30. Complete Boolean Algebras

Measure Algebras

A complete Boolean algebra B is a *measure algebra* if it carries a (strictly positive probabilistic) *measure*, i.e., a real-valued function m on B that satisfies (22.1) (or cf. Definition 30.2 below). In Chapters 26 and 22 we looked at two examples of measure algebras: The algebra B_m of (26.1), and the more general product measure algebra defined in (22.3). We present below a theorem that states that this measure algebra is essentially the only measure algebra that exists.

Throughout this section, we consider measure algebras, and for simplicity assume that all the measure algebras under consideration are atomless. Note that every measure algebra satisfies the countable chain condition, and consequently, all questions of completeness can be reduced to σ-completeness.

If G is a subset of a measure algebra B, we say that G *σ-generates* B if B is the smallest σ-subalgebra containing G. The *weight* of B is the least size of $G \subset B$ that σ-generates B. B is *homogeneous* if each $B{\restriction}u$ (with $u \neq 0$) has the same weight. Note that every measure algebra is the direct sum of ω many homogeneous measure algebras.

The result that we shall prove in this section is the following:

Theorem 30.1 (Maharam). *Every infinite homogeneous measure algebra is the unique measure algebra of its weight.*

If A and B are infinite homogeneous measure algebras of the same weight and if μ and ν are strictly positive probabilistic measures on A and B, then there exists an isomorphism f between A and B such that $\nu(f(a)) = \mu(a)$ for all $a \in A$.

We begin by introducing some terminology and presenting two lemmas that are standard techniques of measure theory.

Definition 30.2. Let B be a complete Boolean algebra. A *measure* on B is a real-valued function μ on B that satisfies

(i) $\mu(0) = 0$,
(ii) $\mu(a) \geq 0$ for all $a \in A$,

(iii) for all pairwise disjoint a_n, $n = 0, 1, \ldots,$

$$\mu(\sum_{n=0}^{\infty} a_n) = \sum_{n=0}^{\infty} (a_n).$$

A measure μ is *strictly positive* if

(iv) $\mu(a) > 0$ for all $a \neq 0$,

and *probabilistic*, if also

(v) $\mu(1) = 1$.

Finally, a function μ that satisfies (i) and (iii) is called a *signed measure.*

Lemma 30.3. *If ν is a signed measure on B that satisfies c.c.c. then there exists an $a \in B$ such that $\nu(x) \geq 0$ for all $x \leq a$ and $\nu(x) \leq 0$ for all $x \leq -a$.*

Proof. First we claim that when $\nu(a) > 0$ then there exists some $b \leq a$ such that

$$\nu(b) > 0, \text{ and } \nu(x) \geq 0 \text{ for all } x \leq b. \tag{30.1}$$

If (30.1) fails then for every $b \leq a$, $b \neq 0$, there exists an $x \leq a$, $x \neq 0$, with $\nu(x) \leq 0$. Thus let W be a maximal antichain below a such that $\nu(x) \leq 0$ for every $x \in W$. Then $\sum W = a$ and we have $\nu(a) \leq 0$, a contradiction.

Now let Z be a maximal antichain such that (30.1) holds for every $b \in Z$. If $\nu(a) \leq 0$ for all $a \in B$ then the lemma holds trivially. Otherwise, Z is nonempty, and let $a = \sum Z$. This a satisfies the lemma. □

Lemma 30.4. *Let μ and ν be measures on B and let $a \in B$ be such that $\nu(a) > 0$. Then there exist a $b \leq a$, $b \neq 0$, and a number $\varepsilon > 0$ such that $\nu(x) \geq \varepsilon \cdot \mu(x)$ for all $x \leq b$.*

Proof. Let $\varepsilon > 0$ be such that $\nu(a) > \varepsilon \cdot \mu(a)$ and consider the signed measure $\nu - \varepsilon\mu$ on $B{\restriction}a$. By Lemma 30.3 there exists a $b \leq a$ such that $(\nu - \varepsilon\mu)(x) \geq 0$ for all $x \leq b$, and $(\nu - \varepsilon\mu)(x) \leq 0$ for all $x \leq a - b$. Since $(\nu - \varepsilon\mu)(b) \geq (\nu - \varepsilon\mu)(a) > 0$, we have $b \neq 0$. □

