28. More Applications of Forcing

In this chapter we present a selection of forcing constructions related to topics discussed earlier in the book.

A Nonconstructible Δ^1_3 Real

By Shoenfield's Absoluteness Theorem, every Π^1_2 or Σ^1_2 real is constructible; on the other hand $0^\sharp$ is a Δ^1_3 real. We now present a model due to Jensen that produces a nonconstructible Δ^1_3 real by forcing over L.

Theorem 28.1 (Jensen). *There is a generic extension $L[a]$ of L such that a is a Δ^1_3 real.*

The construction is a combination of perfect set forcing and arguments using the $\diamondsuit$-principle. Let us consider *perfect trees* $p \subset \mathit{Seq}(\{0,1\})$, cf. (15.24). The *stem* of a perfect tree p is the maximal $s \in \mathit{Seq}(\{0,1\})$ such that for every $t \in p$, either $t \subset s$ or $s \subset t$. If p is a perfect tree and if $s \in p$, we denote by $p{\restriction}s$ the perfect tree $\{t \in p : t \subset s \text{ or } t \supset s\}$.

Assume that P is a set of perfect trees, partially ordered by $\subset$, such that if $p \in P$ and $s \in p$, then $p{\restriction}s \in P$, and let G be an L-generic filter on P. Then there is a unique $f \in \{0,1\}^\omega$ which is a branch in every $p \in G$; and conversely, $G = \{p \in P : f \text{ is a branch in } p\}$. Therefore $L[G] = L[f]$, and we call f P-generic over L. Note that $f \in \{0,1\}^\omega$ is P-generic over L if and only if for every constructible predense set $X \subset P$, f is a branch in some $p \in X$.

Similarly, a generic filter G on $P \times P$ corresponds to a unique pair (a, b) such that for each $(p, q) \in G$, a is a branch in p and b is a branch in q. A pair (a, b) is $(P \times P)$-generic over L if and only if for every constructible predense set $X \subset P \times P$, there exists a pair $(p, q) \in X$ such that a is a branch in p and b is a branch in q.

In Chapter 15 we used the Fusion Lemma for perfect trees. Let $T = \{T(s) : s \in \mathit{Seq}(\{0,1\})\}$ be a collection of perfect trees such that for every s,

(28.1) (i) $T(s)$ is a perfect tree whose stem has length $\geq$ length(s).
(ii) $T(s^\frown 0) \subset T(s)$ and $T(s^\frown 1) \subset T(s)$.
(iii) $T(s^\frown 0)$ and $T(s^\frown 1)$ have incompatible stems.

If T satisfies (28.1), we say that T is *fusionable* and we let

$$\mathcal{F}(T) = \bigcap_{n=0}^{\infty} \bigcup_{s\in\{0,1\}^n} T(s). \tag{28.2}$$

For each fusionable T, $p = \mathcal{F}(T)$ (the *fusion* of T) is a perfect tree; and for each s, if t is the stem of $p_s = T(s)$, then $p{\restriction}t$ is stronger than both p and p_s.

We shall not use the set of all perfect trees as the notion of forcing; rather we shall construct a set P of perfect trees with the property that if $p \in P$ and $s \in p$, then $p{\restriction}s \in P$. We shall construct P such that if a is P-generic over L, then a is the only P-generic set in $L[a]$, and such that $\{n \in \boldsymbol{N} : a(n) = 1\}$ is (in $L[a]$) a Δ^1_3 subset of $\boldsymbol{N}$.

We shall construct P as the union of countable sets

$$P_0 \subset P_1 \subset \ldots \subset P_\alpha \subset \ldots \qquad (\alpha < \omega_1)$$

of perfect trees. The construction uses the $\diamondsuit$-principle. There is a $\diamondsuit$-sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ that is Δ_1 over L_{ω_1}; let us fix such a sequence. Also, let us fix a Δ_1 over L_{ω_1} function τ that is a one-to-one mapping of L_{ω_1} onto ω_1.

We shall now construct the sequence $P_0 \subset P_1 \subset \ldots \subset P_\alpha \subset \ldots$:

(28.3) $P_0 =$ the set of all $p_0{\restriction}s$ where p_0 is the full binary tree, $p_0 = Seq(\{0,1\})$, and $s \in p_0$;

$$P_\alpha = \bigcup_{\beta<\alpha} P_\beta \text{ if } \alpha \text{ is a limit ordinal.}$$

$P_{\alpha+1} = P_\alpha \cup Q_{\alpha+1}$ where $Q_{\alpha+1}$ is a set of perfect trees defined as follows:

Let $P_\alpha = \{p^\alpha_n : n \in \omega\}$; and let us consider the $<_L$-least such enumeration. Let $\mathcal{X}_\alpha$ and $\mathcal{Y}_\alpha$ be the following countable collections of subsets of P_α and and $P_\alpha \times P_\alpha$ respectively:

(28.4) $\mathcal{X}_\alpha$ contains:
- (i) all Q_β, $\beta \le \alpha$,
- (ii) all $X \subset P_\alpha$ such that $\tau``X = S_\beta$ for some $\beta \le \alpha$.

$\mathcal{Y}_\alpha$ contains:
- (i) $Q_\beta \times Q_\beta$ for all $\beta \le \alpha$;
- (ii) all $Y \subset P_\alpha \times P_\alpha$ such that $\tau``Y = S_\beta$ for some $\beta \le \alpha$.

There exists a family $\{T_n : n \in \omega\}$ of fusionable collections of elements of P_α such that:

(28.5)
- (i) $T_n(\emptyset) = p^\alpha_n$ for all n;
- (ii) for every $X \in \mathcal{X}_\alpha$, if X is predense in P_α, then for every $n \in \boldsymbol{N}$ and every $h \in \boldsymbol{N}$ there is $k \ge h$ such that for each $s \in \{0,1\}^k$, there exists an $x \in X$ such that $T_n(s) \le x$;
- (iii) for every $Y \in \mathcal{Y}_\alpha$, if Y is predense in $P_\alpha \times P_\alpha$, then for every n, every m, and every h there is a $k \ge h$ such that for each $s \in \{0,1\}^k$ and each $t \in \{0,1\}^k$; if either $n \ne m$ or $s \ne t$, then there exists $(x,y) \in Y$ such that $(T_n(s), T_m(t)) \le (x,y)$.

