving $f : S \to T$ if and only if either T is ill-founded or $\|S\| \le \|T\|$.* $\square$

Trees used in descriptive set theory are trees on $\omega \times K$ (or on $\omega^r \times K$) where K is some set, usually well-ordered.

Let $Seq(K)$ be the set of all finite sequences in K. A *tree* on $\omega \times K$ is a set of pairs $(s, h) \in Seq \times Seq(K)$ such that length(s) = length(h) and that for each $n \le$ length(s), $(s{\restriction}n, h{\restriction}n) \in T$. For every $x \in \mathcal{N}$, let

$$T(x) = \{h \in Seq(K) : (x{\restriction}n, h) \in T \text{ where } n = \text{length}(h)\}. \tag{25.12}$$

$T(x)$ is a tree on K. Further we let

$$\begin{aligned} p[T] &= \{x \in \mathcal{N} : T(x) \text{ is ill-founded}\} \\ &= \{x \in \mathcal{N} : [T(x)] \ne \emptyset\} \\ &= \{x \in \mathcal{N} : (\exists f \in K^\omega)\, \forall n\, (x{\restriction}n, f{\restriction}n) \in T\}. \end{aligned}$$

Trees on $\omega^r \times K$ are defined analogously.

Definition 25.8. Let κ be an infinite cardinal. A set $A \subset \mathcal{N}$ is *κ-Suslin* if $A = p[T]$ for some tree on $\omega \times \kappa$.

By the Normal Form Theorem for Π^1_1 sets, every $\boldsymbol{\Sigma}^1_1$ set is ω-Suslin. In fact if A is $\Sigma^1_1(a)$ then $A = p[T]$ where T is a tree on $\omega \times \omega$ recursive in a. Let us associate with each $x \in \mathcal{N}$ the following binary relation E_x on $\boldsymbol{N}$:

$$m\ E_x\ n \leftrightarrow x(\Gamma(m, n)) = 0 \tag{25.13}$$

where Γ is a (recursive) pairing function of $\boldsymbol{N} \times \boldsymbol{N}$ onto $\boldsymbol{N}$; we say that x *codes* the relation E_x. We define

$$\begin{aligned} \text{WF} &= \{x \in \mathcal{N} : x \text{ codes a well-founded relation}\}, \\ \text{WO} &= \{x \in \mathcal{N} : x \text{ codes a well-ordering on } \boldsymbol{N}\}. \end{aligned} \tag{25.14}$$

Lemma 25.9. *The sets* WF *and* WO *are* Π^1_1.

Proof. We prove in some detail that WF is Π^1_1. E_x is well-founded if and only if there is no $z : \boldsymbol{N} \to \boldsymbol{N}$ such that $z(k+1)\ E_x\ z(k)$ for all k. Thus

$$x \in \text{WF} \leftrightarrow \forall z\, \exists k\, \neg z(k+1)\ E_x\ z(k).$$

In other words, WF $= \forall z\, A$, where

$$(x, z) \in A \leftrightarrow \exists k\, x(\Gamma(z(k+1), z(k))) \ne 0$$

and it suffices to show that A is arithmetical. But

$$\begin{aligned} (x, z) \in A \leftrightarrow \exists n, m, j, k\, [i = (z{\restriction}n)(k+1) \wedge j = (z{\restriction}n)(k) \wedge \\ m = \Gamma(i, j) \wedge (x{\restriction}n)(m) \ne 0]. \end{aligned}$$

To show that WO is Π^1_1 it suffices to verify that the set

$$\text{LO} = \{x : E_x \text{ is a linear ordering of } \boldsymbol{N}\}$$

is arithmetical. Then WO = WF $\wedge$ LO is Π^1_1. $\square$

We show below that neither WF nor WO is a $\mathbf{\Sigma}^1_1$ set; thus neither is a Borel set.

For each $x \in \mathrm{WF}$, let

$$\|x\| = \text{the height of the well-founded relation } E_x \tag{25.15}$$

(see (2.7)). For each x, $\|x\|$ is a countable ordinal (and for each $\alpha < \omega_1$ there is some $x \in \mathrm{WF}$ such that $\|x\| = \alpha$). If $x \in \mathrm{WO}$, then $\|x\|$ is the order-type of the well-ordering E_x.

Lemma 25.10. *For each $\alpha < \omega_1$, the sets*

$$\mathrm{WF}_\alpha = \{x \in \mathrm{WF} : \|x\| \le \alpha\}, \qquad \mathrm{WO}_\alpha = \{x \in \mathrm{WO} : \|x\| \le \alpha\}$$

are Borel sets.

Proof. Note that the set $\{(x, n) : n \in \operatorname{field}(E_x)\}$ is arithmetical (and hence Borel). Let us prove the lemma first for WO_α.

For each $\alpha < \omega_1$, let

$$\begin{aligned} B_\alpha = \{(x, n) : E_x \text{ restricted to } \{m : m\ E_x\ n\} \\ \text{is a well-ordering of order type} \le \alpha\}. \end{aligned}$$

We prove, by induction on $\alpha < \omega_1$, that each B_α is a Borel set. It is easy to see that B_0 is arithmetical. Thus let $\alpha < \omega_1$ and assume that all B_β, $\beta < \alpha$, are Borel. Then $\bigcup_{\beta<\alpha} B_\beta$ is Borel and hence B_α is also Borel because

$$(x, n) \in B_\alpha \leftrightarrow \forall m \Big(m\ E_x\ n \to (x, m) \in \bigcup_{\beta<\alpha} B_\beta\Big).$$

It follows that each WO_α is Borel because

$$x \in \mathrm{WO}_\alpha \leftrightarrow \forall n \Big(n \in \operatorname{field}(E_x) \to (x, n) \in \bigcup_{\beta<\alpha} B_\beta\Big).$$

To handle WF_α, note that the rank function ρ_E can be defined for any binary relation E; namely:

$$\begin{aligned} \rho_E(u) = \alpha \quad \text{if and only if} \quad & \forall v\, (v\ E\ u \to \rho_E(v) \text{ is defined) and} \\ & \alpha = \sup\{\rho_E(v) + 1 : v\ E\ u\}. \end{aligned}$$

For each $\alpha < \omega_1$, let

$$C_\alpha = \{(x, n) : \rho_{E_x}(n) \text{ is defined and} \le \alpha\}.$$

Again, C_0 is arithmetical, and if we assume that all C_β, $\beta < \alpha$, are Borel, then C_α is also Borel:

$$(x, n) \in C_\alpha \leftrightarrow \forall m \Big(m\ E_x\ n \to (x, m) \in \bigcup_{\beta<\alpha} C_\beta\Big).$$

Hence each C_α is Borel, and it follows that each WF_α is Borel:

$$x \in \mathrm{WF}_\alpha \leftrightarrow \forall n \Big(n \in \operatorname{field}(E_x) \to (x, n) \in \bigcup_{\beta<\alpha} C_\beta\Big). \qquad \square$$

Corollary 25.11. *The sets* $\{x \in \mathrm{WF} : \|x\| = \alpha\}$ *and* $\{x \in \mathrm{WF} : \|x\| < \alpha\}$ *are Borel* (*similarly for* WO).

Proof. $\{x \in \mathrm{WF} : \|x\| < \alpha\} = \bigcup_{\beta<\alpha} \mathrm{WF}_\beta$. □

Theorem 25.12. *If C is a $\mathbf{\Pi}^1_1$ set, then there exists a continuous function $f : \mathcal{N} \to \mathcal{N}$ such that $C = f_{-1}(\mathrm{WF})$, and there exists a continuous function $g : \mathcal{N} \to \mathcal{N}$ such that $C = g_{-1}(\mathrm{WO})$.*

Proof. We shall give the proof for WF; the proof for WO is similar. Let $T \subset Seq_2$ be such that

$$x \in C \leftrightarrow T(x) \text{ is well-founded.}$$

Let $\{t_0, t_1, \ldots, t_n, \ldots\}$ be an enumeration of the set *Seq*. For each $x \in \mathcal{N}$, let $y = f(x)$ be the following element of $\mathcal{N}$:

$$y(\Gamma(m,n)) = \begin{cases} 0 & \text{if } t_m, t_n \in T(x), \text{ and } t_m < t_n, \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that E_y is isomorphic to $(T(x), <)$, and hence $y \in \mathrm{WF}$ if and only if $T(x)$ is well-founded. Thus $C = f_{-1}(\mathrm{WF})$ and it remains to show only that f is continuous. But it should be obvious from the definitions of $T(x)$ and of $y = f(x)$ that for any finite sequence $s = \langle \varepsilon_0, \ldots, \varepsilon_{k-1}\rangle$, there is $\check{s} \in Seq$ such that if $x \supset \check{s}$ and $y = f(x)$, then $y{\restriction}k = s$. Hence f is continuous. □

Corollary 25.13. WF *is not* $\mathbf{\Sigma}^1_1$; WO *is not* $\mathbf{\Sigma}^1_1$.

Proof. Otherwise every $\mathbf{\Pi}^1_1$ set would be the inverse image by a continuous function of an analytic set and hence analytic; however, there are $\mathbf{\Pi}^1_1$ sets that are not analytic. □

Corollary 25.14 (Boundedness Lemma). *If $B \subset \mathrm{WO}$ is $\mathbf{\Sigma}^1_1$, then there is an $\alpha < \omega_1$ such that $\|x\| < \alpha$ for all $x \in B$.*

Proof. Otherwise we would have

$$\mathrm{WO} = \{x \in \mathcal{N} : \exists z\,(z \in B \wedge \|x\| \le \|z\|)\}.$$

Hence $\|x\| \le \|z\|$ for $x, z \in \mathcal{N}$ means: Either $z \notin \mathrm{WO}$ or $\|x\| \le \|z\|$; this relation is Σ^1_1; see Exercise 25.3. This would mean that WO is Σ^1_1, a contradiction. □

Corollary 25.15. *Every $\mathbf{\Pi}^1_1$ set is the union of $\aleph_1$ Borel sets.*

Proof. If C is $\mathbf{\Pi}^1_1$, then $C = f_{-1}(\mathrm{WF})$ for some continuous f. But $\mathrm{WF} = \bigcup_{\alpha<\omega_1} \mathrm{WF}_\alpha$, and hence

$$C = \bigcup_{\alpha<\omega_1} f_{-1}(\mathrm{WF}_\alpha).$$

Each $f_{-1}(\mathrm{WF}_\alpha)$ is the inverse image of a Borel set by a continuous function, hence Borel. □

Corollary 25.16. *Assuming the Axiom of Choice, every $\mathbf{\Pi}^1_1$ set is either at most countable, or has cardinality $\aleph_1$, or cardinality $2^{\aleph_0}$.* $\square$

Theorem 25.19 below improves Corollary 25.15 by showing that every $\mathbf{\Sigma}^1_2$ set is the union of $\aleph_1$ Borel sets. The following lemma is the first step toward that theorem.

