29. More Combinatorial Set Theory

Ramsey Theory

Ramsey's Theorem 9.1 has been generalized in many ways, giving rise to an area of combinatorial mathematics known as Ramsey theory. In this section we present three results involving combinatorics of infinite sets. For a complete account of Ramsey theory we refer the reader to the book [1980] of Graham, Rothschild and Spencer.

Theorem 29.1 (Hindman). *If $\boldsymbol{N}$ is partitioned into finitely many pieces then one of the pieces A contains an infinite set H such that $a_1+\ldots+a_n \in A$ whenever $a_1, \ldots, a_n$ are distinct members of H.*

For the proof, we introduce the concept of an *idempotent ultrafilter*. If U and V are ultrafilters on $\boldsymbol{N}$, let

$$U + V = \{X \subset \boldsymbol{N} : \{m \in \boldsymbol{N} : X - m \in V\} \in U\} \tag{29.1}$$

where $X-m = \{n : m+n \in X\}$. See Exercises 29.1 and 29.2 for an alternative characterization. In the proof we use the following lemma due to S. Glazer:

Lemma 29.2. *There exists a nonprincipal ultrafilter U on $\boldsymbol{N}$ such that $U + U = U$.*

An ultrafilter U such that $U + U = U$ is *idempotent.* While Glazer's Lemma can be proved directly, it can be deduced from a more general result on topological semigroups. Let $\beta\boldsymbol{N}$ be the space of all ultrafilters on $\boldsymbol{N}$, the Stone-Čech compactification on $\boldsymbol{N}$. The operation $U+V$ on $\beta\boldsymbol{N}$ is a continuous function of U for any fixed V, thus making $(\beta\boldsymbol{N},+)$ a left-topological semigroup. It can be shown that every compact left-topological semigroup has an idempotent element (Exercises 29.3 and 29.4).

Proof of Theorem 29.1. Given a partition of $\boldsymbol{N}$ into finitely many pieces, let U be an idempotent ultrafilter, and let A be a piece of the partition such that $A \in U$. We construct a sequence $A = A_0 \supset A_1 \supset A_2 \supset \ldots$ with $A_k \in U$ and $a_0 < a_1 < a_2 < \ldots$ as follows: Let $a_0 \in A_0$. Given $A_k \in U$ and a_{k-1}, we find $a_k > a_{k-1}$ such that $a_k \in A_k$ and that $A_k - a_k \in U$ (since $\{n : A_k - n \in U\} \in U$). Let $A_{k+1} = A_k \cap (A_k - a_k)$.

Now let $H = \{a_k : k < \omega\}$. To verify that all finite sums from H are in A, one shows, by induction on n, that if $k_1 > \ldots > k_n$ then $a_{k_1} + \ldots + a_{k_n} \in A_{k_n}$. □

A similar technique can be used to give a proof of the following classical theorem in Ramsey Theory. An *arithmetic progression* (of length k) is a finite set of the form

$$\{n, n+d, n+2d, \ldots, n+(k-1)d\} \tag{29.2}$$

where d is a positive integer.

Theorem 29.3 (van der Waerden). *If $\boldsymbol{N}$ is partitioned into finitely many pieces then one of the pieces contains arbitrarily long arithmetic progressions.*

For the proof of Theorem 29.3, we fix an integer $k \geq 1$ and consider the space $(\beta\boldsymbol{N})^k$. Let I be the set of all arithmetic progressions of length k, and let $\bar{I}$ be its closure in $(\beta\boldsymbol{N})^k$. Arguments similar to those in Exercise 29.3 can be used to show that if $R \subset \beta\boldsymbol{N}$ is any minimal right ideal and $U \in R$, then $\langle U, \ldots, U\rangle \in \bar{I}$. For details, we refer reader to Todorčević's book [1997].

Now to prove the theorem, let U be a nonprincipal ultrafilter on $\boldsymbol{N}$ that belongs to some minimal right ideal on $\beta\boldsymbol{N}$. Let A be the piece of the given partition such that $A \in U$, and let $A^* = \{V \in \beta\boldsymbol{N} : A \in V\}$. If $k \geq 1$ is any integer, let I and $\bar{I}$ be as above. Since $\langle U, \ldots, U\rangle \in \bar{I}$, it follows that $(A^* \times \ldots \times A^*) \cap \bar{I}$ is nonempty, and hence $I \cap (A \times \ldots \times A) \neq \emptyset$. Therefore A contains an arithmetic progression of length k.

The third result, which we state without a proof, is the Hales-Jewett Theorem. Let Σ be a finite set, called *alphabet*, and let W be the set of all *words* on Σ, the set of all finite sequences in Σ. Let v be a *variable*, an object not in Σ. The set V of all *variable words* over Σ is the set of all words on $\Sigma \cup \{v\}$ in which v occurs. An *instance* of a variable word $x \in V$ is the result of substituting some $a \in \Sigma$ for v in x.

Theorem 29.4 (Hales-Jewett). *If W is partitioned into finitely many pieces then there is a variable word $x \in V$ whose all instances lie in the same piece of the partition.*

We refer the reader to Todorčević's book for a proof using topological semigroups.

Gaps in ω^ω

Consider the partial order on ω^ω by eventual domination: $f < g$ if and only if $f(n) < g(n)$ for all but finitely many n.