A family $\{T_n : n \in \omega\}$ with properties (28.5)(i)–(iii) is easily constructed because $\mathcal{X}_\alpha$ and $\mathcal{Y}_\alpha$ are countable. We denote $\{T_n^\alpha : n \in \omega\}$ the $<_L$-least such family, and let

$$Q_{\alpha+1} = \{p{\restriction}s : p = \mathcal{F}(T_n^\alpha) \text{ for some } n, \text{ and } s \in p\}. \tag{28.6}$$

We let $P = \bigcup_{\alpha<\omega_1} P_\alpha$. The following sequence of lemmas will show that if $a \in \{0,1\}^\omega$ is P-generic over L, then in $L[a]$ the set $\{n : a(n) = 1\}$ is Δ^1_3.

Lemma 28.2. *For each α, $Q_{\alpha+1}$ is dense in $P_{\alpha+1}$, and $Q_{\alpha+1} \times Q_{\alpha+1}$ is dense in $P_{\alpha+1} \times P_{\alpha+1}$.*

Proof. It suffices to show that below each $p \in P_\alpha$ there is some $q \in Q_{\alpha+1}$. If $p \in P_\alpha$, the $p = p_n^\alpha$ for some n, and $\mathcal{F}(T_n) \subset T_n(\emptyset) = p$. $\square$

Lemma 28.3. *For each α, if $X \in \mathcal{X}_\alpha$ is predense in P_α, then X is predense in $P_{\alpha+1}$; if $Y \in \mathcal{Y}_\alpha$ is predense in $P_\alpha \times P_\alpha$, then Y is predense in $P_{\alpha+1} \times P_{\alpha+1}$. Consequently, if $X \in \mathcal{X}_\alpha$ is predense in P_α (if $Y \in \mathcal{Y}_\alpha$ is predense in $P_\alpha \times P_\alpha$), then X is predense in P (Y is predense in $P \times P$).*

Proof. Let $X \in \mathcal{X}_\alpha$ be predense in P_α; we have to show that for each $p{\restriction}n \in Q_{\alpha+1}$ there is a stronger $q \in Q_{\alpha+1}$ such that $q \le x$ for some $x \in X$. Let $p = \mathcal{F}(T_n)$ and let $u \in p$. Let $h = \text{length}(u)$. There is $k \ge h$ such that n and k satisfy (28.5)(ii). There is $s \in \{0,1\}^k$ such that $u \in T_n(s)$; let v be the stem of $T_n(s)$. Then $p{\restriction}v \le T_n(s)$ and $T_n(s) \le x$ for some $x \in X$.

A similar argument, using (28.5)(iii), shows that if $Y \in \mathcal{Y}_\alpha$ is predense in $P_\alpha \times P_\alpha$, then Y is predense in $P_{\alpha+1} \times P_{\alpha+1}$.

Since the sequences $\mathcal{X}_\alpha$, $\alpha < \omega_1$, and $\mathcal{Y}_\alpha$, $\alpha < \omega_1$, are increasing, it follows by induction that X is predense in every P_β, $\beta < \omega_1$, and hence in P. Similarly for Y. $\square$

Lemma 28.4. *$P \times P$ satisfies the countable chain condition (and hence P also satisfies the countable chain condition).*

Proof. Here we use $\diamondsuit$. Let us assume that $Y \subset P \times P$ is a maximal incompatible set of conditions in $P \times P$ and that Y is uncountable. Since each P_α is countable, it is easy to see that the set of all $\alpha < \omega_1$ such that $\tau(Y \cap (P_\alpha \times P_\alpha)) = \tau(Y) \cap \omega_1$ is closed unbounded (τ is the one-to-one function of L_{ω_1} onto ω_1). Then it is not much more difficult to see that the set of all $\alpha < \omega_1$ such that $Y \cap (P_\alpha \times P_\alpha)$ is a maximal antichain in $P_\alpha \times P_\alpha$, is closed unbounded (compare this argument with the $\diamondsuit$-construction of a Suslin tree in L).

By $\diamondsuit$, there exists an α such that $Y' = Y \cap (P_\alpha \times P_\alpha)$ is predense in $P_\alpha \times P_\alpha$ and that $\tau(Y') = S_\alpha$. Therefore $Y' \in \mathcal{Y}_\alpha$ and by Lemma 28.3, Y' is predense in $P \times P$. It follows that $Y' = Y$. Thus Y is countable. $\square$

Lemma 28.5.

(i) *If $a \in \{0,1\}^\omega$, then a is P-generic over L if and only if for every $\alpha < \omega_1$ there is some $n \in \boldsymbol{N}$ such that a is a branch in $\mathcal{F}(T_n^\alpha)$.*
(ii) *If $a \neq b \in \{0,1\}^\omega$, then (a,b) is $(P \times P)$-generic over L if and only if for every $\alpha < \omega_1$ there exist $n, m \in \boldsymbol{N}$ such that a is a branch in $\mathcal{F}(T_n^\alpha)$ and b is a branch in $\mathcal{F}(T_m^\alpha)$.*

Proof. (i) Let a be P-generic and let $\alpha < \omega_1$. Since $Q_{\alpha+1}$ is dense in $P_{\alpha+1}$ and because $Q_{\alpha+1} \in \mathcal{X}_{\alpha+1}$, $Q_{\alpha+1}$ is predense in P. By the genericity of a, there exists a $q \in Q_{\alpha+1}$ such that a is a branch in q. But $q = p{\restriction}s$ where $p = \mathcal{F}(T_n^\alpha)$ for some n and $s \in p$, and clearly a is a branch in p.

Conversely, let us assume that the condition is satisfied. Let $X \subset P$ be a maximal antichain; we wish to show that a is a branch in some $x \in X$. By Lemma 28.4, X is countable, and there is an α such that $X \in \mathcal{X}_\alpha$. Let $n \in \boldsymbol{N}$ be such that a is a branch in $\mathcal{F}(T_n^\alpha)$.

By (28.5)(ii), there is $k \in \boldsymbol{N}$ such that each $T_n(s)$, $s \in \{0,1\}^k$, is stronger than some $x \in X$. Since a is a branch in $\mathcal{F}(T_n)$, it is clear that there is a unique $s \in \{0,1\}^k$ such that a is a branch in $T_n(s)$. But if $x \in X$ is such that $T_n(s) \subset x$, then a is also a branch in x.

(ii) The proof that the condition is necessary is analogous to (i). Thus let us assume that the condition is satisfied and let $Y \subset P \times P$ be a maximal antichain; we want to find $(x,y) \in Y$ such that a is a branch in x and b is a branch in y. Again, there is α such that $Y \in \mathcal{Y}_\alpha$. Let n and $m \in \boldsymbol{N}$ be such that a is a branch in $\mathcal{F}(T_n^\alpha)$ and that b is in $\mathcal{F}(T_m^\alpha)$.