Lemma 25.17. *Every $\mathbf{\Sigma}^1_1$ set is the union of $\aleph_1$ Borel sets.*

Proof. Let A be a $\mathbf{\Sigma}^1_1$ set. Let $T \subset Seq_2$ be a tree such that $A = p[T]$. We prove by induction on α that for each $t \in Seq$ and every $\alpha < \omega_1$, the set

$$\{x \in \mathcal{N} : \|T(x)/t\| \le \alpha\} \tag{25.16}$$

is Borel. Namely, $\{x : \|T(x)/t\| \le 0\} = \{x : (x{\restriction}n, t) \notin T\}$ and if $\alpha > 0$, then $\|T(x)/t\| \le \alpha$ if and only if $\forall n\, (\exists \beta < \alpha)\, \|T(x)/t^\frown n\| \le \beta$.

Let us define, for each α, the set B_α as follows:

$$x \in B_\alpha \leftrightarrow \neg(\|T(x)\| < \alpha) \wedge \forall t\, (\neg \|T(x)/t\| = \alpha).$$

Since the sets in (25.16) are Borel, it follows that each B_α is Borel. We shall prove that $A = \bigcup_{\alpha<\omega_1} B_\alpha$. First let $x \in A$. Thus $T(x)$ is ill-founded; hence $\|T(x)\| \not< \alpha$ for any α, and it suffices to show that there is an α such that $\|T(x)/t\| \ne \alpha$ for all t. If there is no such α, then for every α there is t such that $\|T(x)/t\| = \alpha$, but there are $\aleph_1$ α's and only $\aleph_0$ t's; a contradiction.

Next let $x \notin A$, and let us show that $x \notin B_\alpha$, for all α. Let $\alpha < \omega_1$ be arbitrary. Since $T(x)$ is well-founded, either $\|T(x)\| < \alpha$ and $x \notin B_a$, or $\|T(x)\| \ge \alpha$ and there exists some $t \in T(x)$ such that $\|T(x)/t\| = \alpha$ and again $x \notin B_\alpha$. $\square$

$\mathbf{\Sigma}^1_2$ Sets

The Normal Form Theorem for Π^1_1 sets provides a tree representation for Σ^1_2 sets:

Theorem 25.18. *Every $\mathbf{\Sigma}^1_2$ set is ω_1-Suslin. If A is $\Sigma^1_2(a)$ then $A = p[T]$ where T is a tree on $\omega \times \omega_1$ and $T \in L[a]$.*

Proof. Let A be a $\Sigma^1_2(a)$ subset of $\mathcal{N}$. There is a tree $U \subset Seq_3$, recursive in a such that

$$x \in A \leftrightarrow \exists y\, \forall z\, \exists n\, (x{\restriction}n, y{\restriction}n, z{\restriction}n) \notin U.$$

In other words,

$$x \in A \leftrightarrow \exists y\, U(x, y) \text{ is well-founded.}$$

A necessary and sufficient condition for a countable relation to be well-founded is that it admits an order-preserving mapping into ω_1. Thus

$$\begin{aligned} x \in A &\leftrightarrow \exists y\,(\exists f : U(x,y) \to \omega_1)\text{ if } u \subset v \text{ then } f(u) > f(v) \\ &\leftrightarrow \exists y\,(\exists f : \mathit{Seq} \to \omega_1)\, f{\restriction}U(x,y) \text{ is order-preserving.} \end{aligned}$$

Let $\{u_n : n \in \boldsymbol{N}\}$ be a recursive enumeration of the set Seq such that for every n, $\text{length}(u_n) \le n$. If f is a function on (a subset of) $\boldsymbol{N}$, let f^* be the function on (a subset of) Seq defined by $f^*(u_n) = f(n)$. Thus

$$x \in A \leftrightarrow \exists y\,(\exists f : \omega \to \omega_1)\, f^*{\restriction}U(x,y) \text{ is order-preserving.} \tag{25.17}$$

Now we define a tree T' on $\omega^2 \times \omega_1$ as follows: If $s, t \in \mathit{Seq}$ and $h \in \mathit{Seq}(\omega_1)$ are all of length n, we let

$$(s,t,h) \in T' \leftrightarrow h^*{\restriction}U_{s,t} \text{ is order-preserving} \tag{25.18}$$

where $U_{s,t} = \{u \in \mathit{Seq} : k = \text{length}\,u \le n \text{ and } (s{\restriction}k, t{\restriction}k, u) \in U\}$. Clearly, T' is a tree on $\omega^2 \times \omega_1$.

Let $x, y \in \mathcal{N}$. We claim that if $(x{\restriction}n, (y{\restriction}n, h) \in T'$, then $h^*{\restriction}U(x,y)$ is order-preserving. This is because if $u, v \in \text{dom}(h^*) \cap U(x,y)$, then $u = u_i$, $v = u_j$ for some $i, j < n$, hence $\text{length}(u), \text{length}(v) < n$ and hence $u, v \in U_{s,t}$, where $s = x{\restriction}n$, $t = y{\restriction}n$. Thus

$$f \in T'(x,y) \leftrightarrow \forall n\,(f{\restriction}n)^*{\restriction}U(x,y) \text{ is order-preserving.}$$

But clearly a mapping $f : \omega \to \omega_1$ satisfies the right-hand side if and only if $f^*{\restriction}U(x,y)$ is order-preserving. Hence (25.17) and (25.18) give

$$\begin{aligned} x \in A &\leftrightarrow \exists y\,\exists f : \omega \to \omega_1\, f \in T'(x,y) \\ &\leftrightarrow \exists y\,\exists f : \omega \to \omega_1\, \forall n\,(x{\restriction}n, y{\restriction}n, f{\restriction}n) \in T'. \end{aligned}$$

Now we transform T' (on $\omega^2 \times \omega_1$) into a tree T'' (on $\omega \times K$ where $K = \omega \times \omega_1$) such that we replace triples

$$(\langle s(0), \ldots, s(n-1)\rangle, \langle t(0), \ldots, t(n-1), \rangle, \langle h(0), \ldots, h(n-1)\rangle)$$

by pairs

$$(\langle s(0), \ldots, s(n-1)\rangle, \langle (t(0), h(0)), \ldots, (t(n-1), h(n-1))\rangle)$$

and we get

$$x \in A \leftrightarrow (\exists g : \omega \to K)\,\forall n\,(x{\restriction}n, g{\restriction}n) \in T''.$$

Since $K = \omega \times \omega_1$ is in an obvious one-to-one correspondence with ω_1, it is clear that we can find a tree T on $\omega \times \omega_1$ such that

$$x \in A \leftrightarrow (\exists g : \omega \to \omega_1)\,\forall n\,(x{\restriction}n, g{\restriction}n) \in T, \tag{25.19}$$

that is $A = p[T]$. The tree T so obtained is constructible from the tree U, which in turn is constructible from a. □

One consequence of Theorem 25.18 is the following:

Theorem 25.19 (Sierpiński). *Every* $\mathbf{\Sigma}^1_2$ *set is the union of* $\aleph_1$ *Borel sets.*

It follows that in ZFC, every $\mathbf{\Sigma}^1_2$ set has cardinality either at most $\aleph_1$, or $2^{\aleph_0}$.

Proof. Let A be a $\mathbf{\Sigma}^1_2$ set. By Theorem 25.18 there is a tree T on $\omega \times \omega_1$ such that $A = p[T]$. For each $\gamma < \omega_1$ let $T^\gamma = \{(s,h) \in T : h \in Seq(\gamma)\}$. Since every $f : \omega \to \omega_1$ has the range included in some $\gamma < \omega_1$, it is clear that

$$A = \bigcup_{\gamma<\omega_1} p[T^\gamma].$$

For each $\gamma < \omega_1$, the set $p[T^\gamma]$ is analytic (because $p[T^\gamma] = p[\tilde{T}]$ for some $\tilde{T} \subset Seq_2$) and is the union of $\aleph_1$ Borel sets. In fact, Lemma 25.17 gives a uniform decomposition into $\aleph_1$ Borel sets for any $p[U]$ where U is a tree on $\omega \times S$ with S countable. If we let

$$x \in B^\gamma_\alpha \leftrightarrow \neg(\|T^\gamma(x)\| < \alpha) \wedge (\forall t \in Seq(\gamma))(\neg\|T^\gamma(x)/t\| = \alpha)$$

then $A = \bigcup_{\alpha<\omega_1} \bigcup_{\gamma<\omega_1} B^\gamma_\alpha$. □

The main application of Theorem 25.18 is absoluteness of Σ^1_2 (and Π^1_2) relations.

Theorem 25.20 (Shoenfield's Absoluteness Theorem). *Every* $\Sigma^1_2(a)$ *relation and every* $\Pi^1_2(a)$ *relation is absolute for all inner models* M *of* ZF + DC *such that* $a \in M$. *In particular,* $\mathbf{\Sigma}^1_2$ *and* $\mathbf{\Pi}^1_2$ *relations are absolute for* L.

It is clear from the proof that every $\mathbf{\Sigma}^1_2(a)$ relation is absolute for every transitive model M of a finite fragment of ZF + DC such that $\omega_1 \in M$.

Proof. Let $a \in \mathcal{N}$ and let A be a $\Sigma^1_2(a)$ subset of $\mathcal{N}$; let $A = \{x : A(x)\}$ where $A(x)$ is a $\Sigma^1_2(a)$ property. Let M be an inner model of ZF + DC such that $a \in M$. We shall prove that $M \vDash A$ if and only if A holds.

Let $U \subset Seq_3$ be a tree, arithmetical in a, such that for all $x \in \mathcal{N}$,

$$x \in A \leftrightarrow \exists y\, U(x,y) \text{ is well-founded.}$$

Thus for all $x \in \mathcal{N} \cap M$

$$x \in A^M \leftrightarrow (\exists y \in M)\, M \vDash U(x,y) \text{ is well-founded.}$$

However, for all $x, y \in M$, $U(x,y)$ is the same tree in M as in V; and since well-foundedness is absolute, we have

$$x \in A^M \leftrightarrow (\exists y \in M)\, U(x,y) \text{ is well-founded.}$$

Thus, if $x \in A^M$, then $x \in A$, and it suffices to prove that if $x \in A \cap M$ then $x \in A^M$.