Definition 29.5. Let κ and λ be regular cardinals. A (κ, λ)-*gap* in ω^ω is a pair of transfinite sequences $\langle f_\alpha : \alpha < \kappa \rangle$ and $\langle g_\beta : \beta < \lambda \rangle$ in ω^ω such that

(29.3) (i) $f_{\alpha_1} < f_{\alpha_2}$ if $\alpha_1 < \alpha_2$,
(ii) $g_{\beta_1} > g_{\beta_2}$ if $\beta_1 < \beta_2$,
(iii) $f_\alpha < g_\beta$ for all $\alpha < \kappa$ and $\beta < \lambda$,
(iv) there is no h between $\{f_\alpha\}_\alpha$ and $\{g_\beta\}_\beta$, i.e., no h such that $f_\alpha < h < g_\beta$ for all α and β.

We shall prove a classical theorem of Hausdorff stating that (ω_1, ω_1)-gaps exist.

First we prove that (ω, ω)-gaps do not exist:

Lemma 29.6. *If $f_0 < f_1 < \ldots < f_n < \ldots < g_m < \ldots < g_1 < g_0$, then there exists an h between $\{f_n\}_n$ and $\{g_m\}_m$.*

Proof. For each k there exists an n_k such that for all $n \geq n_k$, $m_k(n) = \max\{f_0(n), \ldots, f_k(n)\} \leq \min\{g_0(n), \ldots, g_k(n)\} = M_k(n)$. Choose such n_k's so that $n_0 < n_1 < \ldots < n_k < \ldots$, and let h be a function such that $m_k(n) \leq h(n) \leq M_k(n)$ when $n_k \leq n < n_{k+1}$. □

Another easily seen fact is that a (κ, λ)-gap exists if and only if a (λ, κ)-gap exists (Exercise 29.5). That some gaps exists follows from Zorn's Lemma. In fact, there exists an $(\omega, \mathfrak{b})$-gap, see Exercise 29.6 (and there are no (ω, λ)-gaps for $\lambda < \mathfrak{b}$, see Exercise 29.7).

Apart from Hausdorff's Theorem 29.7, the existence of specific (κ, λ)-gaps is unprovable: For instance, $(\mathfrak{c}, \mathfrak{c})$-gaps may or may not exist, depending on the model. A detailed account of known consistency results on gaps can be found in Scheepers [1993].

Theorem 29.7 (Hausdorff). *There exists an (ω_1, ω_1)-gap in ω^ω.*

Proof. We construct an increasing $\langle f_\alpha : \alpha < \omega_1 \rangle$ and a decreasing $\langle g_\beta : \beta < \omega_1 \rangle$ such that

(29.4) (i) for all α and β, $\lim_{n\to\infty} g_\beta(n) - f_\alpha(n) = \infty$,
(ii) for every $\alpha < \omega_1$ and every $n \in \omega$, there are only finitely many $\beta < \alpha$ such that $\forall k \geq n \; f_\alpha(k) < g_\beta(k)$.

Let us show first that (29.4) guarantees that $\{f_\alpha\}_\alpha$, $\{g_\beta\}_\beta$ is a gap. Assume that $h \in \omega^\omega$ is between $\{f_\alpha\}_\alpha$, $\{g_\beta\}_\beta$. Then there exist an uncountable $Z \subset \omega_1$ and some $n \in \omega$ such that for all $k \geq n$ and $h(k) < g_\alpha(k)$ for all $k \geq n$. Thus $f_\alpha(k) < g_\beta(k)$ for all $\alpha, \beta \in Z$ and all $k \geq n$. Now if α is the ωth element of Z, the set $\{\beta < \alpha : \forall k \geq n \; f_\alpha(k) < g_\beta(k)\}$ is infinite, contradicting (29.4)(ii).

We construct f_α and g_α by induction on α. Let $f_0(n) = 0$ and $g_0(n) = n$ for all n. Let $\gamma < \omega_1$. If f_α and g_β satisfy (29.4) for all $\alpha, \beta \leq \gamma$, then it is easy to find $f_{\gamma+1}$ and $g_{\gamma+1}$ such that (29.4) remains true.

Thus let γ be a limit ordinal and assume that $\{f_\alpha\}_{\alpha<\gamma}$ $\{g_\beta\}_{\beta<\gamma}$ satisfy (29.4). Let us use the following terminology: If $f < g_\beta$ for all $\beta < \gamma$ and if $C \subset \gamma$, we say that f is *near* C if for every n the set $\{\beta \in C : \forall n \geq k\; f(k) < g_\beta(k)\}$ is finite. Note that if $f < f'$ and f is near C then f' is near C.

We wish to find f and g such that $f_\alpha < f < g < g_\beta$ for all $\alpha, \beta < \gamma$ (and $\lim_n(g(n) - f(n)) = \infty$) and that f is near γ. Let h be some function such that $f_\alpha < h < g_\beta$ for all $\alpha, \beta < \gamma$; such an h exists by Lemma 29.6. As each f_α is near α, it follows that h is near α for all $\alpha < \gamma$. It now suffices to find some $f > h$ such that $f < g_\beta$ for all $\beta < \gamma$ and such that f is near α. Then g is easily found. For each n let $C_n = \{\beta < \gamma : \forall k \geq n\, h(k) < g_\beta(k)\}$. Clearly, $C_0 \subset C_1 \subset \ldots \subset C_n \subset \ldots$. As long as all C_n are finite, h is near γ and we are done. Thus assume that the C_α are eventually infinite.

We construct inductively a sequence $h = h_0 < h_1 < \ldots < h_n < \ldots$ of functions below $\{g_\beta\}_{\beta<\gamma}$ such that for each n, h_{n+1} is near C_n. Then if f is any function between $\{h_n\}_{n<\omega}$ and $\{g_\beta\}_{\beta<\gamma}$ and $f(n) \geq h(n)$ for all n, then for each n, f is near the set $\{\beta < \gamma : \forall k \geq n\, f(k) < g_\beta(k)\} \subset C_n$ and hence f is near γ.