Let $h \in \boldsymbol{N}$ be such that $a{\restriction}h \neq b{\restriction}h$; by (28.5)(iii), there is some $k \in \boldsymbol{N}$ such that for each $s, t \in \{0,1\}^k$, if either $n \neq m$ or $s \neq t$, then $(T_n(s), T_m(t)) \leq (x,y)$ for some $(x,y) \in Y$. There is a unique pair s, t such that a is a branch in x and b is a branch in y where (x,y) is some element of Y such that $(T_n(s), T_m(t)) \leq (x,y)$. □

Corollary 28.6. *If a and b are P-generic over L and $a \neq b$, then (a,b) is $(P \times P)$-generic over L.* □

Corollary 28.7. *If a is P-generic over L, then $L[a] \vDash a$ is the only P-generic over L.*

Proof. If $a \neq b$ and if both a and b are P-generic over L, then by the Product Lemma, b is a P-generic over $L[a]$ and hence $b \notin L[a]$. □

Lemma 28.8. *The set $H = \{a : a$ is P-generic over $L\}$ is Π_1 over HC.*

Proof. It follows from the construction of P that the function $\alpha \mapsto \langle T_n^\alpha : n \in \omega\rangle$ is Δ_1 over $L_{\omega_1^L}$. Since $L_{\omega_1^L}$ is a Σ_1 set over HC, the function is Δ_1 over HC. By Lemma 28.5,

$$a \in H \leftrightarrow (\forall \alpha < \omega_1^L)(\exists n \in \omega)\, a \text{ is a branch in } \mathcal{F}(T_n^\alpha)$$

and hence H is Π_1 over HC. □

Corollary 28.9. *If a is a P-generic over L and $A = \{n \in \boldsymbol{N} : a(n) = 1\}$, then $L[a] \vDash A$ is a Δ^1_3 subset of $\boldsymbol{N}$.*

Proof. We have in $L[a]$

$$n \in A \leftrightarrow (\exists a \in \mathcal{N})(a \in H \text{ and } a(n) = 1) \leftrightarrow (\forall a \in \mathcal{N})(a \in H \rightarrow a(n) = 1).$$

Since H is Π_1 over HC, it is a Π^1_2 subset of $\mathcal{N}$. It follows that A is Δ^1_3. $\square$

Namba Forcing

By Jensen's Covering Theorem if λ is a regular cardinal in L and V is a generic extension of L, then either $\operatorname{cf}\lambda = |\lambda|$ or $\lambda < \aleph_2$. In other words, the only nontrivial change of cofinality is to make $|\lambda| = \aleph_1$ and $\operatorname{cf}\lambda = \omega$. The following model, due to Namba, does exactly that:

Theorem 28.10 (Namba). *Assume* CH. *There is a generic extension $V[G]$ such that $\aleph_1^{V[G]} = \aleph_1$ and $\operatorname{cf}^{V[G]}(\omega_2^V) = \omega$.*

Proof. Let S be the set of all finite sequences of ordinals less than ω_2, $S = \omega_2^{<\omega}$. A *tree* is a set $T \subset S$ such that if $t \in T$ and $s = t{\restriction}n$ for some n, then $s \in T$. A nonempty tree $T \subset S$ is *perfect* if every $t \in T$ has $\aleph_2$ extensions $s \supset t$ in T. (Note that then every $t \in T$ has $\aleph_2$ *incompatible* extensions in T.) In analogy with perfect sets in the Baire space, we have a Cantor-Bendixson analysis of trees $T \subset S$: Let

$$T' = \{t \in T : t \text{ has } \aleph_2 \text{ extensions in } T\}$$

and let $T_0 = T$, $T_{\alpha+1} = T'_\alpha$, $T_\alpha = \bigcap_{\beta<\alpha} T_\beta$ if α is limit. Let $\theta < \omega_3$ be the least θ such that $T'_\theta = T_\theta$. Then T_θ is either empty or perfect.

If T has no perfect $\bar{T} \subset T$, then the above procedure leads to $T_\theta = \emptyset$, and we can associate with each $t \in T$ an ordinal number

$$h_T(t) = \text{the least } \alpha \text{ such that } t \notin T_{\alpha+1}. \tag{28.7}$$

If $s \subset t$, then $h_T(s) \geq h_T(t)$, and for every $t \in T$,

$$|\{s \in T : t \subset s \text{ and } h_T(s) = h_T(t)\}| < \aleph_2. \tag{28.8}$$

Now let us describe the notion of forcing.

Let P be the set of all perfect trees $T \subset S$, partially ordered by inclusion. We shall show that in the generic extension, ω_2 has cofinality ω and ω_1 is preserved.

If G is a generic set of conditions, we define in $V[G]$ a function $f : \omega \to \omega_2^V$ as follows:

$$f(n) = \alpha \leftrightarrow \forall T \in G \ \exists s \in T \text{ such that } s(n) = \alpha.$$

An easy argument using genericity of G shows that $f(n)$ is uniquely defined for each n and that the function f maps ω cofinally into ω_2^V.

We shall prove now that ω_1 is preserved in the extension by showing that every $f : \omega \to \{0,1\}$ in $V[G]$ is in the ground model. Thus let T be a condition, and let $\dot{f}$ be a name such that

$$T \Vdash \dot{f} \text{ is a function from } \omega \text{ into } \{0,1\}.$$

We shall find a stronger condition that decides each $\dot{f}(n)$; that is, we shall find a function $g : \omega \to \{0,1\}$ such that some condition stronger than T forces $\dot{f}(n) = g(n)$, for all n.

We proceed as follows. By induction on length of s, we construct, for each $s \in S$, conditions T_s and numbers $\alpha_s \in \{0,1\}$ such that:

(28.9)
- (i) if $s_1 \subset s_2$, then $T_{s_1} \supset T_{s_2}$,
- (ii) if $\text{length}(s) = n$, then $T_s \Vdash \dot{f}(n) = \alpha_s$,
- (iii) for every n, the conditions T_s, $s \in \omega_2^n$, are mutually incompatible, and moreover, there are mutually incompatible sequences $t_s \in S$, $s \in \omega_2^n$, such that for each $s \in \omega_2^n$, $t_s \in T_s$ and for all $t \in T_s$, either $t \subset t_s$ or $t_s \subset t$.

The "moreover" clause in (iii) is stronger than incompatibility of the conditions and implies that any condition stronger than $\bigcup_{s\in\omega_2^n} T_s$ is compatible with some T_s, $s \in \omega_2^n$.