We use the tree representation of $\mathbf{\Sigma}^1_2$ sets. Let T be the tree on $\omega \times \omega_1$ constructed in the proof of Theorem 25.18. Hence $T \in L[a]$ and for every $x \in \mathcal{N}$,

$$x \in A \leftrightarrow T(x) \text{ is ill-founded.}$$

Now if $x \in M$ is such that $x \in A$, then $T(x)$ is ill-founded, and by absoluteness of well-foundedness,

$$M \vDash T(x) \text{ is ill-founded.}$$

In other words, there exists a function $g \in M$ from $\mathcal{N}$ into the ordinals such that $\forall n\,(x{\restriction}n, g{\restriction}n) \in T$. Now following the proof of Theorem 25.18 backward, from (25.19) to the beginning, and working inside M, one finds a $y \in M$ such that

$$M \vDash U(x, y) \text{ is well-founded.}$$

Hence if $x \in A \cap M$, then $x \in A^M$ and we are done. □

With only notational changes Theorem 25.18 gives a tree representation of subsets of ω (or ω^k) and we have:

Corollary 25.21. *If $A \subset \omega$ is $\Sigma^1_2(a)$ then $A \in L[a]$. In particular, every Σ^1_2 real* (*and every Π^1_2 real*) *is constructible.*

The following lemma is an interesting application of Shoenfield's Absoluteness.

Lemma 25.22. *Let S be a set of countable ordinals such that the set $A = \{x \in \mathrm{WO} : \|x\| \in S\}$ is Σ^1_2. Then S is constructible.* (*And more generally, if A is $\Sigma^1_2(a)$, then $S \in L[a]$.*)

Proof. Let $A(x)$ be the Σ^1_2 property such that $A = \{x : A(x)\}$. For each countable ordinal α, let P_α be the notion of forcing that collapses α; i.e., the elements of P_α are finite sequences of ordinals less than α. Each P_α is constructible; let us consider, in L, the forcing languages associated with the P_α, and the corresponding Boolean-valued models L^{P_α}.

We shall show that for every $\alpha < \omega_1$, α belongs to S if and only if

$$L \vDash \text{every } p \in P_\alpha \text{ forces } \exists x\,(A(x) \wedge \|x\| = \alpha). \tag{25.20}$$

This will show that S is constructible.

In order to prove that $\alpha \in S$ is equivalent to (25.20), let us consider a generic extension N of V in which ω_1^V is countable. Let us argue in N.

The notion of forcing P_α has only countably many constructible dense subsets, and hence for every $p \in P_\alpha$ there exists a $G \subset P_\alpha$ such that G is L-generic and $p \in G$. It follows that for every α, every φ and every $p \in P_\alpha$,

$$L \vDash (p \Vdash \varphi) \quad \text{if and only if} \quad \text{for every } L\text{-generic } G \ni p,\ L[G] \Vdash \varphi. \tag{25.21}$$

Let $\alpha < \omega_1^V$, and let $z \in V$ be such that $\|z\| = \alpha$. Clearly, α belongs to S if and only if V satisfies

$$\exists x\,(A(x) \wedge \|x\| = \|z\|). \tag{25.22}$$

The property (25.22) is Σ^1_2 and by absoluteness, it holds in V if and only if it holds in N.

Let G be an arbitrary L-generic filter on P_α, and let $u \in L[G]$ be such that $\|u\| = \alpha$. Since N satisfies (25.22) if and only if it satisfies the Σ^1_2 property

$$\exists x\,(A(x) \wedge \|x\| = \|u\|), \tag{25.23}$$

it follows that $\alpha \in S$ if and only if $L[G]$ satisfies (25.23). Since an L-generic filter on P_α exists in N, we conclude (still in N), that $\alpha \in S$ is equivalent to:

$$\text{For every } L\text{-generic } G \subset P_\alpha,\ L[G] \vDash \exists x\,(A(x) \wedge \|x\| = \alpha).$$

But in view of (25.21) this last statement is equivalent to (25.20). □

Another application of the tree representation of $\mathbf{\Sigma}^1_2$ sets is the Perfect Set Theorem of Mansfield and Solovay:

Theorem 25.23 (Mansfield-Solovay). *Let A be a $\Sigma^1_2(a)$ set in $\mathcal{N}$. If A contains an element that is not in $L[a]$, then A has a perfect subset.*

The theorem follows from this more general lemma:

Lemma 25.24. *Let T be a tree on $\omega \times K$ and let $A = p[T]$. Either $A \subset L[T]$, or A contains a perfect subset; moreover, in the latter case there is a perfect tree $U \in L[T]$ on ω such that $[U] \subset A$.*

Proof. The proof follows the Cantor-Bendixson argument. If T is a tree on $\omega \times K$, let

$$T' = \{(s,h) \in T : \text{there exist } (s_0,h_0),(s_1,h_1) \in T \text{ such that } s_0 \supset s,\ s_1 \supset s,\ h_0 \supset h,\ h_1 \supset h, \text{ and that } s_0 \text{ and } s_1 \text{ are incompatible}\} \tag{25.24}$$

and then, inductively,

$$T^{(0)} = T, \qquad T^{(\alpha+1)} = (T^{(\alpha)})',$$
$$T^{(\alpha)} = \bigcap_{\beta<\alpha} T^{(\beta)} \quad \text{if } \alpha \text{ is limit.}$$

The definition (25.24) is absolute for all models that contain T, and hence $T^{(\alpha)} \in L[T]$ for all α. Let α be the least ordinal such that $T^{(\alpha+1)} = T^{(\alpha)}$.

Let us assume first that $T^{(\alpha)} = \emptyset$; we shall show that $A \subset L[T]$. Let $x \in A$ be arbitrary. There exists an $f \in K^\omega$ such that $(x,f) \in [T]$. Let $\gamma < \alpha$ be such that $(x,f) \in [T^{(\gamma)}]$ but $(x,f) \notin [T^{(\gamma+1)}]$. Thus there is some

$(s,h) \in T^{(\gamma)}$ such that $s \subset x$, $h \subset f$, and $(s,h) \notin T^{(\gamma+1)}$; this means that for any $(s',h') \in T^{(\gamma)}$, if $s' \supset s$ and $h' \supset h$, then $s' \subset x$. Now it follows that $x \in L[T]$; in $L[T]$, x is the unique $x = \bigcup\{s' \supset s : (s',h') \in T^{(\gamma)}$ for some $h \supset h'\}$.

Now let us assume that $T^{(\alpha)} \neq \emptyset$. The tree $T^{(\gamma)}$ has the property that for every $(s,h) \in T^{(\alpha)}$ there exist two extensions (s_0,h_0) and (s_1,h_1) of (s,h) that are incompatible in the first coordinate. Let us work in $L[T]$. Let (s_0,h_0) and (s_1,h_1) be some elements of $T^{(\alpha)}$ such that s_0 and s_1 are incompatible. Then let $(s_{0,0},h_{0,0})$, $(s_{0,1},h_{0,1})$, $(s_{1,0},h_{1,0})$, and $(s_{1,1},h_{1,1})$ be elements of $T^{(\alpha)}$ such that $s_{i,j} \supset s_i$, $h_{i,j} \supset h_i$ and that the $s_{i,j}$ are incompatible. In this fashion we construct $(s_t,h_t) \in T^{(\alpha)}$ for each 0–1 sequence t. The s_t generate a tree $U = \{s : s \subset s_t$ for some $t\}$. It is clear that U is a perfect three, that $U \in L[T]$, and that $[U] \subset p[T] = A$. □

The following observation establishes a close connection between the projective hierarchy and the Lévy hierarchy of Σ_n properties of hereditarily countable sets:

Lemma 25.25. *A set $A \subset \mathcal{N}$ is Σ^1_2 if and only if it is Σ_1 over $(HC, \in)$.*

Proof. If A is Σ_1 over HC, there exists a Σ_0 formula φ such that

$$x \in A \leftrightarrow HC \vDash \exists u\, \varphi(u,x) \leftrightarrow (\exists u \in HC)\, HC \vDash \varphi[u,x].$$

Since φ is Σ_0, it is absolute for transitive models and we have

$$x \in A \leftrightarrow (\exists \text{ transitive set } M)(\exists u \in M)\, M \vDash \varphi[u,x]$$

(e.g., $M = \mathrm{TC}(\{u,x\})$). By the Principle of Dependent Choices every $\mathrm{TC}(\{u,x\})$ is countable and we have

$$\begin{aligned} x \in A &\leftrightarrow (\exists \text{ countable transitive set } M)(\exists u \in M)\, M \vDash \varphi[u,x] \\ &\leftrightarrow (\exists \text{ well-founded extensional relation } E \text{ on } \omega) \\ &\qquad \exists n\, \exists m\, (\pi_E(m) = x \text{ and } (\omega,E) \vDash \varphi[n,m]) \end{aligned}$$

where π_E is the transitive collapse of (ω,E) onto $(M,\in)$. Recalling the definition (25.13) of E_x for $z \in \mathcal{N}$ we have

$$\begin{aligned} x \in A \leftrightarrow (\exists z \in \mathcal{N})(&z \in \mathrm{WF} \text{ and } (\omega,E_x) \vDash \text{Extensionality}, \\ &\exists n\, \exists m\, (\pi_{E_x}(m) = x \text{ and } (\omega,E_x) \vDash \varphi[n,m])). \end{aligned} \tag{25.25}$$

We shall verify that (25.25) gives a Σ^1_2 definition of A. Since WF is Π^1_1, it suffices to show that the relation "$(\omega,E) \vDash \varphi[n_1,\ldots,n_k]$" and "$\pi_E(m) = x$" are arithmetical in E. It is easy to see that $(\omega,E) \vDash \varphi$ is a property arithmetical in E. As for the transitive collapse, we notice first that if $k \in \boldsymbol{N}$, then

$$\pi_E(m) = k \leftrightarrow \exists\langle r_0, \ldots, r_k\rangle \text{ such that } m = r_k \text{ and } (\omega, E) \vDash r_0 = \emptyset$$
$$\text{and } (\forall i < k)\,(\omega, E) \vDash (r_{i+1} = r_i \cup \{r_i\}).$$

Then for $x \subset \omega$ we have

$$\pi_E(m) = x \leftrightarrow \forall n\,(n \mathrel{E} m \leftrightarrow \pi_E(n) \in x)$$

and a similar formula, arithmetical in E, defines $\pi_E(m) = x$ for $x \in \mathcal{N}$.

Hence $A \in \Sigma^1_2$.