Let $n \geq 0$. If C_n is finite, any h_{n+1} is near C_n; thus assume that C_n is infinite. Since for each $\alpha < \gamma$, the set $C_n \cap \alpha$ is finite (because h is near α), the order-type of C_n is ω, and C_n is cofinal in γ. Let $\beta_0 < \beta_1 < \ldots < \beta_i < \ldots$ be the enumeration of C_n. It suffices to find $h_{n+1} > h_n$ such that $h_{n+1} < g_{\beta_i}$ for all i, and that for every m,

$$\{i < \omega : \forall k \geq m\, h_{n+1}(k) < g_{\beta_i}(k)\} \text{ is finite.} \tag{29.5}$$

Let $m_0 < m_1 < \ldots < m_i < \ldots$ be such that for every i,

$$h_n(k) < g_{\beta_i}(k) < g_{\beta_{i-1}}(k) < \ldots < g_{\beta_0}(k) \text{ for all } k \geq m_i.$$

Then the function h_{n+1} defined by

$$h_{n+1}(k) = \begin{cases} h_n(k) & \text{if } k < m_0, \\ g_{\beta_i}(k) & \text{if } m_i \leq k < m_{i+1} \end{cases}$$

satisfies (29.5) and hence is near C_n. □

The Open Coloring Axiom

We shall now discuss the axiom OCA (Open Coloring Axiom) that has a number of applications in combinatorial set theory. Let X be a set of reals (or $X \subset \mathcal{N}$, or $X \subset P(\omega)$, etc.) and let $K \subset [X]^2$. We say that K is *open* if the set $\{(x, y) : \{x, y\} \in K\}$ is an open set in the space $X \times X$. The Open Coloring Axiom (OCA) states:

(29.6) Let X be a subset of $\boldsymbol{R}$. For any partition $[X]^2 = K_0 \cup K_1$ with K_0 open, either there is an uncountable $Y \subset X$ such that $[Y]^2 \subset K_0$, or there exist sets H_n, $n \in \omega$, such that $X = \bigcup_{n=0}^\infty H_n$ and $[H_n]^2 \subset K_1$ for all n.

The axiom OCA is consistent with ZFC; we discuss this in Chapter 31. It should be noted that its dual version is false; Exercise 29.9. OCA has a number of consequences; see Todorčević [1989]. (One example is Exercise 29.10.) The most notable is the following result:

Theorem 29.8 (Todorčević). *If* OCA *holds then* $\mathfrak{b} = \aleph_2$.

First we show that under OCA, $\mathfrak{b} > \omega_1$.

Lemma 29.9. *Assume* OCA. *Then every subset of* ω^ω *of size* $\aleph_1$ *is bounded.*

Proof. In order to show that every subset $X \subset \omega^\omega$ of size $\aleph_1$ is bounded, it is clearly enough to show this for every increasing $X = \{f_\alpha\}_{\alpha<\omega_1}$ (i.e., $f_\alpha < f_\beta$ if $\alpha < \beta$), and assume that each f_α is an increasing function from ω to ω. Let $X = \{f_\alpha\}_\alpha$ be such and let $[X]^2 = K_0 \cup K_1$ where K_0 consists of all $\{f_\alpha, f_\beta\}$ with $\alpha < \beta$ such that $f_\alpha(k) > f_\beta(k)$ for some k.

First assume that $X = \bigcup_{n=0}^\infty H_n$ and $[H_n]^2 \subset K_1$ for all n. Then for some n, H_n is uncountable, and if $\alpha < \beta$ are such that $f_\alpha, f_\beta \in H_n$ then $f_\alpha(k) \le f_\beta(k)$ for all k, and $f_\alpha(k) < f_\beta(k)$ for some k. Then if we let $S_\alpha = \{(m, k) : m \le f_\alpha(k)\}$, we have an ω_1-chain of subsets of $\omega\times\omega$, a contradiction.

Thus, assuming OCA, there is an uncountable $Y \subset X$ such that $[Y^2] \subset K_0$. We claim that Y is bounded (and it follows that X is bounded). To prove the claim, assume that Y is not bounded and let $\{g_\alpha : \alpha < \omega_1\}$ be the increasing enumeration of Y. For each $t \in \omega^{<\omega}$ that is an initial segment of some $g \in Y$, choose α_t such that $t \subset g_{\alpha_t}$. Then let $\gamma > \sup_t \alpha_t$, and let k_0 be such that for uncountably many β, $g_\gamma(k) < g_\beta(k)$ for all $k \ge k_0$. Thus there is an uncountable $Z \subset \omega_1 - \gamma$ such that $g_\gamma(k) < g_\beta(k)$ for all $\beta \in Z$ and all $k \ge k_0$ and that $g_{\beta_1}\restriction k_0 = g_{\beta_2}\restriction k_0$ whenever $\beta_1, \beta_2 \in Z$.

Now let $m \ge k_0$ be the least m such that the set $\{g_\beta(m) : \beta \in Z\}$ is infinite (m exists because $\{g_\beta\}_{\beta\in Z}$ is not bounded). There exist some $t \in \omega^m$ and some $W \subset Z$ such that $g_\beta\restriction m = t$ for all all $\beta \in W$ and $\{g_\beta(m) : \beta \in W\}$ is infinite.