The construction of conditions satisfying (28.9) is straightforward: We let $t_\emptyset = \emptyset$ and $T_\emptyset \subset T$ be any condition that decides $\dot{f}(0)$: $T_\emptyset \Vdash \dot{f}(0) = \alpha_\emptyset$. Having defined T_s, t_s, and α_s for $s \in \omega_2^n$ we first pick $\aleph_2$ incompatible extensions $t_{s^\frown i}$, $i < \omega_2$, of t_s in T_s, and then find $T_{s^\frown i} \subset T_s$ and $\alpha_{s^\frown i}$, $i < \omega_2$, such that $T_{s^\frown i} \Vdash \dot{f}(n+1) = \alpha_{s^\frown i}$ and that each $t \in T_{s^\frown i}$ is compatible with $t_{s^\frown i}$. Note that if $s_1 \subset s_2$, then $t_{s_1} \subset t_{s_2}$.

For any function $g : \omega \to \{0,1\}$, we define a tree $T(g) \subset S$ (not necessarily a perfect tree) as follows: If $\bar{\beta} = \langle \beta_0, \ldots, \beta_n \rangle$ is a finite sequence of zeros and ones, we let

$$T(\bar{\beta}) = \bigcup\{T_s : s \in \omega_2^n \text{ and } \langle \beta_0, \ldots, \beta_n \rangle = \langle \alpha_\emptyset, \ldots, \alpha_{s\restriction k}, \ldots, \alpha_s \rangle\} \tag{28.10}$$

and

$$T(g) = \bigcap_{n=1}^{\infty} T(g\restriction n). \tag{28.11}$$

Each $T(\bar{\beta})$ is a condition (a perfect tree) and by the remark following (28.9), we have

$$T(\bar{\beta}) \Vdash \dot{f}(k) = \beta_k \qquad (k = 0,\ \ldots,\ n).$$

Thus if we show that there is at least one $g : \omega \to \{0,1\}$ such that the tree $T(g)$ contains a perfect subtree, our proof will be complete.

Lemma 28.11. *There exists some $g : \omega \to \{0,1\}$ such that $T(g)$ contains a perfect subtree.*

Proof. Let us assume that no $T(g)$ has a perfect subtree. Then by (28.8) there exists, for each $g : \omega \to \{0,1\}$, a function $h_g : T(g) \to \omega_3$ such that $h_g(s) \geq h_g(t)$ whenever $s \subset t$, and that for each $t \in T(g)$, there are at most $\aleph_1$ elements $s \supset t$ in $T(g)$ such that $h_g(s) = h_g(t)$.

By induction, we construct a sequence $s_0 \subset s_1 \subset \ldots \subset s_n \subset \ldots$ such that for all n, $s_n \in \omega_2^n$. At stage n we consider the node t_{s_n} of T_{s_n}. Since there are only $\aleph_1$ functions $g : \omega \to \{0,1\}$, there exists an $i < \omega_2$ such that $h_g(t_{s_n^\frown i}) < h_g(t_{s_n})$ for all g for which $h_g(t_{s_n^\frown i})$ is defined. We let $s_{n+1} = s_n^\frown i$.

Given the sequence s_n, $n = 0, 1, \ldots$, we consider the function $g(n) = \alpha_{s_n}$, $n < \omega$. By (28.10) and (28.11), each t_{s_n} belongs to $T(g)$, and so $h_g(t_{s_n})$ is defined for all n. However, then the sequence $h_g(t_{s_0}) > h_g(t_{s_1}) > \ldots$ of ordinals is descending, a contradiction. $\square$

A Cohen Real Adds a Suslin Tree

We proved earlier that Suslin trees exist in L, and that adding generically a subset of ω_1 with countable conditions adds a Suslin tree. It turns out that adding a Cohen real also adds a Suslin tree. This result is due to Shelah; the following proof is due to Todorčević.

Theorem 28.12 (Shelah). *If r is a Cohen real over V then in $V[r]$ there exists a Suslin tree.*

Proof. We start with an alternative construction of an Aronszajn tree, a modification of the construction in Theorem 9.16.

Lemma 28.13. *There exists an ω_1-sequence of functions $\langle e_\alpha : \alpha < \omega_1 \rangle$ such that*

(28.12) (i) *e_α is a one-to-one function from α into ω, for each $\alpha < \omega_1$;*
(ii) *for all $\alpha < \beta < \omega_1$, $e_\alpha(\xi) = e_\beta(\xi)$ for all but finitely many $\xi < \alpha$.*

Proof. Exercise 28.1 (or see Kunen [1980], Theorem II.5.9). $\square$

The set $\{e_\alpha \restriction \beta : \alpha, \beta \in \omega_1\}$ ordered by inclusion is a tree. Since every node at level α is a finite change of e_α, all levels are countable; there are no uncountable branches and so the tree is an Aronszajn tree (Exercise 28.2).

For any function $r : \omega \to \omega$, consider the tree

$$T_r = \{r \circ (e_\alpha \restriction \beta) : \alpha, \beta \in \omega_1\}; \tag{28.13}$$

again, T_r is an ω_1-tree whose all levels are countable (but need not be Aronszajn in general). We prove Theorem 28.12 by showing that if $\langle e_\alpha : \alpha < \omega_1 \rangle$ is, in V, a sequence that satisfies (28.12) and if r is a Cohen real over V, then in $V[G]$, T_r is a Suslin tree.

We show that T_r has no uncountable antichains; this, and an easy argument using genericity of r, also shows that T_r has no uncountable branches. If T_r has an uncountable antichain then, because every uncountable subset of ω_1 in $V[r]$ has an uncountable subset in V (Exercise 28.3), there exist in V, an uncountable set $W \subset \omega_1$ and a function $\langle \alpha(\beta) : \beta \in W \rangle$ such that

$$\{r \circ (e_{\alpha(\beta)} \restriction \beta) : \beta \in W\} \tag{28.14}$$

is an antichain.

For each $\beta \in W$, let $t_\beta = e_{\alpha(\beta)} \restriction \beta$, and let p be a Cohen forcing condition; we shall find a stronger condition q and $\beta_1, \beta_2 \in W$ that forces that $\dot{r} \circ t_{\beta_1}$ and $\dot{r} \circ t_{\beta_2}$ are compatible functions; therefore no condition forces that (28.14) is an antichain in T_r.

Let $p = \langle p(0), \ldots, p(n-1) \rangle$. For each $\beta \in W$, let X_β be the finite set $\{\xi < \beta : t_\beta(\xi) < n\}$. By the Δ-Lemma (Theorem 9.18) there exist a finite set $S \subset \omega_1$ and an uncountable $Z \subset W$ such that when $\beta_1, \beta_2 \in Z$, then $X_{\beta_1} \cap X_{\beta_2} = S$ and that $t_{\beta_1} \restriction S = t_{\beta_2} \restriction S$.