The next lemma is due to Fremlin:

Lemma 30.5 (Fremlin [1989]). *Let A be a measure algebra and let μ be a strictly positive measure on A. Let B be a complete subalgebra of A and let ν be a measure on B such that $\nu(b) \leq \mu(b)$ for all $b \in B$. Assume that*

$$A{\restriction}a \neq \{a \cdot b : b \in B\} \text{ for every } a \in A^+. \tag{30.2}$$

Then there exists some $a \in A$ such that

$$\nu(b) = \mu(a \cdot b) \text{ for all } b \in B. \tag{30.3}$$

Proof. For each $a \in A$, let ν_a denote the measure on B defined by (30.3): $\nu_a(b) = \mu(a \cdot b)$. We first prove the following consequence of (30.2): For every $a \in A^+$ and every $\varepsilon > 0$ there exists a $c \in (A{\restriction}a)^+$ such that $\nu_c(b) \le \varepsilon \cdot \nu_a(b)$ for all $b \in B$.

It is enough to prove this claim for $\varepsilon = \frac{1}{2}$, as the general case follows by a repeated application of the special case.

Thus let $a \in A^+$. By (30.2) there exists some $d < a$ such that $d \ne a \cdot b$ for every $b \in B$. Consider the signed measure $\frac{1}{2}\nu_a - \nu_d$ on B. By Lemma 30.3 there exists some $b \in B$ such that $\nu_d(x) \le \frac{1}{2}\nu_a(x)$ for all $x \in B{\restriction}b$ and $\nu_d(x) \ge \frac{1}{2}\nu_a(x)$ for all $x \in B{\restriction}(-b)$.

If $b \cdot d > 0$, we let $c = b \cdot d$, and we have $\nu_c(x) \le \frac{1}{2}\nu_a(x)$ for all $x \in B$.

If $b \cdot d = 0$ then $d \le a - b$, and we let $c = (a - b) \cdot (a - d)$. Since $d \ne a - b$ (by (30.2)), we have $c \ne 0$. For all $x \in B$, $\nu_c(x) \le \nu_a(x) - \nu_d(x) \le \frac{1}{2}\nu_a(x)$.

This proves the claim for $\varepsilon = \frac{1}{2}$ and the general case follows. To prove the lemma, let $a \in A$ be a maximal (in the partial order $\le$ on A) element such that $\nu_a(b) \le \nu(b)$ for all $b \in B$. We finish the proof by showing that $\nu_a = \nu$.

By contradiction, assume that there exists some $b_1 \in B$ such that $\nu_a(b_1) < \nu(b_1)$. By Lemma 30.4 there exist some $b_2 \le b_1$, $b_2 \ne 0$, and $\varepsilon > 0$ such that $(\nu - \nu_a)(x) \ge \varepsilon\mu(x)$ for all $x \in B{\restriction}b_2$. Note that $b_2 \not\le a$, since otherwise we would have $\nu_a(b_2) = \mu(b_2) \ge \nu(b_2)$.

Now we apply the earlier claim to $b_2 - a$, and get some $c \le b_2 - a$, $c \ne 0$, such that $\nu_c(x) \le \varepsilon\nu_{b_2-a}(x) \le \nu(x) - \nu_a(x)$ for all $x \in B$. Since $c \cdot a = 0$, we have $\nu_{a+c} = \nu_a + \nu_c \le \nu$, contradicting the maximality of a. ☐

Lemma 30.5 allows one to extend partial measure-preserving isomorphisms between homogeneous measure algebras. If μ and ν are probabilistic measures on measure algebras A and B, then an isomorphism f of A onto B is *measure-preserving* if $\nu(f(a)) = \mu(a)$ for all $a \in A$.