Let $\alpha = \alpha_t$; since $\alpha < \gamma$, there exists a $k_1 \ge m$ such that $g_\alpha(k) < g_\gamma(k)$ for all $k \ge k_1$. Let $\beta \in W$ be such that $g_\beta(m) \ge g_\alpha(k_1)$. Since $g_\beta\restriction m = t = g_\alpha\restriction m$, and since g_α and g_β are increasing, we have $g_\alpha(k) \le g_\beta(k)$ for all $k \le k_1$. But for $k \ge k_1$ we have $g_\alpha(k) \le g_\gamma(k) < g_\beta(k)$; hence $g_\alpha(k) \le g_\beta(k)$ for all k. This contradicts the assumption that $\{g_\alpha, g_\beta\} \in K_0$. Hence Y is bounded, and so X is bounded. $\square$

Toward the proof of $\mathfrak{b} \le \aleph_2$ we prove the following result on gaps:

Lemma 29.10. *Assume* OCA. *There is no* (κ, λ)*-gap in* ω^ω *such that* κ *and* λ *are regular uncountable, and* $\kappa > \omega_1$.

Proof. Let $\kappa \ge \lambda$ be regular uncountable with $\kappa > \omega_1$, and assume that $\{f_\alpha\}_{\alpha<\kappa}$, $\{g_\beta\}_{\beta<\lambda}$ is a gap. In order to define an open partition, we first modify the gap. For each $\alpha < \kappa$ there exists an m_α such that for λ many β's,

$f_\alpha(k) < g_\beta(k)$ for all $k \ge m_\alpha$; for κ many α's, this m_α is the same. Therefore there is a gap for which $m_\alpha = 0$ for all $\alpha < \kappa$, and we assume that the given gap is such. For each $\alpha < \kappa$, let $S_\alpha = \{\beta < \lambda : \forall k\, f_\alpha(k) < g_\beta(k)\}$; $|S_\alpha| = \lambda$.

Let $X = \{(f_\alpha, g_\beta) : \alpha < \kappa \text{ and } \beta \in S_\alpha\}$, a subspace of $\mathcal{N} \times \mathcal{N}$. Consider the partition $[X]^2 = K_0 \cup K_1$ where $\{(f_\alpha, g_\beta), (f_\gamma, g_\delta)\} \in K_0$ when for some k, either $f_\alpha(k) > g_\delta(k)$ or $f_\gamma(k) > g_\beta(k)$. Because of the additional assumption on the gap, K_0 is open.

First assume that $X = \bigcup_{n=0}^\infty H_n$ with $[H_n]^2 \subset K_1$ for each n. Since κ and λ are uncountable, there exist $A \subset \kappa$ of size κ and for each $\alpha < \kappa$ some $T_\alpha \subset S_\alpha$ of size λ such that all (f_α, g_β) with $\alpha \in A$ and $\beta \in T_\alpha$ are in the same H_n. Since $[H_n]^2 \subset K_1$, we have $\forall k\, f_\alpha(k) < g_\delta(k)$ whenever $\alpha, \gamma \in A$ and $\delta \in T_\gamma$. Thus fix $\gamma \in A$ and let $B = T_\gamma$. A is cofinal in κ, B is cofinal in λ, and if $\alpha \in A$ and $\beta \in B$ then $\forall k\, f_\alpha(k) < g_\beta(k)$. But then the function h defined by $g(k) = \min_{\beta \in B} g_\beta(k)$ is between $\{f_\alpha\}_\alpha$ and $\{g_\beta\}_\beta$, a contradiction.

Next assume that there exists an uncountable $Y \subset X$ such that $[Y]^2 \subset K_0$. If (f_α, g_β) and (f_γ, g_δ) are distinct elements of Y, then because $\beta \in S_\alpha$, $\delta \in S_\gamma$ and $\{(f_\alpha, g_\beta), (f_\gamma, g_\delta)\} \in K_0$, we have $\alpha \ne \gamma$ and $\delta \ne \beta$; thus Y is one-to-one. Therefore there exist increasing ω_1-sequences $\langle \alpha_\nu : \nu < \omega_1 \rangle$ and $\langle \beta_\nu : \nu < \omega_1 \rangle$ such that $\{(f_{\alpha_\nu}, g_{\beta_\nu}) : \nu < \omega_1\} \subset Y$.

Now since $\kappa > \omega_1$, let $h = f_\delta$ where $\delta > \sup_\nu \alpha_\nu$. The function h is between $\{f_{\alpha_\nu}\}_\nu$ and $\{g_{\beta_\nu}\}_\nu$. Now we can find an uncountable $Z \subset \omega_1$ and some m such that for all $\nu, \eta \in Z$, $f_{\alpha_\nu}(k) < h(k) < g_{\beta_\eta}(k)$ for all $k \ge m$, and $f_{\alpha_\nu} \restriction m = f_{\alpha_\eta} \restriction m$, $g_{\beta_\nu} \restriction m = g_{\beta_\eta} \restriction m$. Since $\beta_\nu \in S_{\alpha_\nu}$ for each ν, it follows that $f_{\alpha_\nu}(k) < g_{\beta_\eta}(k)$ for all k, contrary to the assumption that $\{(f_{\alpha_\nu}, g_{\beta_\nu}), (f_{\alpha_\eta}, g_{\beta_\eta})\} \in K_0$. $\square$

Proof of Theorem 29.8. Assuming $\mathfrak{b} > \omega_2$, we shall construct an (ω_2, λ)-gap with λ regular uncountable. Then OCA and Lemma 29.10 complete the proof.