Now let $\beta_1 < \beta_2$ be two elements of Z. We claim that there exists a condition $q \supset p$ such that $q \circ (t_{\beta_2} \restriction \beta_1) = q \circ t_{\beta_1}$ (q obliterates the disagreement). Such a condition q forces $\dot{r} \circ t_{\beta_1} \subset \dot{r} \circ t_{\beta_2}$.

To construct q, let m be greater than $t_{\beta_i}(\xi)$, $i = 1, 2$, for each $\xi < \beta_1$ such that $t_{\beta_1}(\xi) \neq t_{\beta_2}(\xi)$. Let k be such that $n \leq k < m$. If there exist $\xi, \eta < \beta_1$ such that $t_{\beta_2}(\eta) = k$ and $t_{\beta_1}(\eta) = t_{\beta_2}(\xi)$, let $l = t_{\beta_1}(\xi)$ and let $q(k) = p(l)$. More generally, let $f = t_{\beta_1}^{-1} \circ t_{\beta_2}$ and let f^i, $i < \omega$, denote the i-th iterate of f. If there exist $\xi, \eta < \beta_1$ such that $t_{\beta_2}(\eta) = k$ and $\eta = f^i(\xi)$ for some i, let $l = t_{\beta_1}(\xi)$ and let $q(k) = p(l)$. Otherwise, let $q(k) = 0$. Verify that q obliterates the disagreement. $\square$

Consistency of Borel's Conjecture

A set X of real numbers has strong measure zero if for every sequence $\langle \varepsilon_n : n < \omega \rangle$ of positive real numbers there is a sequence $\langle I_n : n < \omega \rangle$ of intervals with $\text{length}(I_n) \leq \varepsilon_n$ such that $X \subset \bigcup_{n=0}^{\infty} I_n$.

Borel's Conjecture. *All strong measure zero sets are countable.*

Borel's Conjecture fails under CH—see Exercise 26.18. The following theorem shows that it is consistent with ZFC:

Theorem 28.14 (Laver). *Assuming* GCH *there is a generic extension* $V[G]$ *in which* $2^{\aleph_0} = \aleph_2$ *and Borel's Conjecture holds.*

Laver's proof uses the countable support iteration (of length ω_2) of a forcing notion that adds a *Laver real.* We shall now describe this forcing. (Subsequently, Laver proved that an iteration of Mathias forcing also yields Borel's Conjecture).

Definition 28.15. A tree $p \subset Seq$ is a *Laver tree* if it has a *stem*, i.e., a maximal node $s_p \in p$ such that $s_p \subset t$ or $t \subset s_p$ for all $t \in p$, and

(28.15) $\forall t \in p$ if $t \supset s_p$ then the set $S^p(t) = \{a \in \omega : t^\frown a \in p\}$ is infinite.

Laver forcing has as forcing conditions Laver trees, partially ordered by inclusion. If G is a generic set of Laver trees, let

$$f = \bigcup\{s_p : p \in G\}; \tag{28.16}$$

the function $f : \omega \to \omega$ is a *Laver real.* Since

$$G = \{p : s_p \subset f \text{ and } \forall n \geq |s|\, f(n) \in S^p(f{\restriction}n)\}$$

we have $V[G] = V[f]$.

Consider a canonical enumeration of Seq in which s appears before t if $s \subset t$ and $s^\frown a$ appears before $s^\frown(a+1)$. If p is a Laver tree, then the part of p above the stem is isomorphic to Seq and we have an enumeration $s_0^p = s_p$, $s_1^p, \ldots, s_n^p, \ldots$ of $\{t \in p : t \supset s_p\}$, for every Laver tree p. Let

$$q \leq_n p \text{ if } q \leq p \text{ and } s_i^p = s_i^q \text{ for all } i = 0, \ldots, n \tag{28.17}$$

(in particular $q \leq_0 p$ means that $q \leq p$ and p and q have the same stem). A *fusion sequence* is a sequence of Laver trees such that

$$p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 \cdots \geq_n \cdots.$$

Lemma 28.16. *If $\{p_n\}_{n=0}^\infty$ is a fusion sequence then $p = \bigcap_{n=0}^\infty p_n$ is a Laver tree* (*the* fusion *of* $\{p_n\}_{n=0}^\infty$), *and $p \leq_n p_n$ for all n.*

Proof. Let s_0 bet the stem of p_0. Then s_0 is the stem of p, and the set $S^p(s_0) = \bigcap_n S^{p_n}(s_0)$ is infinite. For every $a \in S^p(s_0)$, the set $S^p(s_0{}^\frown a) = \bigcap_n S^{p_n}(s_0{}^\frown a)$ is infinite, and so on. □

If p is a Laver tree and $s \in p$, then $p{\restriction}s$ is the Laver tree $\{t \in p : t \subset s$ or $t \supset s\}$. Let p be a Laver tree and let $n \geq 0$. For each $i \leq n$, let p_i be the tree with stem s_i^p that is the union of all $p{\restriction}(s_i^p{}^\frown a)$ where $a \in S^p(s_i^p)$ and $s_i^p{}^\frown a$ is not one of the s_j^p, $j \leq n$. The trees $p_0, \ldots, p_n$ (the *n-components* of p) form a maximal set of incompatible subtrees of p.

Let $q_0, \ldots, q_n$ be the Laver trees such that $q_i \leq_0 p_i$ for all $i = 0, \ldots, n$. The *amalgamation* of $\{q_0, \ldots, q_n\}$ into p is the tree

$$r = q_0 \cup \ldots \cup q_n; \tag{28.18}$$

we have $r \leq_n p$.

Lemma 28.17. *If $p \Vdash \dot{X} : \omega \to V$ then there exists a $q \leq_0 p$ and a countable A such that $q \Vdash \dot{X} \subset A$.*

Proof. Let $\{u_n\}_n$ be a sequence of natural numbers such that each number appears infinitely often. We shall construct a fusion sequence $\{p_n\}_n$ with $p_0 = p$, and finite sets A_n so that the fusion forces $\dot{X} \subset \bigcup_n A_n$. At stage n, let $p^0, \ldots, p^n$ be the n-components of the Laver tree p_n. For each $i = 0, \ldots, n$ if there exist a condition $q_i \leq_0 p^i$ and some a_n^i such that

$$q_i \Vdash \dot{X}(u_n) = a_n^i \tag{28.19}$$

we choose such q_i and a_n^i (otherwise let $q_i = p^i$). Let A_n be the collection of the a_n^i, and let p_{n+1} be the amalgamation of $\{q_0, \ldots, q_n\}$ into p_n. We have $p_{n+1} \leq_n p_n$.