Lemma 30.6. *Let A_1 and A_2 be homogeneous measure algebras, both of the same weight κ, and let μ_1 and μ_2 be probabilistic measures on A_1 and A_2. Let B_1 and B_2 be complete subalgebras of A_1 and A_2, let f be a measure-preserving isomorphism of B_1 onto B_2, and assume that B_1 is σ-generated by fewer than κ generators. Then for every $a_1 \in A_1$ there exist an $a_2 \in A_2$ and a measure-preserving isomorphism $g \supset f$ of $\langle B_1 \cup \{a_1\}\rangle$, the subalgebra generated by $B_1 \cup \{a_1\}$, onto $\langle B_2 \cup \{a_2\}\rangle$.*

Proof. First we note that since every $A_1{\restriction}a$ has weight κ, the subalgebra B_1 satisfies (30.2); similarly for A_2 and B_2. Let $a_1 \in A_1$; if we let $\nu(f(b)) = \mu_1(a_1 \cdot b)$ for every $b \in B_1$, then ν is a measure on B_2 with $\nu \le \mu_2$. By Lemma 30.5 there exists some $a_2 \in A_2$ such that $\nu(f(b)) = \mu_2(a_2 \cdot f(b))$ for every $b \in B_1$.

The algebra $\langle B_1 \cup \{a_1\}\rangle$ consists of all elements of the form $b \cdot a_1 + c \cdot (-a_1)$ where $b, c \in B_1$. Thus we let

$$g(b \cdot a_1 + (c - a_1)) = f(b) \cdot a_2 + (f(c) - a_2). \tag{30.4}$$

We have to verify that g is well-defined. If $b \in B_1$ and $b \le a_1$ then $\mu_1(b) = \mu_1(a_1 \cdot b)$, and we have $\mu_2(f(b)) = \mu_1(b) = \mu_1(a_1 \cdot b) = \nu(f(b)) = \mu_2(a_2 \cdot f(b))$, and so $f(b) \le a_2$. It follows that $b \cdot a_1 = b' \cdot a_1$ implies $f(b) \cdot a_2 = f(b') \cdot a_2$. Similarly, one proves that if $c \in B_1$ and $c \le -a_1$ then $f(c) \le -a_2$, and therefore $c - a_1 = c' - a_1$ implies $f(c) - a_2 = f(c') - a_2$. Thus g is well-defined.

Since $\mu_2(f(b) \cdot a_2 + (f(c) - a_2)) = \mu_1(b \cdot a_1 + (c - a_1))$, g is measure-preserving, and a one-to-one homomorphism of $\langle B_1 \cup \{a_1\}\rangle$ onto $\langle B_2 \cup \{a_2\}\rangle$. $\square$

Proof of Theorem 30.1*.* The construction proceeds by induction. Let A and B be homogeneous measure algebras of weight κ and let μ and ν be probabilistic measures on A and B. Let $\{a_\alpha : \alpha < \kappa\}$ and $\{b_\alpha : \alpha < \kappa\}$ be generators of A and B. Inductively, we construct $A_0 \subset A_1 \subset \ldots \subset A_\alpha \subset \ldots$ and $B_0 \subset B_1 \subset \ldots \subset B_\alpha \subset \ldots$ and measure-preserving isomorphisms $f_0 \subset f_1 \subset \ldots \subset f_\alpha \subset \ldots$ such that for every α, A_α is a complete subalgebra of A of weight $< \kappa$ and $a_\alpha \in A_\alpha$, similarly for B_α, and $f_\alpha(A_\alpha) = B_\alpha$.

At successor stages we apply Lemma 30.6 to either $\langle A_\alpha \cup \{a_{\alpha+1}\}\rangle$ or $\langle B_\alpha \cup \{b_{\alpha+1}\}\rangle$. At a limit stage α, we consider the algebras $\tilde{A}_\alpha = \bigcup_{\beta<\alpha} A_\beta$ and $\tilde{B}_\alpha = \bigcup_{\beta<\alpha} B_\beta$. These are subalgebras of A and B, not necessarily complete. However, the completion A_α of $\tilde{A}_\alpha$ can be described as follows: The elements of A_α are limits of convergent countable sequences in $\tilde{A}_\alpha$ (see Exercise 30.1). The measure-preserving isomorphism $\tilde{f} = \bigcup_{\beta<\alpha} f_\beta$ between $\tilde{A}_\alpha$ and $\tilde{B}_\alpha$ extends to a unique measure-preserving isomorphism between A_α and the completion B_α of $\tilde{B}_\alpha$ (use Exercise 30.2). $\square$

Cohen Algebras

Let κ be an infinite cardinal. We consider the notion of forcing P_κ that adds κ Cohen reals: conditions in P_κ are finite 0–1 functions with domain $\subset \kappa$. Let

$$\boldsymbol{C}_\kappa = B(P_\kappa) \tag{30.5}$$

denote the complete Boolean algebra corresponding to P_κ. Throughout this section, $\overline{B}$ denotes the completion of a Boolean algebra B.