Let $\langle f_\alpha : \alpha < \omega_2 \rangle$ be an increasing sequence of increasing functions. Since $\mathfrak{b} > \omega_2$, there exists some g_0 such that $g_0 > f_\alpha$ for all α. Let $\langle g_\beta : \beta < \vartheta \rangle$ be a maximal decreasing sequence of functions such that $g_\beta > f_\alpha$ for all α. At successor stages we can let $g_{\beta+1}(k) = g_\beta(k) - 1$ and so ϑ is a limit ordinal. We complete the proof by showing that $\operatorname{cf} \vartheta > \omega$.

Thus assume that $\vartheta = \lim_n \beta_n$. Given $\alpha < \omega_2$ let $m_\alpha(0) < m_\alpha(1) < \ldots < m_\alpha(n) < \ldots$ be such that for all $i = 0, \ldots, n$, $f_\alpha(k) < g_{\alpha_i}(k)$ for all $k \ge m_\alpha(n)$. Since $\mathfrak{b} > \omega_2$, there exists a function h such that $h > m_\alpha$ for all α. Now if we let $g(n) = \min_{i \le n} g_{\alpha_i}(h(n))$, then $g > f_\alpha$ for all α and $g < g_{\alpha_n}$ for all n, contrary to the maximality of ϑ. $\square$

Almost Disjoint Subsets of ω_1

Let κ be a regular uncountable cardinal, and let X and Y be unbounded subsets of κ. The sets X and Y are almost disjoint if $|X \cap Y| < \kappa$ (cf. Definition 9.20). Similarly, two functions f and g on κ are *almost disjoint* if for some $\gamma < \kappa$, $f(\alpha) \neq g(\alpha)$ for all $\alpha > \gamma$ (cf. Definition 9.22). Unlike in the case $\kappa = \omega$, it is a nontrivial question how large a set of almost disjoint sets of functions can be; clearly, the maximal size of an almost disjoint family of subsets of κ is equal to the maximal size of an almost disjoint family of functions from κ to κ.

For simplicity, we consider the case $\kappa = \omega_1$. This can be generalized to any regular uncountable κ.

First, there exists an almost disjoint family of size $\aleph_2$ (Lemma 9.23), and if $2^{\aleph_0} = \aleph_1$ then there exists one of size $2^{\aleph_1}$. We shall prove the following:

Theorem 29.11. *If $2^{\aleph_0} < 2^{\aleph_1}$ and $2^{\aleph_0} < \aleph_{\omega_1}$ then there exists an almost disjoint family of $2^{\aleph_1}$ uncountable subsets of ω_1.*

Compare this with Theorem 22.16: If I is the ideal of bounded subsets of ω_1, then Theorem 29.11 states that $\mathrm{sat}(I) = (2^{\aleph_1})^+$; by Theorem 22.16, $\mathrm{sat}(I) \geq 2^{\aleph_1}$. As $\mathrm{sat}(I)$ is regular (by Theorem 7.15), the following lemma implies the theorem:

Lemma 29.12. *Assume $2^{\aleph_0} < \aleph_{\omega_1}$. If κ is a regular cardinal such that $2^{\aleph_0} < \kappa \leq 2^{\aleph_1}$, then there exists a family of κ almost disjoint functions from ω_1 into ω_1.*

Proof. Let $\mathcal{F}$ be a family of almost disjoint functions on ω_1; we call $\mathcal{F}$ a *branching family* if whenever $f, g \in \mathcal{F}$ and α is such that $f(\alpha) = g(\alpha)$, then $f(\xi) = g(\xi)$ for all $\xi \leq \alpha$.

For each $X \subset \omega_1$, let $f_X = \langle X \cap \alpha : \alpha < \omega_1 \rangle$. The family $\mathcal{F} = \{f_X : X \in P(\omega_1)\}$ is a branching family of functions on ω_1, $|\mathcal{F}| = 2^{\aleph_1}$; and for each $\alpha < \omega_1$, the functions in $\mathcal{F}$ take values in $P(\alpha)$. Thus there exists a branching family of $2^{\aleph_1}$ functions from ω_1 into $2^{\aleph_0}$.

Let κ be a regular cardinal such that $2^{\aleph_0} < \kappa \leq 2^{\aleph_1}$. We shall show that for every $\aleph_\gamma$ such that $\aleph_1 < \aleph_\gamma \leq 2^{\aleph_0}$, if there is a branching family of κ functions from ω_1 into ω_γ, then there is a branching family of κ functions from ω_1 into some $\omega_\delta < \omega_\gamma$. Then the lemma clearly follows.

First let $\aleph_\gamma = \aleph_{\delta+1}$ where $\aleph_1 \leq \aleph_\delta$, and let $\mathcal{F}$ be a branching family of κ functions from ω_1 into $\omega_{\delta+1}$. Each $f \in \mathcal{F}$ is bounded below $\omega_{\delta+1}$ (because $\omega_{\delta+1} > \omega_1$), and because κ is regular and $\kappa > \omega_{\delta+1}$, there exists $\alpha < \omega_{\delta+1}$ such that $\mathrm{ran}(f) \subset \alpha$ for κ functions in $\mathcal{F}$. Thus there exists a branching family of κ functions from ω_1 into α; and since $|\alpha| \leq \aleph_\delta$, there is also a branching family of κ functions from ω_1 into ω_δ.