Conversely, if A is a Σ^1_2 set then for some Π^1_1 property P, $A = \{x : \exists y\, P(x, y)\}$. By Mostowski's Absoluteness, $x \in A$ if and only if for some countable transitive model $M \ni x$ adequate for P there exists a $y \in M$ such that $M \vDash P(x, y)$. But this gives a Σ_1 definition of A over $(HC, \in)$. □

As a consequence, Σ^1_{n+1} sets are exactly those that are Σ_n over HC.

Projective Sets and Constructibility

We now compute the complexity of the set of all constructible reals:

Theorem 25.26 (Gödel). *The set of all constructible reals is a Σ^1_2 set. The ordering $<_L$ is a Σ^1_2 relation.*

The field of $<_L$ is $\boldsymbol{R} \cap L$. If all reals are constructible, then $<_L$ is also Π^1_2 (because $x <_L y$ if and only if $y \not\leq_L x$) and hence $<_L$ is then a Δ^1_2 relation.

The theorem easily generalizes to $L[a]$: If $a \in \boldsymbol{R}$ (or $a \subset \omega$ or $a \in \mathcal{N}$), then the set $\boldsymbol{R} \cap L[a]$ is $\Sigma^1_2(a)$; also, the relation "x is constructible from y" is a Σ^1_2 relation.

We proved in Chapter 13 that "x is constructible" and "$x <_L y$" are Σ_1 relations over the model $(HC, \in)$. Thus Theorem 25.26 follows from Lemma 25.25.

The following lemma tells even more than $<_L$ is a Σ^1_2 relation. For any $z \in \mathcal{N}$, let z_m, $m \in \boldsymbol{N}$, be defined by $z_m(n) = z(\Gamma(m, n))$ (the canonical homeomorphism between $\mathcal{N}$ and $\mathcal{N}^\omega$).

Lemma 25.27. *The following relation R on $\mathcal{N}$ is Σ^1_2:*

$$(z, x) \in R \leftrightarrow \{z_n : n \in \boldsymbol{N}\} = \{y : y <_L x\}.$$

Proof. Since the relation $\{z_n : n \in \boldsymbol{N}\} \subset \{y : y <_L x\}$ is clearly Σ^1_2, it suffices to show that

$$\forall y <_L x\, \exists n\,(y = z_n) \tag{25.26}$$

is Σ^1_2. There is a sentence Θ (provable in ZF) such that if M is a transitive model of Θ, then $<_L$ is absolute for M; and if $x \in M$ is constructible, then every $y <_L x$ is in M. Thus (25.26) is equivalent to

$$\begin{array}{l}\exists \text{ countable transitive model } M \text{ that contains } x,\, z, \text{ and all } z_n, \\ \text{and } M \vDash (\Theta \text{ and } \forall y <_L x \, \exists n\, (y = z_n)).\end{array}$$

This last property is Σ^1_2 by a proof similar to Lemma 25.25. □

Every $\boldsymbol{\Sigma}^1_1$ set is Lebesgue measurable, has the Baire property and if uncountable, has a perfect subset. The following results show that this is best possible.

Corollary 25.28. *If $V = L$ then there exists a Δ^1_2 set that is not Lebesgue measurable and does not have the Baire property.*

Proof. Let $A = \{(x, y) : x <_L y\}$. For every y, the set $\{x : (x, y) \in A\}$ is countable, hence null and meager, and by Lemmas 11.12 and 11.16, if A is measurable, then it is null; and if it has the Baire property, then it is meager.

Let B be the complement of A in $\boldsymbol{R}^2$, $B = \{(x, y) : y \le_L x\}$. Again, for every x, the set $\{y : (x, y) \in B\}$ is countable, and hence null if measurable, and meager if has the Baire property.

It clearly follows that A neither is Lebesgue measurable nor has the property of Baire □

Corollary 25.29. *If $V = L$ then there exists an uncountable Σ^1_2 set without a perfect subset.*

Proof. Let

$$x \in A \leftrightarrow x \in \mathrm{WO} \wedge \forall y <_L x\, (\neg \|y\| = \|x\|).$$

The set A is uncountable: A is a subset of WO and for every $\alpha < \omega_1$ there is exactly one x in A such that $\|x\| = \alpha$. Let us show that A is Σ^1_2: Let R be the Σ^1_2 relation from Lemma 25.27; thus

$$x \in A \leftrightarrow x \in \mathrm{WO} \wedge \exists z\, (R(z, x) \wedge \forall n\, (\neg \|z_n\| = \|x\|)),$$

and since $\neg \|z_n\| = \|x\|$ is Π^1_1, A is Σ^1_2.

The set A does not have a perfect subset; in fact, it does not have an uncountable analytic subset. This follows from the Boundedness Lemma: For every analytic set $X \subset A$, the set $\{\|x\| : x \in X\}$ is bounded, and hence countable (because of the definition of A). □

Below (Corollary 25.37) we improve this by showing that in L there exists an uncountable Π^1_1 set without a perfect subset.

By Shoenfield's Absoluteness Theorem, every Σ^1_2 real is constructible. In Part III we show that it is consistent that a nonconstructible Δ^1_3 real exists. In the presence of large cardinals, an example of a nonconstructible Δ^1_3 real is $0^\sharp$:

Lemma 25.30. *If $0^\sharp$ exists then $0^\sharp$ is a Δ^1_3 real, and the singleton $\{0^\sharp\}$ is a Π^1_2 set.*

Proof. We identify $0^\sharp$ with the set of Gödel numbers of the sentences in $0^\sharp$. We claim that the property $\Sigma = 0^\sharp$ is Π_1 over $(HC, \in)$, and therefore Π^1_2. We use the description (18.24) of $0^\sharp$ and note that the quantifiers $\forall\alpha$ can be replaced by $\forall\alpha < \omega_1$, thus making it a Π_1 property over HC.

Thus $\{0^\sharp\}$ is a Π^1_2 set, and

$$n \in 0^\sharp \leftrightarrow \exists z\,(z \in \{0^\sharp\} \text{ and } z(n) = 1) \leftrightarrow \forall z\,(z \in \{0^\sharp\} \to z(n) = 1)$$

shows that $0^\sharp$ is a Δ^1_3 subset of ω. □

Scales and Uniformization

The tree analysis of Σ^1_2 sets can be refined; an analysis of Kondô's proof of the Uniformization Theorem (Theorem 25.36) led Moschovakis to introduce the concept of *scale* that pervades the modern descriptive set theory.

We start with the definition of *norm* and *prewellordering*. While in the present chapter these concepts are applied to $\boldsymbol{\Pi}^1_1$ and $\boldsymbol{\Sigma}^1_2$ sets, the theory applies to more general collection of definable sets of reals.

Definition 25.31. A *norm* on a set A is an ordinal function φ on A. A *prewellordering* of A is a transitive relation $\preccurlyeq$ such that $a \preccurlyeq b$ or $b \preccurlyeq a$ for all $a, b \in A$, and that $\prec$ is well-founded.

A prewellordering of a set A induces an equivalence relation $(a \preccurlyeq b \wedge b \preccurlyeq a)$ and a well-ordering of its equivalence classes. Its rank function is a norm, and conversely, a norm φ defines a prewellordering

$$a \preccurlyeq_\varphi b \quad \text{if and only if} \quad \varphi(a) \le \varphi(b). \tag{25.27}$$

The tree analysis of Π^1_1 and Σ^1_2 sets produces well behaved prewellorderings of Π^1_1 and Σ^1_2 sets:

Theorem 25.32. *For every Π^1_1 set A there exists a norm φ on A with the property that there exist a Π^1_1 relation $P(x,y)$ and a Σ^1_1 relation $Q(x,y)$ such that for every $y \in A$ and all x,*

$$x \in A \text{ and } \varphi(x) \le \varphi(y) \leftrightarrow P(x,y) \leftrightarrow Q(x,y). \tag{25.28}$$

A norm φ with the above property is called a Π^1_1-norm and the statement "every Π^1_1 set has a Π^1_1-norm" is called the *prewellordering property* of Π^1_1.

A relativization of Theorem 25.32 shows that every $\Pi^1_1(a)$ set has a $\Pi^1_1(a)$-norm. A modification of the proof of Theorem 25.32 yields the prewellordering property of Σ^1_2: every Σ^1_2 set has a Σ^1_2 norm, i.e., a norm for which exist a Σ^1_2 P and a Π^1_2 Q that satisfy (25.28) (cf. Exercises 25.5 and 25.6).

Proof. Let A be a Π^1_1 and let T be a recursive tree on $\omega \times \omega$ such that

$$A(x) \leftrightarrow T(x) \text{ is well-founded.}$$

For each $x \in A$ let $\varphi(x) = \|T(x)\|$ be the height of the well-founded tree.

To define the Σ^1_1 relation Q, let

$$(25.29) \qquad Q(x,y) \leftrightarrow \text{there exists an order-preserving function } f : T(x) \to T(y).$$

It is not difficult to see that Q is Σ^1_1, and the equivalence in (25.28) follows from Corollary 25.7. For the Π^1_1 relation, let

$$(25.30) \qquad P(x,y) \leftrightarrow \forall s \neq \emptyset \text{ there exists no order-preserving } f : T(y) \to T(x)/s.$$

This is Π^1_1 and says that $T(x)$ is well-founded and it is not the case that $\|T(y)\| < \|T(x)\|$. □

The prewellordering property of Π^1_1 implies the *reduction principle* for Π^1_1 and the *separation principle* for Σ^1_1—see Exercises. This in turn implies Suslin's Theorem that every $\boldsymbol{\Delta}^1_1$ set is Borel.

The prewellordering property has an important strengthening, the *scale property* which we now introduce.

Let A be a Π^1_1 set. Following the proof of Theorem 25.18 we obtain a tree T on $\omega \times \omega_1$ such that $A = p[T]$. In detail, let U be a recursive tree on $\omega \times \omega$ such that

$$x \in A \leftrightarrow U(x) \text{ is well-founded} \leftrightarrow \exists g : U(x) \to \omega_1 \text{ order preserving.}$$

Let $\{u_n : n \in \boldsymbol{N}\}$ be a recursive enumeration of *Seq* such that $\text{length}(u_n) \leq n$, and let T be the tree on $\omega \times \omega_1$ defined by

$$(25.31) \quad (s,h) \in T \leftrightarrow \forall m,n < \text{length}(s)(\text{if } u_m \supset u_n \text{ and } (s{\restriction}k, s{\restriction}u_m) \in U \text{ where } k = \text{length}(u_m), \text{ then } h(m) < h(n)).$$

The relevant observation is that not only that $A = p[T]$, i.e.,

$$x \in A \leftrightarrow \exists \text{ a branch } g \text{ in } T(x)$$

but that for every $x \in p[T]$ there exists a (pointwise) *least* branch g in $T(x)$, i.e., for every $f \in p[T]$, $g(n) \leq f(n)$ for all n. To see this, let

$$g_x(n) = \begin{cases} \rho_{T(x)}(u_n) & \text{if } u_n \in U(x), \\ 0 & \text{otherwise.} \end{cases}$$

That g_x is the least branch in $T(x)$ holds because the rank function is the least order-preserving function.