Let p_∞ be the fusion of $\{p_n\}_{n=0}^\infty$ and let $A = \bigcup_{n=0}^\infty A_n$. We have $p_\infty \leq_0 p$; to prove that $p_\infty \Vdash \dot{X} \subset A$, let $q \leq p_\infty$ and let $u \in \omega$. Let $\bar{q} \leq q$ and a be such such that $\bar{q} \Vdash \dot{X}(n) = a$. Let n be large enough so that $u = u_n$ and that the stem of $\bar{q}$ is in the set $\{s_0^{p_n}, \ldots, s_n^{p_n}\}$, say $s = s_i^{p_n}$.

Let p^i be the ith n-component of p_n. As $\bar{q} \cap p^i \leq_0 p^i$ and decides $\dot{X}(u_n)$, we have chosen $a_n^i = a$ at that stage, and therefore $a \in A$, and $\bar{q} \Vdash \dot{X}(u) \in A$. Hence $p_\infty \Vdash \dot{X} \subset A$. □

Corollary 28.18. *The Laver forcing preserves $\aleph_1$.* □

The following property of the Laver forcing is reminiscent of Prikry and Mathias forcings:

Lemma 28.19. *Let $p \Vdash \varphi_1 \vee \ldots \vee \varphi_k$. Then there exists some $q \leq_0 p$ such that*

$$\exists i \leq k \; q \Vdash \varphi_i. \tag{28.20}$$

Proof. Assume to the contrary that the lemma fails. Let s be the stem of p; there are only finitely many $a \in S^p(s)$ such that some $q \leq_0 p{\restriction}(s^\frown a)$ satisfies (28.20). By removing the part of p above these finitely many nodes we obtain $p_1 \leq_0 p$. For every $s^\frown a \in p_1$ there are only finitely many $b \in S^p(s^\frown a)$ such that $\exists q \leq_0 p_1{\restriction}(s^\frown a^\frown b)$ with property (28.20). By removing all such b's (and the nodes above them) we get $p_2 \leq_1 p_1$. Continuing in this way we construct a fusion sequence $p \geq_0 p_1 \geq_1 p_2 \geq_2 \ldots$ and $r = \bigcap_{n=0}^\infty p_n$. If $t \in r$, then there is no $q \leq_0 r{\restriction}t$ with property (28.20). But then no $q \leq r$ forces $\exists i \leq k \; \varphi_i$, a contradiction. □

The main idea of Laver's proof is the following property of the Laver forcing. It shows that forcing with Laver trees kills uncountable strong measure zero sets.

Lemma 28.20. *Let G be a generic set for the Laver forcing. Every set of reals in the ground model that has strong measure zero in $V[G]$ is countable in $V[G]$.*

We begin by proving two technical lemmas:

Lemma 28.21. *Let p be a Laver tree with stem s and let $\dot{x}$ be a name for a real in $[0,1]$. Then there exist a condition $q \le_0 p$ and a real u such that for every $\varepsilon > 0$,*

$$q{\restriction}(s^\frown a) \Vdash |\dot{x} - u| < \varepsilon$$

for all but finitely many $a \in S^q(s)$.

Proof. Let $\{t_n\}_n$ be an enumeration of $\{s^\frown a : a \in S^p(s)\}$. For each n we find, by Lemma 28.19, a condition $q_n \le_0 p{\restriction}t_n$ and an interval $J_n = [\frac{m}{n}, \frac{m+1}{n}]$ such that $q_n \Vdash \dot{x} \in J_n$. There is a sequence $\langle k_n : n < \omega\rangle$ so that the J_{k_n} form a decreasing sequence converging to a unique real u. Let $q = \bigcup_{n=0}^\infty q_{k_n}$. □

Lemma 28.22. *Let p be a condition with stem s and let $\langle \dot{x}_n : n < \omega\rangle$ be a sequence of names for reals. Then there exist a condition $q \le_0 p$ and a set of reals $\{u_t : t \in q,\ t \supset s\}$ such that for every $\varepsilon > 0$ and every $t \in q$, $t \supset s$, for all but finitely many $a \in S^q(t)$,*

$$q{\restriction}(t^\frown a) \Vdash |\dot{x}_k - u_t| < \varepsilon$$

where $k = \operatorname{length}(t) - \operatorname{length}(s)$.

Proof. Using Lemma 28.21 we get $p_1 \le_0 p$ and u_s. For every immediate successor t of s in p_1, we get $q_t \le_0 p_1{\restriction}t$ and u_t, and let $p_2 = \bigcup_t q_t$. By repeating this argument, we build a fusion sequence $p \ge_0 p_1 \ge_1 p_2 \ge_2 \ldots$, and let $q = \bigcap_{n=0}^\infty p_n$. □

Proof of Lemma 28.20. Let f be the Laver real, and let

$$\varepsilon_n = 1/f(n). \tag{28.21}$$

We shall show that if $X \in V$ is uncountable, then for some n, the sequence $\langle \varepsilon_k : k \ge n\rangle$ witness that X does not have strong measure zero.

Thus let $X \in V$ be a subset of $[0,1]$ and let p be such that $p \Vdash X$ has strong measure zero. Let s be the stem of p of length n. Let $\langle \dot{x}_k : k \ge n\rangle$ be a sequence of names of reals, and for each $k \ge n$ let $\dot{I}_k$ be the interval of length $\dot{\varepsilon}_k$ centered at $\dot{x}_k$. Let us assume that $p \Vdash X \subset \bigcup_{k\ge n} \dot{I}_k$. We shall find a stronger condition that forces that X is countable.

Let $q \le_0 p$ and $\{u_t : t \in q,\ t \supset s\}$ be a condition and a countable set of reals obtained in Lemma 28.22. We will show that $q \Vdash X \subset \{u_t\}_t$.

Let $v \notin \{u_t : t \in q,\ t \supset s\}$; we shall find some $r \le q$ such that $r \Vdash v \notin \dot{I}_k$, for all $k \ge n$. We construct r by induction on the levels of q; at stage $k \ge n$ we ensure that $r \Vdash v \notin \dot{I}_k$.