If $\aleph_\gamma$ is a limit cardinal, then $\mathrm{cf}(\omega_\gamma) = \omega$ because $\aleph_\gamma < \aleph_{\omega_1}$. Let $\mathcal{F}$ be a branching family of κ functions from ω_1 into ω_γ. For each $f \in \mathcal{F}$ there exists an ordinal $\eta_f < \omega_\gamma$ such that $f(\alpha) < \eta_f$ for uncountably many α's. Since κ is a regular cardinal and $\kappa > \aleph_\gamma$, there exists $\aleph_\delta$ such that $\aleph_1 \leq \aleph_\delta < \aleph_\gamma$, and a family $\mathcal{G} \subset \mathcal{F}$ of size κ such that for every $f \in \mathcal{G}$, $f(\alpha) < \omega_\delta$ for uncountably many α's.

For each $\alpha < \omega_1$, let $S_\alpha = \{f(\alpha) : f \in \mathcal{G}\}$. Since $\mathcal{G}$ is a branching family and $\mathcal{G} \subset \prod_{\alpha<\omega_1} S_\alpha$, it suffices to show that $|S_\alpha| \le \aleph_\delta$ for all $\alpha < \omega_1$. Thus let $\alpha < \omega_1$. We define a function $t : S_\alpha \to \omega_1 \times \omega_\delta$ as follows: For each $x \in S_\alpha$, we first pick some $f \in \mathcal{G}$ such that $x = f(\alpha)$. Then there exists some $\xi > \alpha$ such $f(\xi) < \omega_\delta$, and we let

$$t(x) = (\xi, f(\xi)).$$

We shall now complete the proof by showing that the function t is one-to-one, and hence $|S_\alpha| \le \aleph_\delta$. Let $x, y \in S_\alpha$ be such $t(x) = t(y)$. Let $\xi > \alpha$ and $f, g \in \mathcal{G}$ be such that $x = f(\alpha)$, $y = g(\alpha)$, and $t(x) = t(y) = (\xi, f(\xi)) = (\xi, g(\xi))$. Since $\mathcal{G}$ is a branching family and $f(\xi) = g(\xi)$, we have $f(\alpha) = g(\alpha)$ and hence $x = y$. □

The assumption $2^{\aleph_0} < 2^{\aleph_1}$ in Theorem 29.11 is necessary; see Exercise 29.11.

Functions from ω_1 into ω

Consider the set ω^{ω_1} of all functions from ω_1 into ω, partially ordered by eventual domination:

(29.7) $$f < g \quad \text{if and only if} \quad \exists\gamma\,\forall\alpha \ge \gamma\, f(\alpha) < g(\alpha).$$

Let $\operatorname{cof}(\omega^{\omega_1})$ be the smallest size of a cofinal family $\mathcal{F} \subset \omega^{\omega_1}$, i.e., for every g there exists some $f \in \mathcal{F}$ such that $g < f$. It is an open problem whether $\operatorname{cof}(\omega^{\omega_1}) < 2^{\aleph_1}$ is possible.

Theorem 29.13.

(i) *If* $\operatorname{cof}(\omega^{\omega_1}) < 2^{\aleph_1}$ *then* $2^{\aleph_0} \ge \aleph_3$.
(ii) *If* $2^{\aleph_0} < 2^{\aleph_1}$ *and* $2^{\aleph_0} < \aleph_{\omega_1}$ *then* $\operatorname{cof}(\omega^{\omega_1}) = 2^{\aleph_1}$.

The theorem is a consequence of this lemma:

Lemma 29.14. *If there exist* $2^{\aleph_1}$ *almost disjoint functions from* ω_1 *into* ω_2 *then* $\operatorname{cof}(\omega^{\omega_1}) = 2^{\aleph_1}$.

Then Theorem 29.13 follows: If $2^{\aleph_0} \le \aleph_2$ then use Exercise 29.12(ii); for (ii), use Theorem 29.11.

Toward the proof of Lemma 29.14, let I be an ideal on a set S. We say that two functions f, g on S are *I-disjoint* if $\{x \in S : f(x) = g(x)\} \in I$. If I and J are ideals on S and T, then $I \times J$ is the ideal on $S \times T$

(29.8) $$X \in I \times J \quad \text{if and only if} \quad \{x \in S : \{y \in T : (x, y) \in X\} \notin J\} \in I.$$

Lemma 29.15. *There exists a σ-ideal I on ω_1 such that there exist $\aleph_2$ I-disjoint functions from ω_1 into ω.*

Proof. We find such an I on $\omega_1 \times \omega_1$: Let $I = I_0 \times I_0$ where I_0 is the σ-ideal of all countable subsets of ω_1 (each $X \in I$ is included in the union of ω vertical lines and the set under the graph of a function from ω_1 into ω_1). Let $\{g_\alpha : \alpha < \omega_2\}$ be a family of $\aleph_2$ almost disjoint functions from ω_1 into ω_1 (cf. Lemma 9.23), and $\{f_\beta : \beta < \omega_1\}$ a family of $\aleph_1$ almost disjoint functions from ω_1 into ω (Exercise 29.12(i)). For $\alpha < \omega_2$, let $h_\alpha(\xi, \eta) = f_{g_\alpha(\xi)}(\eta)$, for all $(\xi, \eta) \in \omega_1 \times \omega_1$. It is easy to verify that h_α, $\alpha < \omega_2$, are I-disjoint functions from ω_1 into ω. □

Lemma 29.16. *If there exist $2^{\aleph_1}$ almost disjoint functions from ω_1 into ω_2 then there exists a σ-ideal J on ω_1 such that there are $2^{\aleph_1}$ J-disjoint functions from ω_1 into ω.*