Definition 25.33. A *scale* on a set A is a sequence of norms $\langle \varphi_n : n \in \omega \rangle$ such that: If $\langle x_i : i \in \omega \rangle$ is a sequence of points in A with $\lim_{i\to\infty} x_i = x$ and such that

(25.32) for every n, the sequence $\langle \varphi_n(x_i) : i \in \omega \rangle$ is eventually constant, with value α_n,

then $x \in A$, and for every n, $\varphi_n(x) \le \alpha_n$.

It is easy to see that every Π^1_1 set A has a scale: Let A be a Π^1_1 set and let T be the tree in (25.31). We have $A = p[T]$ and for each $x \in A$, $T(x)$ has a least branch g_x. Let $\langle \varphi_n : n \in \omega \rangle$ be the sequence of norms on A defined by

$$\varphi_n(x) = g_x(n). \tag{25.33}$$

If $\langle x_i : i \in \omega \rangle$ is a sequence in A with $\lim_{i\to\infty} x_i = x$ that satisfies (25.32) then $\langle \alpha_n : n \in \omega \rangle$ is a branch in $T(x)$ witnessing $x \in p[T]$, and for every n, $g_x(n) \le \alpha_n$.

The norms defined in (25.33) are Π^1_1-norms; this can be verified as in the proof of Theorem 25.32. To be precise, the scale $\langle \varphi_n : n \in \omega \rangle$ is a Π^1_1-*scale*:

Theorem 25.34. *For every Π^1_1 set A there exists a scale $\langle \varphi_n : n \in \omega \rangle$ on A with the property that there exist a Π^1_1 relation $P(n, x, y)$ and a Σ^1_1 relation $Q(n, x, y)$ such that for every n, every $y \in A$, and all x,*

$$x \in A \text{ and } \varphi_n(x) \le \varphi_n(y) \leftrightarrow P(n, x, y) \leftrightarrow Q(n, x, y). \tag{25.34}$$ □

The statement "every Π^1_1 set has a Π^1_1-scale" is called the *scale property* of Π^1_1. A relativization shows that every $\Pi^1_1(a)$ set has a $\Pi^1_1(a)$-scale, and a modification of the above construction yields the scale property for Σ^1_2: every $\Sigma^1_2(a)$ set has a $\Sigma^1_2(a)$-scale; cf. Exercises 25.12 and 25.13.

A major application of scales is the *uniformization property.*

Definition 25.35. A set $A \subset \mathcal{N} \times \mathcal{N}$ is *uniformized* by a function F if $\mathrm{dom}(F) = \{x : \exists y\,(x, y) \in A\}$, and $(x, F(x)) \in A$ for all $x \in \mathrm{dom}(F)$.

[Equivalently, $F \subset A$ and $\mathrm{dom}(F) = \mathrm{dom}(A)$.]

Theorem 25.36 (Kondô). *Every Π^1_1 relation $A \subset \mathcal{N} \times \mathcal{N}$ is uniformized by a Π^1_1 function.*

The statement of Theorem 25.36 (the *Uniformization Theorem*) is called the *uniformization property* of Π^1_1. A relativization shows that every $\Pi^1_1(a)$ relation is uniformized by a $\Pi^1_1(a)$ function, and a modification of the proof yields the uniformization property of Σ^1_2; see Exercise 25.15.

Proof. We give a proof of the following statement that easily generalizes to a proof of Kondô's Theorem: If A is a nonempty Π^1_1 subset of $\mathcal{N}$ then there exists an $a \in A$ such that $\{a\}$ is Π^1_1.

Thus let A be a nonempty Π^1_1 subset of $\mathcal{N}$. Given a scale $\langle \varphi_n : n \in \omega \rangle$ on A, we select an element $a \in A$ as follows: We let $A_0 = A$, and for each n let

$$\begin{aligned} A_{2n+1} &= \{x \in A_{2n} : \varphi_n(x) \text{ is least}\}, \\ A_{2n+2} &= \{x \in A_{2n+1} : x(n) \text{ is least}\}. \end{aligned}$$

Then $A_0 \supset A_1 \supset \ldots \supset A_n \supset \ldots$ and the intersection has at most one element. Definition 25.33 guarantees that the limit a is in A and so $\bigcap_{n=0}^{\infty} A_n = \{a_n\}$.

If the scale $\langle \varphi_n : n \in \omega \rangle$ is Π^1_1 then using (25.34) one verifies that the set $\{a\}$ is Π^1_1. □

Theorem 25.36 can be used to improve the result in Corollary 25.29:

Corollary 25.37. *If $V = L$ then there exists an uncountable Π^1_1 set without a perfect subset.*

Proof. Let A be a Σ^1_2 set without a perfect subset (by 25.29). Now A is the projection of some Π^1_1 set $B \subset \mathcal{N}^2$. By the Uniformization Theorem, B has a Π^1_1 subset f that is a function and has the same projection A. The set f is uncountable; we claim that f does not have a perfect subset. Assume that $P \subset f$ is perfect. The projection of P is an analytic subset of A. Since $P \subset f$, P is itself a function and because P is uncountable, the projection $\mathrm{dom}(P)$ is also uncountable. This is a contradiction since we proved that every analytic subset of A is countable. □

Combining this result with Theorem 25.23 we obtain:

Theorem 25.38. *The following are equivalent:*

(i) *For every $a \subset \omega$, $\aleph_1^{L[a]}$ is countable.*
(ii) *Every uncountable $\mathbf{\Pi}^1_1$ set contains a perfect subset.*
(iii) *Every uncountable $\mathbf{\Sigma}^1_2$ set contains a perfect subset.*

Proof. Obviously, (iii) implies (ii). In order to show that (i) implies (iii), let us assume (i) and let A be an uncountable $\mathbf{\Sigma}^1_2$ set. Let $a \in \mathcal{N}$ be such that $A \in \Sigma^1_2(a)$. Since $\aleph_1^{L[a]}$ is countable, there are only countably many reals in $L[a]$, and hence A has an element that is not in $L[a]$. Thus A contains a perfect subset, by Theorem 25.23.

The remaining implication uses the same argument as Corollaries 25.29 and 25.37. Assume that there exists an $a \subset \omega$ such that $\aleph_1^{L[a]} = \aleph_1$. We claim that there exists an uncountable $\mathbf{\Pi}^1_1$ set without a perfect subset. Let

$$x \in A \leftrightarrow x \in L[a] \wedge x \in \mathrm{WO} \wedge \forall y <_{L[a]} x\, (\neg \|y\| = \|x\|).$$

A is a $\Sigma^1_2(a)$ subset of WO and for all $\alpha < \omega_1$, A has exactly one element x such that $\|x\| = \alpha$. The rest of the proof proceeds as before, and we obtain a $\Pi^1_1(a)$ set of cardinality $\aleph_1$ without a perfect subset. □

$\boldsymbol{\Sigma^1_2}$ Well-Orderings and $\boldsymbol{\Sigma^1_2}$ Well-Founded Relations

The canonical well-ordering of constructible reals is Σ^1_2, and so if $V = L$ then there exists a Σ^1_2 well-ordering of the set $\boldsymbol{R}$ (and of $\mathcal{N}$). We now prove the converse: If there exists a Σ^1_2 well-ordering of $\boldsymbol{R}$ then all reals are constructible.

Theorem 25.39 (Mansfield). *If $<$ is a Σ^1_2 well-ordering of $\mathcal{N}$ then every real is constructible. More generally, if $<$ is $\Sigma^1_2(a)$ then $\mathcal{N} \subset L[a]$.*

Proof. Let $<$ be a Σ^1_2 well-ordering of $\mathcal{N}$ and let us assume that there is a nonconstructible real. Let $T_0 = Seq(\{0,1\})$, and let $\boldsymbol{C} = [T_0] = \{0,1\}^\omega$ be the Cantor space. Let us consider trees $T \subset T_0$ and functions $f : T \to T_0$ such that $s \subset t$ implies $f(s) \subset f(t)$ and for every $x \in [T]$, $\bigcup_{n=0}^\infty f(x{\restriction}n) \in \boldsymbol{C}$. Every such function induces a continuous function from $[T]$ into $\boldsymbol{C}$, which we denote by f^*.

Lemma 25.40. *If $T \subset T_0$ is a constructible perfect tree and if $f : T \to T_0$ is a constructible function such that f^* is one-to-one, then there exist a constructible perfect tree $U \subset T$ and a constructible $g : U \to T_0$ such that g^* is one-to-one, and $g^*(x) < f^*(x)$ for every $x \in [U]$.*

It suffices to prove this lemma because then we can construct a sequence of trees $T_0 \supset T_1 \supset \ldots \supset T_n \supset \ldots$ and functions $f_0, f_1, \ldots, f_n, \ldots$ where f_0 is the identity such that $f^*_{n+1}(x) < f^*_n(x)$ for all $x \in T_{n+1}$. Since all $[T_n]$ are compact sets, their intersection is nonempty and therefore there exists an x such that $f^*_0(x) > f^*_1(x) > \ldots > f^*_n(x) > \ldots$ contrary to the assumption that $<$ is a well-ordering.

Proof of Lemma 25.40. Let $T \subset T_0$ be a constructible tree and let $f : T \to T_0$ be constructible, such that f^* is one-to-one.

Since T is perfect, there exists a constructible function $h : T \to T_0$ such that $h^* : [T] \to \boldsymbol{C}$ is one-to-one and onto. For each $s \in T_0$, let $\bar{s}$ be the "mirror image" of s, namely if $s = \langle s(0), \ldots, s(k)\rangle$, let $\bar{s} = \langle 1 - s(0), \ldots, 1 - s(k)\rangle$; for $x \in \boldsymbol{C}$, $\bar{x}$ is defined similarly.

We claim that at least one of the sets

$$A = \{x \in [T] : f^*(x) > h^*(x)\}, \qquad B = \{x \in [T] : f^*(x) > \overline{h^*(x)}\}$$

contains a nonconstructible element. Let z be the least nonconstructible element of $\boldsymbol{C}$, and let $x, y \in [T]$ be such that $h^*(x) = z$ and $h^*(y) = \bar{z}$. Then both x and y are nonconstructible and hence $f^*(x) \geq z$ and $f^*(y) \geq z$. Thus either $f^*(x) > z$ or $f^*(y) > z$ and so either A or B contains a nonconstructible element. For instance, assume that A does.