We describe the construction for $k = n$; this can be repeated for all $k \ge n$. Let $\varepsilon = |v - u_s|/2$. For all but finitely many $a \in S^q(s)$, $q{\restriction}(s^\frown a) \Vdash |\dot{x}_n - u_s| < \varepsilon$. Since $q{\restriction}(s^\frown a) \Vdash \dot{\varepsilon}_n = 1/\dot{f}(n) = 1/a$, we have $q{\restriction}(s^\frown a) \Vdash |\dot{x}_n - v| > \dot{\varepsilon}_n$, or $v \notin \dot{I}_n$, for all but finitely many a. Thus we ensure $r \Vdash v \notin \dot{I}_n$ by removing finitely many successors of s. □

Laver's model for the consistency of Borel's Conjecture is obtained by iteration with countable support of length ω_2. At each stage of the iteration, one adds a Laver real by forcing with Laver trees. If the ground model satisfies GCH, then the iteration preserves cardinals and cofinalities, makes $2^{\aleph_0} = \aleph_2$, and the resulting model satisfies Borel's Conjecture.

We state the relevant properties of Laver's model without proof:

Firstly, for every countable set X of ordinals in $V[G]$ there is a set $Y \in V$, countable in V, such that $X \subset Y$. This is the analog of Lemma 28.17 (see Lemma 6(iii) of Laver [1976]) and implies that $\aleph_1$ is preserved by the iteration. In Chapter 31 we prove a more general result, showing that this property is preserved by countable support iteration of *proper* forcing.

Secondly, the iteration satisfies the $\aleph_2$-chain condition (Lemma 10(ii) of Laver [1976]). This can be proved as in Exercise 16.20, or Lemma 23.11, by first showing that for every $\alpha < \omega_2$, the Laver iteration of length α has a dense subset of cardinality $\aleph_1$. Again, this is a general property of countable support iteration of proper forcing, when at each stage, the βth iterate $\dot{Q}_\beta$ has cardinality $\aleph_1$.

The key property of Laver's iteration is that there are no uncountable strong measure zero sets in $V[G]$. If X is a set of reals of size $\aleph_1$ in $V[G]$, then because of the $\aleph_2$-chain condition, X appears at some stage $V[G_\alpha]$, and by forcing a Laver real, one makes X not to have strong measure zero in $V[G_{\alpha+1}]$. However, one has to show that X fails to have strong measure zero in $V[G]$, not just in $V[G_{\alpha+1}]$. The main technical lemma (Laver's Lemma 15) proves that, and is analogous to Lemma 28.20, working with iteration of Laver forcing rather than with Laver trees only.

In his paper [1983] Baumgartner gives the consistency proof of Borel's Conjecture using the countable support iteration of Mathias forcing. His Theorem 7.1 shows that the iteration of either Laver or Mathias forcing preserves $\aleph_1$, and if CH holds in the ground model then iteration of length ω_2 satisfies the $\aleph_2$-chain condition. He also gives a detailed proof of Borel's Conjecture in the iteration of Mathias forcing.

κ^+-Aronszajn Trees

Theorem 9.16 states that there exists an Aronszajn tree, i.e., a tree of length ω_1 with countable levels and no branch of length ω_1. In Chapter 9 we also defined what it means for an infinite cardinal κ to have the *tree property*: Every tree of height κ and levels of size $< \kappa$ has a branch of length κ. When κ is inaccessible then the tree property is equivalent to weak compactness.

Let κ^+ be a successor cardinal. A tree of height κ^+ is a κ^+-*Aronszajn* tree if its levels have size at most κ and it has no branch of length κ^+. When κ is singular, the tree property of κ^+ is related to large cardinals; we shall now address the case when κ is regular. We discuss the case of $\aleph_2$ as it easily generalizes to any successor of a regular. The construction in Theorem 9.16 generalizes to $\aleph_2$ under the assumption that $2^{\aleph_0} = \aleph_1$ (see Exercises 28.5 and 28.6). It follows that an $\aleph_2$-Aronszajn tree exists unless there is a weakly compact cardinal in L:

Theorem 28.23 (Silver). *If there exists no $\aleph_2$-Aronszajn tree then $\aleph_2$ is a weakly compact cardinal in L.*

Proof. If $\aleph_2$ is a successor cardinal in L, then there exists some $A \subset \omega_1$ such that $\aleph_1^{L[A]} = \aleph_1$ and $\aleph_2^{L[A]} = \aleph_2$. In $L[A]$, $2^{\aleph_0} = \aleph_1$ holds and therefore there exists a special $\aleph_2$-Aronszajn tree T. But then T is a special $\aleph_2$-Aronszajn tree in V. Thus if there are no $\aleph_2$-Aronszajn trees, $\aleph_2$ is inaccessible in L.

To show that $\lambda = \aleph_2$ is weakly compact in L if λ has the tree property, let $B \in L$ be (in L) a λ-complete algebra of subsets of λ and $|B| = \lambda$. We shall find a λ-complete nonprincipal ultrafilter U on B with $U \in L$. (Then, by the argument in Lemma 10.18, it follows that λ is weakly compact in L.)

Let $\alpha < (\lambda^+)^L$ be a limit ordinal such that $B \in L_\alpha$ and $L_\alpha \vDash |B| = \lambda$. Let $\{X_\xi : \xi < \lambda\}$ be an enumeration, in L, of $P(\lambda) \cap L_\alpha$, and let T be the set of all constructible functions $f \in \{0,1\}^{<\lambda}$ such that

$$\left|\bigcap\{X_\xi : f(\xi) = 1\} \cap \bigcap\{\lambda - X_\xi : f(\xi) = 0\}\right| = \lambda.$$

Since λ is inaccessible in L, T is a λ-tree with levels of size $< \lambda$.

Since λ has the tree property, T has a branch of length λ, a function $F : \lambda \to \{0,1\}$ such that $F{\restriction}\nu \in T$ for all $\nu < \lambda$. If we let $D = \{X_\xi : F(\xi) = 1\}$ then D is (in V) a λ-complete nonprincipal ultrafilter on $P(\lambda) \cap L_\alpha$. Let $\mathrm{Ult} = \mathrm{Ult}_D\, L_\alpha$ be the ultrapower of L_α by D (using functions in L_α), let L_β be its transitive collapse and let $j : L_\alpha \to L_\beta$ be the corresponding elementary embedding.

If $e \in L_\alpha$ is an enumeration of B, $e : \lambda \to B$, then $E = j(e) \in L_\beta$ and $U = \{e(\xi) : \lambda \in E(\xi)\}$ is a constructible λ-complete nonprincipal ultrafilter on B. $\square$

The following theorem shows that it is consistent (relative to a weakly compact cardinal) that there exist no $\aleph_2$-Aronszajn trees.