Proof. We find such a J on $\omega_1 \times \omega_1$: Let $J = I_0 \times I$ where I_0 is the ideal of countable sets and I is the ideal given by Lemma 29.15. Let $\{g_\alpha : \alpha < 2^{\aleph_1}\}$ be a family of almost disjoint functions from ω_1 into ω_2, and $\{f_\beta : \beta < \omega_2\}$ a family of I-disjoint functions from ω_1 into ω. For $\alpha < \omega_2$, let $h_\alpha(\xi, \eta) = f_{g_\alpha(\xi)}(\eta)$, for all $(\xi, \eta) \in \omega_1 \times \omega_1$. The functions h_α, $\alpha < 2^{\aleph_1}$, are J-disjoint. □

Proof of Lemma 29.14. By Lemma 29.16 there exist a σ-ideal J on ω_1 and a family $\mathcal{H} = \{h_\alpha : \alpha < 2^{\aleph_1}\}$ of J-disjoint functions from ω_1 into ω. Let $\mathcal{F}$ be a cofinal family in ω^{ω_1} such that $|\mathcal{F}| < 2^{\aleph_1}$. There exists an $f \in \mathcal{F}$ that eventually dominates infinitely many h_α; then let $A \subset 2^{\aleph_1}$ be a countable infinite set such that $h_\alpha < f$ for all $\alpha \in A$. The set $\{\xi < \omega_1 : h_\alpha(\xi) = h_\beta(\xi)$ for some distinct $\alpha, \beta \in A\}$ is the union of countably many sets in J, hence belongs to J, and hence its complement is uncountable. Thus for uncountably many $\xi < \omega_1$, the set $\{h_\alpha(\xi) : \alpha \in A\}$ is an infinite subset of ω. This contradicts the fact that there exists a $\gamma < \omega_1$ such that for all $\xi \geq \gamma$, $h_\alpha(\xi) < f(\xi)$ for all $\alpha \in A$. □

Exercises

29.1. If U is an ultrafilter on $\boldsymbol{N}$ and φ a formula, let $(Un)\,\varphi$ be an abbreviation for $\{n : \varphi(n)\} \in U$. Then $(U+V)k\,\varphi(k)$ if and only if $(Un)(Vm)\,\varphi(m+n)$.

29.2. If $\{x_k\}_{k=0}^{\infty}$ is a sequence of real numbers then $\lim_{U+V} x_k = \lim_U y_m$ where $y_m = \lim_V x_{m+n}$.

29.3. Let S be a minimal closed subsemigroup of a compact left-topological semigroup and let $u \in S$. Then $u + u = u$.

[$S + u$ is a continuous image of S, hence closed and $S + u = S$. Then $\{v \in S : v + u = u\} \subset S$ is closed and hence equals S; $u + u = u$ follows.]

29.4. $\beta\boldsymbol{N} - \boldsymbol{N}$ contains an idempotent element.

[By Zorn's Lemma and by compactness, $\beta\boldsymbol{N} - \boldsymbol{N}$ has a nonempty minimal closed subsemigroup.]

29.5. If a (κ, λ)-gap exists then a (λ, κ)-gap exists.

[Given $\{f_\alpha\}_\alpha$ and $\{g_\beta\}_\beta$, consider $\{g_0 - g_\beta\}_\beta$ and $\{g_0 - f_\alpha\}_\alpha$.]

29.6. There exists an $(\omega, \mathfrak{b})$-gap.

[Take constant functions as the ω-part of the gap. Then the $\mathfrak{b}$-part of the gap can be constructed in the family M of monotone unbounded functions. For $f \in M$ let $\varphi(f) = g$ in M be defined by $g(n) = \min\{k : f(k) \geq n\}$ and let $M' = \varphi``M$. $\varphi : (M, >) \to (M', <)$ is an order isomorphism and $(M', <)$ is cofinal in ω^ω while $(M, >)$ is cofinal in the family of all functions $f \in \omega^\omega$ which are above the constant functions ordered by $>$.]

29.7. There are no (ω, λ)-gaps for $\lambda < \mathfrak{b}$.

[The constant functions in the preceding exercise can be replaced by any $<$-increasing ω-sequence of functions which shows that $\mathfrak{b}$ is the minimal cardinal κ such that there exists an (ω, κ)-gap.]

29.8. $\mathcal{N}$ is the union of an increasing ω_1-sequence of G_δ sets.

[Let $\{f_\alpha\}_\alpha$, $\{g_\beta\}_\beta$ be an (ω_1, ω_1)-gap and let A_α be the complement of $\{h \in \mathcal{N} : f_\alpha < h < g_\alpha\}$.]

29.9. Let X be the set of all increasing transfinite sequences of rationals (a subspace of $P(\boldsymbol{Q})$), and let K_0 be the set of all $\{s, t\}$ such that $s \subset t$ or $t \subset s$. The set K_0 is closed and has no uncountable homogeneous subset. Show that there are no H_n with $[H_n]^2 \subset X - K_0$ such that $X = \bigcup_{n=0}^\infty H_n$.

[Let H_n be such that $[H_n]^2 \subset X - K_0$. Construct $q_0 > q_1 > \ldots$ and $t_0 \subset t_1 \subset \ldots$ such that $\sup t_n < q_n$, and if possible, $t_n \in H_n$. Then $t = \bigcup_n t_n$ is not a member of any H_n.]

29.10. Assuming OCA, every uncountable subset of $P(\omega)$ contains an uncountable chain or antichain.

[$\{A, B\} \in K_0$ if and only if A and B are incomparable.]