Since $<$ is Σ_2, and T, f, and h are constructible subsets of HF, the set A is $\Sigma^1_2(a)$ for some $a \in L$. By Lemma 25.24 there exists a constructible perfect tree U such that $[U] \subset A$. If we let $g = h{\restriction}U$, then U and g satisfy the lemma. $\square$

The set WO is Π^1_1 but not Σ^1_1. One consequence of this fact, related to the Boundedness Lemma, is that there is no $\boldsymbol{\Sigma}^1_1$ well-ordering of the reals, in fact every $\boldsymbol{\Sigma}^1_1$ well-ordering of a set of reals is countable. A more general statement holds:

Lemma 25.41. *Every $\boldsymbol{\Sigma}^1_1$ well-founded relation on $\mathcal{N}$ has countable height.*

Proof. Assuming that some $\boldsymbol{\Sigma}^1_1$ well-founded relation on $\mathcal{N}$ has height $\geq \omega_1$, we reach a contradiction by describing the set WO in a $\boldsymbol{\Sigma}^1_1$ way.

First consider the special case of well-orderings. Let E be a $\boldsymbol{\Sigma}^1_1$ well-ordering and let us assume that its order-type is $\geq \omega_1$. Then for every $\alpha < \omega_1$ there is an order-preserving mapping of $(\alpha, <)$ into $(\mathcal{N}, E)$. Conversely, if a countable linearly ordered set $(Q, <)$ can be embedded in $(\mathcal{N}, E)$, then $(Q, <)$ is a well-ordering. Hence let E_x be, for each $x \in \mathcal{N}$, the relation coded by x (see (25.13)), and let LO be the arithmetical set of all x that code a linear ordering of $\boldsymbol{N}$. Then

$$\begin{aligned}(25.35)\quad x \in \mathrm{WO} &\leftrightarrow x \in \mathrm{LO} \wedge (\exists f : \omega \to \mathcal{N})\, \forall n\, \forall m \\ &\qquad\qquad (n\ E_x\ m \to (f(n), f(m)) \in E) \\ &\leftrightarrow x \in \mathrm{LO} \wedge \exists z \in \mathcal{N}\, \forall n\, \forall m\, (n\ E_x\ m \to (z_n, z_m) \in E),\end{aligned}$$

where for each $z \in \mathcal{N}$ and each n, z_n is the element of $\mathcal{N}$ defined by $z_n(k) = z(\Gamma(n,k))$ for all $k \in \boldsymbol{N}$, where Γ is the pairing function. Now (25.35) gives a $\boldsymbol{\Sigma}^1_1$ description of WO, a contradiction.

In the general case when E is a $\boldsymbol{\Sigma}^1_1$ well-founded relation we observe that if α is a countable ordinal less than the height of E, then there exist a countable set $S \subset \mathcal{N}$ and a function f of S onto α such that for every $u \in S$ and every $\beta < f(u)$ there exists a $v \in S$ such that $v\ E\ u$ and $\beta \leq f(v)$ (namely $f(x) = \rho_E(x)$, and the countable set S is constructed with the help of the Principle of Dependent Choices). Conversely, if $(Q, <)$ is a linearly ordered set and if there is a function f from a subset of $\mathcal{N}$ onto Q such that for every $u \in \mathrm{dom}(f)$ and every $q < f(u)$ there is $v \in \mathrm{dom}(f)$ such that $v\ E\ u$ and $q \leq f(v)$, then $(Q, <)$ is a well-ordering. Thus if E has height $\geq \omega_1$, we have

$$\begin{aligned}(25.36)\quad x \in \mathrm{WO} \leftrightarrow x \in \mathrm{LO} \wedge &(\exists \text{ countable } S = \{z_n : n \in \boldsymbol{N}\}) \\ &(\exists f : S \overset{\text{onto}}{\to} \boldsymbol{N})\, \forall n\, \forall k\, [\text{if } (k, f(z_n)) \in E_x, \text{ then} \\ &\exists m \text{ such that } (z_m, z_n) \in E \text{ and} \\ &\text{either } k = f(z_m) \text{ or } (k, f(z_m)) \in E_x].\end{aligned}$$

Again, (25.36) can be written in a $\boldsymbol{\Sigma}^1_1$ manner, and we get a contradiction. $\square$

The next theorem gives an upper bound on heights of $\boldsymbol{\Sigma}^1_2$ well-founded relations.

Theorem 25.42 (Martin). *Every $\mathbf{\Sigma}^1_2$ well-founded relation on $\mathcal{N}$ has length $< \omega_2$.*

Note that since every prewellordering is a well-founded relation, the theorem implies that $\boldsymbol{\delta}^1_2 \le \omega_2$, where

$$\boldsymbol{\delta}^1_2 = \sup\{\alpha : \alpha \text{ is the length of a } \mathbf{\Sigma}^1_2 \text{ prewellordering}\}.$$

Proof. Let $E \subset \mathcal{N} \times \mathcal{N}$ be a $\mathbf{\Sigma}^1_2$ relation. Let T be a tree on $\omega^2 \times \omega_1$ such that for all $x, y \in \mathcal{N}$,

$$(x, y) \in E \leftrightarrow (\exists f : \omega \to \omega_1)\, \forall n\, (x{\restriction}n, y{\restriction}n, f{\restriction}n) \in T. \tag{25.37}$$

As usual, for each $z \in \mathcal{N}$ and each $n \in \boldsymbol{N}$, let $z_n \in \mathcal{N}$ be such that $z_n(k) = z(\Gamma(n, k))$ for all k; similarly, for each $f : \omega \to \omega_1$ and each n, let $f_n : \omega \to \omega_1$ be such that $f_n(k) = f(\Gamma(n, k))$ for all k. (Here Γ is the pairing function.)

Each of the following formulas is equivalent to the statement that the relation E is not well-founded:

$$\begin{aligned}
&\exists x\, \forall m\, (x_{m+1}, x_m) \in E,\\
&\exists x\, \forall m\, \exists f\, \forall n\, (x_{m+1}{\restriction}n, x_m{\restriction}n, f{\restriction}n) \in T,\\
&\exists x\, \exists f\, \forall m\, \forall n\, (x_{m+1}{\restriction}n, x_m{\restriction}n, f_m{\restriction}n) \in T.
\end{aligned}$$

It is easy to construct a tree U on $\omega \times \omega_1$ such that for all $x \in \mathcal{N}$ and all $f : \omega \to \omega_1$,

$$\forall m\, \forall n\, (x_{m+1}{\restriction}n, x_m{\restriction}n, f_m{\restriction}n) \in T \;\text{ if and only if }\; \forall k\, (x{\restriction}k, f{\restriction}k) \in U. \tag{25.38}$$

It follows from (25.38) that

$$E \text{ is well-founded if and only if } U \text{ is well-founded.} \tag{25.39}$$

Now let $E \subset \mathcal{N} \times \mathcal{N}$ be a Σ^1_2 well-founded relation; we want to show that its height is $< \omega_2$. Let T be a tree on $\omega^2 \times \omega_1$ such that (25.37) holds for all $x, y \in \mathcal{N}$ and let U be the tree on $\omega \times \omega_1$ constructed from T as above; since E is well-founded, U is well-founded.

Let us consider a generic extension $V[G]$ of the universe in which ω_1^V is countable and $\omega_2^V = \omega_1^{V[G]}$. Let us argue in $V[G]$.

Let E^* be the relation on $\mathcal{N}$ defined by (25.37). First we observe that $E \subset E^*$: If $x, y \in V$, then

$$\begin{aligned}
(x, y) \in E &\leftrightarrow V \vDash T(x, y) \text{ is ill-founded}\\
&\leftrightarrow V[G] \vDash T(x, y) \text{ is ill-founded}\\
&\leftrightarrow (x, y) \in E^*.
\end{aligned}$$

(because well-foundedness is absolute). We notice further that E^* is well-founded: This is because by the construction of U (which is absolute) and the definition of E^*, $V[G]$ satisfies (25.39), i.e.,

$$E^* \text{ is well-founded if and only if } U \text{ is well-founded.}$$

Hence E^* is well-founded, and $\text{height}(E) \leq \text{height}(E^*)$.

The tree T is a tree on $\omega \times \omega_1^V$ and ω_1^V is a countable ordinal. Since $E^* = p[T]$, it follows that E^* is a $\mathbf{\Sigma}_1^1$ relation. By Lemma 25.41, the height of E^* is countable. It follows that $\text{height}(E) < \omega_1^{V[G]} = \omega_2^V$.

Now we can step back into the ground model and look at the result of the above argument: $\text{height}(E) < \omega_2$. □

Both Theorem 25.42 and Lemma 25.41 are special cases of the more general Kunen-Martin Theorem:

Theorem 25.43. *Let κ be an infinite cardinal. Every κ-Suslin well-founded relation on $\mathcal{N}$ has height $< \kappa^+$.*

Proof. Let $<$ be a κ-Suslin well-founded relation on $\mathcal{N}$. We first associate with $<$ a tree $\mathcal{T}$ on $\mathcal{N}$ as follows:

$$\mathcal{T} = \{\langle x_0, \ldots, x_{n-1}\rangle : x_{n-1} < x_{n-2} < \ldots < x_0\}, \tag{25.40}$$

(and $\langle x\rangle \in \mathcal{T}$ for all $x \in \mathcal{N}$). $\mathcal{T}$ is well-founded and it suffices to prove that the height of $\mathcal{T}$ is $< \kappa^+$.

As $<$ is κ-Suslin, there exists a tree T on $\omega \times \omega \times \kappa$ such that

$$x < y \quad \text{if and only if} \quad \exists f\, (x, y, f) \in [T].$$

Let W be the set of ill sequences (of nodes at the same level of T)

$$w = \langle (s_1, s_0, h_0), \ldots, (s_{i+1}, s_i, h_i), \ldots, (s_k, s_{k-1}, h_{k-1})\rangle$$

with $(s_{i+1}, s_i, h_i) \in T$, and let

$$\begin{aligned} w' \prec w \quad \text{if and only if} \quad & k = \text{length}(w) < \text{length}(w') = k' \\ & \text{length}(s_0) < \text{length}(s_0'), \text{ and} \\ & \forall i < k\, s_i \subset s_i' \text{ and } h_i \subset h_i'. \end{aligned} \tag{25.41}$$

We claim that the relation $\prec$ is well-founded. Otherwise, let $w_n = \langle (s_{i+1}^n, s_i^n, h_i^n) : i < k_n\rangle$ be such that $w_{n+1} \prec w_n$ for all n. For each $i \in \omega$, let $x_i = \bigcup_{n=0}^\infty s_i^n$, and $f_i = \bigcup_{n=0}^\infty h_i^n$ (these exist by (25.41)). It follows that $(x_{i+1}, x_i, f_i) \in [T]$ for all i, hence $x_{i+1} < x_i$, and therefore $x_0 > x_1 > \ldots > x_i > \ldots$, a contradiction.