Theorem 28.24 (Mitchell). *If κ is a weakly compact cardinal then there is a generic extension in which $\kappa = \aleph_2$, $2^{\aleph_0} = \aleph_2$, and there exists no $\aleph_2$-Aronszajn tree.*

The model is obtained by a two-stage iteration $P * \dot{Q}$. The forcing $P = P_\kappa$ adds κ Cohen reals to the ground model; let $G = G_\kappa$ be generic on P; for each $\alpha < \kappa$, let P_α be the forcing for adding α Cohen reals, and let $G_\alpha = G \cap P_\alpha$.

In $V[G]$, consider the forcing conditions q for adding κ Cohen subsets of ω_1: q is a 0–1 function on a countable subset of κ. Let Q be the set of all such q that satisfy, in addition, the requirement that

$$q{\restriction}\alpha \in V[G_\alpha] \qquad (\text{all } \alpha < \kappa).$$

This amounts to forcing with pairs (p, q) where $p \in P$ and q is a countable function on a subset of κ with values $q(\alpha) \in B(P_\alpha)$ (then if G is generic on P, we have $\bar{q} \in Q$ where $\bar{q}(\alpha) = 1$ if $q(\alpha) \in G$ and $\bar{q}(\alpha) = 0$ if $q(\alpha) \notin G$).

We list some properties of $P * \dot{Q}$ which are not difficult to verify. Let G be generic on P and let H be $V[G]$-generic on Q.

First, every countable set of ordinals in $V[G][H]$ is in $V[G]$. Hence $\aleph_1$ is preserved.

Second, every cardinal between $\aleph_1$ and κ is collapsed: If $\aleph_1 \le \delta < \kappa$, let t be the following function on ω_1:

$$t(\alpha) = \{n \in \omega : \exists f \in H\, f(\delta + \omega \cdot \alpha + n) = 1\}.$$

The function maps ω_1 onto $P(\omega)^{V[G_{\delta+\omega_1}]}$ which has cardinality δ.

Third, $P * \dot{Q}$ satisfies the κ-chain condition. This is proved similarly to the κ-chain condition of the Lévy collapse.

Finally, it is clear that $2^{\aleph_0} = \kappa$ in $V[G][H]$. The main technical lemma (Lemma 3.8 of Mitchell [1972/73]) asserts the following: For $\alpha < \kappa$ let $Q_\alpha = \{q \in Q : \operatorname{dom}(q) \subset \alpha\}$, and $H_\alpha = H \cap Q_\alpha$. If $\gamma < \kappa$ is a regular uncountable cardinal and if $t \in V[G][H]$ is an ordinal function on γ such that $t{\restriction}\alpha \in V[G_\gamma][H_\gamma]$ for all $\alpha < \gamma$, then $t \in V[G_\gamma][H_\gamma]$.

Now one shows that κ has the tree property in $V[G][H]$ as follows: Let $B = B(P * \dot{Q})$ and let $\dot{T}$ be a B-name for a binary relation on κ that is in $V[G][H]$ a tree of height κ with levels of size $< \kappa$. There is a closed unbounded set $C \subset \kappa$ such that if $\gamma \in C$ is an inaccessible cardinal then $B_\gamma = B(P_\gamma * \dot{Q}_\gamma)$ is a complete Boolean subalgebra of $B(P * \dot{Q})$ and that $\dot{T} \cap (\gamma \times \gamma)$ is a B_γ-valued name for $\dot{T}{\restriction}\gamma$, the first γ levels of $\dot{T}$.

To show that $\dot{T}$ has a branch of length κ, assume that it has none; that this is so in V^B is a Π^1_1 sentence true in $(\kappa, B, \dot{T})$ and since κ is Π^1_1-indescribable, the same is true in V^{B_γ}: $\dot{T}{\restriction}\gamma$ has no branch of length γ in V^{B_γ}. But any node in the γth level of T produces an ordinal function on γ whose initial segments are in V^{B_γ}; by the technical lemma alluded to above, the function itself is in V^{B_γ}, and is a branch in $\dot{T}{\restriction}\gamma$. A contradiction. □

A related result is the following theorem that we state without proof:

Theorem 28.25 (Laver-Shelah). *If there exists a weakly compact cardinal then there exists a generic extension in which $2^{\aleph_0} = \aleph_1$ and there exists no $\aleph_2$-Suslin tree.* □

(In the Laver-Shelah model, $2^{\aleph_1}$ is greater than $\aleph_2$.)

Exercises

28.1. Find $\langle e_\alpha : \alpha < \omega_1 \rangle$ such that each $e_\alpha : \alpha \to \omega$ is one-to-one and if $\alpha < \beta$ then e_α and $e_\beta \restriction \alpha$ differ at only finitely many places.

[Construct the e_α by induction on α, such that for every α, $\omega - \operatorname{ran}(e_\alpha)$ is infinite.]

28.2. Given $\langle e_\alpha : \alpha < \omega_1 \rangle$ as in Exercise 28.1, show that the set $\langle e_\alpha \restriction \beta : \alpha, \beta \in \omega_1 \rangle$ is an Aronszajn tree.

28.3. If r is a Cohen real over V, then for every uncountable $X \subset \omega_1$ in $V[r]$ there exists an uncountable $Y \subset X$ in V.

[The notion of forcing is countable.]

28.4. A Laver real eventually dominates every $g : \omega \to \omega$ in V.

28.5. If $2^{\aleph_0} = \aleph_1$, then there exists an $\aleph_2$-Aronszajn tree.

[Imitate the proof of Theorem 9.16. Let Q be the lexicographically ordered set $\omega_1^{<\omega}$; every $\alpha < \omega_2$ embeds in any interval of Q. Construct T using bounded increasing sequences in Q of length $< \omega_2$. At limit steps of cofinality ω extend all branches that represent bounded sequences in Q; here we use $2^{\aleph_0} = \aleph_1$.]

28.6. The tree constructed in Exercise 28.5 is special, i.e., the union of $\aleph_1$ antichains.

[Compare with Exercises 9.8 and 9.9.]

Historical Notes

The construction of a nonconstructible Δ^1_3 real in Theorem 28.1 is as in Jensen [1970]. Namba's forcing appeared in Namba [1971]; in [1976] Bukovský obtains the same result by a somewhat different forcing construction. The result that adding a Cohen real adds a Suslin tree is due to Shelah [1984]; the present proof is due to Todorčević [1987] (for details see Bagaria [1994]).

The consistency of Borel's Conjecture is due to Laver [1976].

For the construction of a κ^+-Aronszajn tree if $2^{<\kappa} = \kappa$, see Specker [1949]. The consistency proof of the tree property for $\aleph_2$, as well as the proof of Silver's Theorem 28.23 appeared in Mitchell [1972/73]. Theorem 28.25 is in Laver-Shelah [1981].