29.11. It is consistent that the ideal of countable subsets of ω_1 is ω_3-saturated while $2^{\aleph_1}$ is large.

[Adjoin κ Cohen reals to a model of GCH. Assume that $\{A_i : i < \omega_3\}$ are almost disjoint. For each pair i, j there is a $\gamma_{i,j}$ such that $A_i \cap A_j \subset \gamma_{i,j}$ is forced by all conditions. By the Erdős-Rado Theorem (in the ground model), there exists a subfamily of $\{A_i\}_i$ of size $\aleph_2$ for which $\gamma_{i,j}$ is the same γ. This gives (in $V[G]$) a family of $\aleph_2$ disjoint subsets of γ, a contradiction.]

29.12. (i) There exist $\aleph_1$ almost disjoint functions from ω_1 into ω.

(ii) There exist $2^{\aleph_1}$ almost disjoint functions from ω_1 into $2^{\aleph_0}$.

[(i) For $\xi \leq \alpha < \omega_1$, let $f_\xi(\alpha) = \xi$; this gives $\aleph_1$ almost disjoint functions in $\prod_{\alpha<\omega_1} \alpha$.

(ii) For $X \subset \omega_1$, let $f_X(\alpha) = X \cap \alpha$; this gives $2^{\aleph_1}$ almost disjoint functions in $\prod_{\alpha<\omega_1} P(\alpha)$.]

29.13. If there is a family $\mathcal{F}$ of $\aleph_2$ almost disjoint functions $f : \omega_1 \to \omega$ then Chang's Conjecture fails.

[Consider a model $\mathfrak{A}$ with the universe $\mathcal{F} \cup \omega_1$ and the designated predicate ω_1. If $(\mathcal{G} \cup B, B) \prec \mathfrak{A}$ with $|\mathcal{G}| = \aleph_1$ and $|B| = \aleph_0$, then $B \subset \alpha$ for some $\alpha < \omega_1$. Show that $f(\alpha) \neq g(\alpha)$ for all $f, g \in \mathcal{G}$, a contradiction.]

29.14. Assume that there exists a cofinal $\mathcal{F} \subset \omega^{\omega_1}$ such that the set of all initial segments of all $f \in \mathcal{F}$ has size $\aleph_1$. Then $2^{\aleph_0} = \aleph_1$.

[Let $\langle t_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of all the initial segments, and $\operatorname{dom} t_\alpha \leq \alpha$. By induction on $\alpha < \omega_1$, we construct closed subsets $K_{n,\alpha}$ of $[0,1]$ $(n \in \omega)$ such that $K_{0,\alpha} \subset K_{1,\alpha} \subset \ldots \subset K_{n,\alpha} \subset \ldots$, and the union is $[0,1]$. At stage α, consider the set $K = \bigcap_{\xi \in \operatorname{dom} t_\alpha} K_{t_\alpha(\xi),\xi}$. If K is countable, let $K_{n,\alpha} = [0,1]$ for all n; if K is uncountable, choose a limit point x of K and let $K_{n,\alpha} = \{x\} \cup \{y : |x - y| \geq 1/n\}$. Now if $f \in \mathcal{F}$ then $K_f = \bigcap_\alpha K_{f(\alpha),\alpha}$ is countable, and there exists some $\alpha_f < \omega_1$ such that $K_f = \bigcap_{\alpha < \alpha_f} K_{f(\alpha),\alpha}$. It follows that

$$\begin{aligned}
[0,1] &= \bigcap_{\alpha<\omega_1} \bigcup_{n=0}^{\infty} K_{n,\alpha} = \bigcup_{f:\omega_1\to\omega} \bigcap_{\alpha<\omega_1} K_{f(\alpha),\alpha} \\
&= \bigcup_{f\in\mathcal{F}} \bigcap_{\alpha<\omega_1} K_{f(\alpha),\alpha} = \bigcup_{f\in\mathcal{F}} \bigcap_{\alpha<\alpha_f} K_{f(\alpha),\alpha} = \bigcup_{\gamma<\omega_1} \bigcap_{\alpha\in\operatorname{dom} t_\gamma} K_{t_\gamma(\alpha),\alpha},
\end{aligned}$$

which is a union of $\aleph_1$ countable sets.]

Historical Notes

Hindman's Theorem appeared in [1974]. The present proof is due to Glazer and can be found e.g. in the book [1980] by Graham et al. The book also contains van der Waerden's Theorem and its generalizations. The topological proof presented here is as in Todorčević's book [1997]. For the Hales-Jewett Theorem, see Hales and Jewett [1963].

Hausdorff's Theorem appeared in Hausdorff [1909]. We follow the construction presented in Scheepers [1993], which gives a comprehensive account of the subject of gaps. The Open Coloring Axiom was isolated by Todorčević in [1989]; related partition axioms were previously introduced by Abraham, Rubin and Shelah in [1985]. Theorem 29.8 is due to Todorčević [1989].

Theorem 29.11 as well as Exercise 29.11 are results of Baumgartner [1976]; for almost disjoint functions see Jech and Prikry [1979]. Part (i) of Theorem 29.13 is due to Galvin. For part (ii) and Lemma 29.14, see Jech and Prikry [1984].

Exercise 29.3 is attributed to R. Ellis; see Todorčević [1997].

Exercises 29.6 and 29.7: Rothberger [1941].

Exercise 29.8: Hausdorff [1936a].

Exercise 29.9: Todorčević.

Exercise 29.10: Abraham, Rubin and Shelah [1985]; Baumgartner [1980].

Exercise 29.13: Silver.

Exercise 29.14: Gödel.