The set W has cardinality κ and it suffices to find an order preserving mapping from $\mathcal{T} - \{\emptyset\}$ into $(W, \prec)$. For every pair (x, y) such that $x < y$, the tree $T(x, y)$ on κ is not well-founded and has a branch h; let $h_{x,y}$ be the leftmost branch of the tree $T(x,y)$. Now let $\pi : \mathcal{T} - \{\emptyset\} \to W$ be as follows: $\pi(\langle x\rangle) = \emptyset$ for every $x \in \mathcal{N}$, and for $k \geq 2$,

$$\pi(\langle x_0,\ldots,x_{k-1}\rangle) = \langle (x_1{\restriction}k, x_0{\restriction}k, h_{x_1,x_0}{\restriction}k),\ldots,(x_k{\restriction}k, x_{k-1}{\restriction}k, h_{x_k,x_{k-1}}{\restriction}k)\rangle.$$

As $\pi(\langle x_0,\ldots,x_{k-1},x_k\rangle) \prec \pi(\langle x_0,\ldots,x_{k-1}\rangle)$, the mapping is order-preserving, completing the proof. □

Borel Codes

Every Borel set of reals is obtained, in fewer than ω_1 steps, from open intervals by taking complements and countable unions. We shall show how this procedure can be coded by a function $c \in \omega^\omega$. We shall define the set BC of *Borel codes* and assign to each $c \in \mathrm{BC}$ a unique Borel set A_c. The code c not only describes the Borel set A_c but also describes the procedure by which the set A_c is constructed from basic open sets.

Let $I_1, I_2, \ldots, I_k, \ldots$ be a recursive enumeration of open intervals with rational endpoints (i.e., the sequence of the pairs of endpoints is rercursive). For each $c \in \mathcal{N}$, let

$$u(c) \quad \text{and} \quad v_i(c) \qquad (i \in \boldsymbol{N}) \tag{25.42}$$

be elements of $\mathcal{N}$ defined as follows: If $d = u(c)$, then $d(n) = c(n+1)$ for all n; if $d = v_i(c)$, then $d(n) = c(\Gamma(i,n)+1)$ for all n (where Γ is the canonical one-to-one correspondence between $\boldsymbol{N} \times \boldsymbol{N}$ and $\boldsymbol{N}$).

For $0 < \alpha < \omega_1$, we define sets Σ_α and $\Pi_\alpha \subset \mathcal{N}$ as follows:

$$\begin{aligned}
&c \in \Sigma_1 && \text{if } c(0) > 1;\\
&c \in \Pi_\alpha && \text{if either } c \in \Sigma_\beta \cup \Pi_\beta \text{ for some } \beta < \alpha\\
& && \quad \text{or } c(0) = 0 \text{ and } u(c) \in \Sigma_\alpha;\\
&c \in \Sigma_\alpha\ (\alpha > 1) && \text{if either } c \in \Sigma_\beta \cup \Pi_\beta \text{ for some } \beta < \alpha\\
& && \quad \text{or } c(0) = 1 \text{ and } v_i(c) \in \textstyle\bigcup_{\beta<\alpha}(\Sigma_\beta \cup \Pi_\beta) \text{ for all } i.
\end{aligned} \tag{25.43}$$

If $c \in \Sigma_\alpha$ (if $c \in \Pi_\alpha$), we call c a $\mathbf{\Sigma}^0_\alpha$-*code* (a $\mathbf{\Pi}^0_\alpha$-*code*). Let BC, the set of all *Borel codes*, be

$$\mathrm{BC} = \bigcup_{\alpha<\omega_1} \Sigma_\alpha = \bigcup_{\alpha<\omega_1} \Pi_\alpha.$$

For every $c \in \mathrm{BC}$, we define a Borel set A_c as follows (we say that *c codes A_c*):

$$\begin{aligned}
&\text{if } c \in \Sigma_1 && \text{then } A_c = \textstyle\bigcup\{I_n : c(n) = 1\};\\
&\text{if } c \in \Pi_\alpha \text{ and } c(0) = 0 && \text{then } A_c = \boldsymbol{R} - A_{u(c)};\\
&\text{if } c \in \Sigma_\alpha \text{ and } c(0) = 1 && \text{then } A_c = \textstyle\bigcup_{i=0}^\infty A_{v_i(c)}.
\end{aligned} \tag{25.44}$$

It is clear that for every $\alpha > 0$, if $c \in \Sigma_\alpha$ (if $c \in \Pi_\alpha$), then $A_c \in \mathbf{\Sigma}^0_\alpha$ ($A_c \in \mathbf{\Pi}^0_\alpha$). Conversely, if B is a $\mathbf{\Sigma}^0_\alpha$ set (a $\mathbf{\Pi}^0_\alpha$ set), then there exists $c \in \Sigma_\alpha$ ($c \in \Pi_\alpha$) such that $B = A_c$. This is proved by induction on α using facts like: If c_i, $i \in \omega$ are elements of $\bigcup_{\beta<\alpha} \Pi_\beta$, then there is $c \in \Sigma_\alpha$ such that $c_i = v_i(c)$ for all $i \in \omega$.

Thus $\{A_c : c \in \mathrm{BC}\}$ is the collection of all Borel sets.

Lemma 25.44. *The set* BC *of all Borel codes is* Π^1_1.

Proof. Let us consider the following relation E on $\mathcal{N}$:

$$\text{(25.45)} \quad x \mathrel{E} y \quad \text{if and only if} \quad \begin{array}{l} \text{either } y(0) = 0 \text{ and } x = u(y), \\ \text{or } y(0) = 1 \text{ and } x = v_i(y) \text{ for some } i \in \omega. \end{array}$$

The relation E is arithmetical. If $y \in \Sigma_1$, then y is E-minimal (i.e., $\mathrm{ext}_E(y) = \emptyset$) and vice versa; if $y \in \Pi_\alpha$ and $x \mathrel{E} y$, then $x \in \Sigma_\alpha$, and if $y \in \Sigma_\alpha$ ($\alpha > 1$) and $x \mathrel{E} y$, then $x \in \bigcup_{\beta<\alpha}(\Sigma_\beta \cup \Pi_\beta)$.

We claim that

$$\text{(25.46)} \qquad \begin{aligned} y \in \mathrm{BC} &\leftrightarrow E \text{ is well-founded below } y \\ &\leftrightarrow \text{there is no } \langle z_0, z_1, \ldots z_n, \ldots\rangle \text{ such that } z_0 = y \\ &\qquad \text{and that } \forall n\, (z_{n+1} \mathrel{E} z_n). \end{aligned}$$

By the remark following (25.45), if $y \in \mathrm{BC}$, then there can be no infinite descending sequence $z_0 = y$, $z_1 \mathrel{E} z_0$, $z_2 \mathrel{E} z_1$, etc. Conversely, if E is well-founded below y, let ρ denote the rank function for E on $\mathrm{ext}_E(y)$. By induction on $\rho(x)$, one can see that every $x \in \mathrm{ext}_E(y)$ is a Borel code, and finally that y is itself a Borel code.

Now (25.46) gives a Π^1_1 definition of the set BC and the lemma follows. □

Lemma 25.45. *The properties $A_c \subset A_d$, $A_c = A_d$, and $A_c = \emptyset$ are Π^1_1 properties of Borel codes.*

Proof. We shall show that there are properties $P, Q \subset \boldsymbol{R} \times \mathcal{N}$ such that P is Π^1_1 and Q is Σ^1_1 and such that for every $c \in \mathrm{BC}$,

$$\text{(25.47)} \qquad a \in A_c \leftrightarrow (a, c) \in P \leftrightarrow (a, c) \in Q.$$

Then

$$\begin{aligned} A_c \subset A_d &\leftrightarrow c, d \in \mathrm{BC} \wedge \forall a\, ((a, c) \in Q \rightarrow (a, d) \in P), \\ A_c = A_d &\leftrightarrow c, d \in \mathrm{BC} \wedge A_c \subset A_d \wedge A_d \subset A_c, \\ A_c = \emptyset &\leftrightarrow c \in \mathrm{BC} \wedge \forall a\, (a, c) \notin Q. \end{aligned}$$

To find P and Q, let $x \in \mathcal{N}$ be fixed. Let T be the smallest set $T \subset \mathcal{N}$ such that

$$\text{(25.48)} \qquad x \in T, \text{ and if } y \in T \text{ and } z \mathrel{E} y, \text{ then } z \in T.$$

The set T is countable. Let $h : T \to \{0,1\}$ be a function with the following property: For all $y \in T$,

$$\begin{aligned}(25.49)\quad &\text{if } y(0) > 1\text{, then } h(y) = 1 \text{ if and only if}\\ &\qquad\qquad\text{for some } n,\ y(n) = 1 \text{ and } a \in I_n;\\ &\text{if } y(0) = 0\text{, then } h(y) = 1 \text{ if and only if } h(u(y)) = 0;\\ &\text{if } y(0) = 1\text{, then } h(y) = 1 \text{ if and only if for some } i,\ h(v_i(y)) = 1.\end{aligned}$$

Note that if x is a Borel code then there is a unique smallest countable set $T \subset \mathcal{N}$ with the property (25.48), and a unique function h with the property (25.49); moreover, for every $y \in T$ we have $h(y) = 1$ if and only if $a \in A_y$. Thus we let

$$(25.50)\quad (a,x) \in P \leftrightarrow (\forall \text{ countable } T \subset \mathcal{N})(\forall h : T \to \{0,1\})\\ \text{if (25.48) and (25.49) then } h(x) = 1,$$

and

$$(25.51)\quad (a,x) \in Q \leftrightarrow (\exists \text{ countable } T \subset \mathcal{N})(\exists h : T \to \{0,1\})\\ (25.48) \wedge (25.49) \wedge h(x) = 1,$$

and it is clear that if $c \in \mathrm{BC}$, then $a \in A_c$ if and only if $(a,c) \in P$ if and only if $(a,c) \in Q$.

It is a routine matter to verify that (25.50) can be written in Π^1_1 way and (25.51) in a Σ^1_1 way. (The quantifiers $\forall T$, $\forall h$, and $\exists T$, $\exists h$ are the only ones for which one needs quantifiers over $\mathcal{N}$; note that for instance, $\forall z\,(z \mathrel{E} y \to y \in T)$ in (25.48) can be written as

$$(y(0) = 0 \to u(y) \in T) \wedge (y(0) = 1 \to \forall i\,(v_i(y) \in T)).) \qquad \square$$

We shall now show that certain properties of Borel codes are absolute for transitive models of ZF + DC. (As usual, full ZF + DC is not needed, and the absoluteness holds for adequate transitive models.) If M is a transitive model of ZF + DC and $c \in \omega^\omega$ is in M, then because the set BC is Π^1_1, c is a Borel code if and only if $M \vDash c$ is a Borel code. By Lemma 25.45 the properties of the codes $A_c \subset A_d$, $A_c = A_d$, and $A_c = \emptyset$ are Π^1_1 and therefore absolute: $A_c = A_d$ holds if and only if $A^M_c = A^M_d$, etc., where A^M_c denotes the Borel set in M coded by c. Moreover, since $a \in A_c$ is Π^1_1, it follows that $A^M_c = A_c \cap M$ for every Borel code $c \in M$.

Lemma 25.46. *The following properties* (*of codes*) *are absolute for all transitive models M of* ZF + DC:

$$A_e = A_c \cup A_d, \qquad A_e = A_c \cap A_d,$$
$$A_e = \boldsymbol{R} - A_c, \qquad A_e = A_c \mathbin{\triangle} A_d, \qquad A_e = \bigcup_{n=0}^{\infty} A_{c_n}$$

(*we assume that the codes c, d, e are in M, as is the sequence $\langle c_n : n \in \omega\rangle$*).

We say that the operations $\cup$, $\cap$, $-$, $\triangle$, $\bigcup_{n=0}^{\infty}$ on Borel sets with codes in M are absolute for M.

Proof. If $c_0, c_1, \ldots, c_n, \ldots$ is a sequence of Borel codes in M, let $c \in \mathcal{N}$ be such that $c(0) = 1$ and that $v_i(c) = c_i$ for all $i \in \omega$. Clearly, c is a Borel code, $c \in M$, and c codes (both in the universe and in M) the Borel set $\bigcup_{n=0}^{\infty} A_{c_n}$. Hence for any Borel code $e \in M$, we have

$$A_e^M = \bigcup_{n=0}^{\infty} A_{c_n}^M \leftrightarrow A_e^M = A_c^M \leftrightarrow A_e = A_c \leftrightarrow A_e = \bigcup_{n=0}^{\infty} A_{c_n}$$

because $A_e = A_c$ is absolute for M. Thus $A_e = \bigcup_{n=0}^{\infty} A_{c_n}$ is absolute.

An analogous argument shows that $\boldsymbol{R} - A_c$ is absolute, and the rest of the lemma follows easily because the operations $\cap$, and $\triangle$ can be defined from $\cup$ and $-$. $\square$

Exercises

25.1. If $A \subset Seq \times \omega$ is arithmetical then $\{(x,n) : (x{\restriction}n, n) \in A\}$ is Δ_1^1.

25.2. (i) Every arithmetical relation is Δ_1^1.
(ii) If $A \subset \mathcal{N} \times \mathcal{N}$ is arithmetical then $\exists x\, A$ is Σ_1^1 and $\forall x\, A$ is Π_1^1.

25.3. The set $A = \{(x,z) : z \notin \mathrm{WO} \vee \|x\| \le \|z\|\}$ is Σ_1^1. Hence for each α, WO_α is $\Sigma_1^1(z)$ for each $z \in \mathrm{WO}$ such that $\|z\| = \alpha$.

[$(x,z) \in A \leftrightarrow z \notin \mathrm{WO} \vee (\exists h : \boldsymbol{N} \to \boldsymbol{N})\, \forall m\, \forall n\, (m\ E_x\ n \to h(m)\ E_x\ h(n))$.]

25.4. Every Σ_1 sentence is absolute for all inner models; in fact for all transitive models $M \supset L_\vartheta$ where $\vartheta = \omega_1^L$.

[Use Shoenfield's Absoluteness Lemma and Lemma 25.25.]

25.5. Modify the proof of Theorem 25.32 to show that Σ_2^1 has the prewellordering property.

25.6. Prove the prewellordering property of Σ_2^1 from the prewellordering property of Π_1^1.

A collection $\mathcal{C}$ of subsets of $\mathcal{N}$ satisfies the *reduction principle* if for every pair $A, B \in \mathcal{C}$ there are disjoint $A', B' \in \mathcal{C}$ such that $A' \subset A$, $B' \subset B$, and $A' \cup B' = A \cup B$. $\mathcal{C}$ satisfies the *separation principle* if for every pair of disjoint sets $A, B \in \mathcal{C}$ there is a set E such that both E and $\neg E$ are in $\mathcal{C}$, and that $A \subset E$ and $B \subset \neg E$. Lemma 11.11 proves that the collection of all analytic sets satisfies the separation principle.

25.7. The collection of Π_1^1 sets satisfies the reduction principle.

[Let φ and ψ be Π_1^1 norms on the Π_1^1 sets A and B and let $A' = \{x \in A : \psi(x) \not< \varphi(x)\}$ and $B' = \{x \in B : \varphi(x) \not\le \psi(x)\}$.]

25.8. The collection of Σ_2^1 sets satisfies the reduction principle.

The two exercises above hold also for $\Pi_1^1(a)$ and $\Sigma_2^1(a)$.

25.9. If a collection $\mathcal{C}$ satisfies the reduction principle then the collection $\mathcal{C}^* = \{A : \neg A \in \mathcal{C}\}$ satisfies the separation principle.

[If $A, B \in \mathcal{C}^*$ are disjoint, then $\neg A \cup \neg B = \mathcal{N}^r$ and so if $A', B' \in \mathcal{C}$ are disjoint such that $A' \subset \neg A$, $B' \subset \neg B$ and $A' \cup B' = \neg A \cup \neg B$, then $B' = \neg A'$ and both A' and B' are in $\mathcal{C}^*$.]

Hence the separation principle holds for Σ^1_1 and for Π^1_2 (and $\Sigma^1_1(a)$ and $\Pi^1_2(a)$).

25.10. There is no universal $\boldsymbol{\Delta}^1_n$ set, for any $n \in \boldsymbol{N}$, i.e., no $D \subset \mathcal{N}^2$ such that D is $\boldsymbol{\Delta}^1_n$ and that for every $\boldsymbol{\Delta}^1_n$ set $A \subset \mathcal{N}$ there is $v \in \mathcal{N}$ such that $A = \{x : (x, v) \in D\}$.

[Assume there is such a D and let $A = \{x : (x, x) \notin D\}$.]

25.11. The collection of $\boldsymbol{\Pi}^1_1$ sets (or $\boldsymbol{\Sigma}^1_2$ sets) does not satisfy the separation principle.

[The reason is that $\boldsymbol{\Pi}^1_1$ satisfies the reduction principle ($\boldsymbol{\Sigma}^1_2$ is similar). Let h be a homeomorphism of $\mathcal{N} \times \mathcal{N}$ onto $\mathcal{N}$, and let $U \subset \mathcal{N}^2$ be a universal $\boldsymbol{\Pi}^1_1$ set. Let $(x, h(u, v)) \in A$ if and only if $(x, u) \in U$, $(x, h(u, v)) \in B$ if and only if $(x, v) \in V$, and let A', B' be disjoint $\boldsymbol{\Pi}^1_1$ sets such that $A' \subset A$, $B' \subset B$, and $A' \cup B' = A \cup B$. If there existed $E \in \boldsymbol{\Delta}^1_1$ such that $A' \subset E$ and $B' \subset \neg E$, then E would be a universal $\boldsymbol{\Delta}^1_1$ set.]

25.12. Modify the proof of Theorem 25.34 to show that Σ^1_2 has the scale property.

25.13. Prove the scale property of Σ^1_2 from the scale property of Π^1_1.

25.14. Let $\langle \varphi_n : n \in \omega \rangle$ be a scale on A and let T be the tree $\{(s, \langle \alpha_0, \ldots, \alpha_{n-1} \rangle) : (\exists x \in A)\, x{\restriction}n = s \text{ and } \forall i < n\, \alpha_i = \varphi_i(x)\}$. Show that $A = p[T]$ and that for each $x \in A$, $T(x)$ has a least branch.

25.15. Using the scale property of Σ^1_2 prove the uniformization property of Σ^1_2.

Historical Notes

For classical descriptive set theory, see the books of Luzin [1930] and Kuratowski [1966]; the terminology is that of modern descriptive set theory based on the analogy with Kleene's hierarchies ([1955]).

The basic facts on $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_1$ sets are all in Luzin's book [1930] and some are of earlier origin: Lemma 25.10 was in effect proved by Lebesgue in [1905], and Corollary 25.13 and Lemma 25.17 were proved by Luzin and Sierpiński in [1923].

Theorem 25.19 appeared in Sierpiński [1925]. Theorem 25.36 is due to Kondô [1939].

Theorem 25.20 is due to Shoenfield [1961]. Previously, Mostowski had established absoluteness of Σ^1_1 and Π^1_1 predicates (Theorem 25.4). Lemma 25.25: Lévy [1965b].

The tree representation of Σ^1_2 sets is implicit in Shoenfield's proof in [1961]. Lemma 25.22 is due to Kechris and Moschovakis [1972].

Theorem 25.23 is due to Mansfield [1970] and Solovay [1969]. Lemma 25.24 was formulated and first proved by Mansfield.

Theorem 25.26 and corollaries: In his announcement [1938] Gödel stated that the Axiom of Constructibility implies that there exists a nonmeasurable $\boldsymbol{\Delta}^1_2$ set and an uncountable $\boldsymbol{\Pi}^1_1$ set without a perfect subset. Gödel did not publish the proof but gave an outline in the second printing (in 1951) of his monograph [1940].

Novikov in [1951] gave a proof of the corollaries (Kuratowski's paper [1948] contains somewhat weaker results) and Addison in [1959] worked out the details of Gödel outline of the proof of the theorem.

Lemma 25.30: Solovay [1967].

For scales and uniformization, see Moschovakis' book [1980]. Moschovakis introduced scales in [1971].

Theorem 25.38: Solovay [1969].

Theorem 25.39: Mansfield [1975].

Theorem 25.42 (as well as the present proof) is due to Martin; and Theorem 25.43 is due to Kunen and Martin, the present proof is Kunen's.

Borel codes are as in Solovay [1970].

The reduction and separation principles were introduced by Kuratowski; they are discussed in detail in Kuratowski's book [1966] and in Addison [1959].

Exercise 25.7: Kuratowski [1936].