Definition 30.7. A Boolean algebra B is a *Cohen algebra* if $\overline{B} = \boldsymbol{C}_\kappa$ for some infinite cardinal κ.

In Theorem 30.10 below we give a combinatorial characterization of Cohen algebras.

Definition 30.8. A subalgebra A of a Boolean algebra B is a *regular subalgebra*,

$$A \le_{\text{reg}} B,$$

if for any $X \subset A$, if $\sum^A X$ exists then $\sum^A X = \sum^B X$.

The following is easily established:

Lemma 30.9. *The following are equivalent*:

(i) $A \leq_{\mathrm{reg}} B$.
(ii) *Every maximal antichain in* A *is maximal in* B.
(iii) *For every* $b \in B^+$ *there exists an* $a \in A^+$ *such that for every* $x \in A^+$, *if* $x \leq a$ *then* $x \cdot b \neq 0$. □

See Exercises 30.3–30.9 for further properties of $\leq_{\mathrm{reg}}$.

If A is a subalgebra of B and $b \in B$, then the *projection* of b to A, $\mathrm{pr}^A(b)$, is the smallest element $a \in A$, if it exists, such that $b \leq a$. (Similarly, $\mathrm{pr}_A(b)$ is the greatest $a \in A$ such that $a \leq b$.)

The *density* of a Boolean algebra B is the least size of a dense subset of B. B has *uniform density* if for every $a \in B^+$, $B{\restriction}a$ has the same density.

If X is a subset of a Boolean algebra B, we denote

$$\langle X \rangle = \text{the subalgebra generated by } X, \tag{30.6}$$

and if A is a subalgebra of B and $b_1, \ldots, b_n \in B$,

$$A(b_1, \ldots, b_n) = \langle A \cup \{b_1, \ldots, b_n\} \rangle. \tag{30.7}$$

Theorem 30.10. *Let* B *be an infinite Boolean algebra of uniform density.* B *is a Cohen algebra if and only if the set* $\{A \in [B]^\omega : A \leq_{\mathrm{reg}} B\}$ *contains a closed unbounded set* C *with the property*

$$\text{if } A_1, A_2 \in C \text{ then } \langle A_1 \cup A_2 \rangle \in C. \tag{30.8}$$

If B is countable, the condition is trivially satisfied as $C = \{B\}$ is a closed unbounded subset of $[B]^\omega$.

First we prove the forward direction of the theorem: If B is a dense subalgebra of $\boldsymbol{C}_\kappa$, then B has the property stated in Theorem 30.10. (In particular, $\boldsymbol{C}_\kappa$ itself has the property.) Let B be a dense subalgebra of $\boldsymbol{C}_\kappa$. For every $S \subset \kappa$, consider the forcing P_S consisting of finite 0–1 functions with domain $\subset S$, and let $\boldsymbol{C}_S = B(P_S)$. Note that $\boldsymbol{C}_S \leq_{\mathrm{reg}} \boldsymbol{C}_\kappa$.

Now let C be the set of all countable subalgebras A of B with the property that there exists a countable $S \subset \kappa$ such that

$$A \text{ is dense in } B \cap \boldsymbol{C}_S \text{ and } B \cap \boldsymbol{C}_S \text{ is dense in } \boldsymbol{C}_S. \tag{30.9}$$

The following lemma will establish the forward direction.

Lemma 30.11. *The set* C *is closed unbounded in* $[B]^\omega$, *satisfies* (30.8), *and every* $A \in C$ *is a regular subalgebra of* B.

Proof. Let $A \in C$ and let S be a countable subset of κ such that (30.9) holds. Since $B \cap \boldsymbol{C}_S$ is dense in $\boldsymbol{C}_S$ and $\boldsymbol{C}_S \le_{\mathrm{reg}} \boldsymbol{C}_\kappa$, we have $B \cap \boldsymbol{C}_S \le_{\mathrm{reg}} \boldsymbol{C}_\kappa$, and since B is dense in $\boldsymbol{C}_\kappa$, we have $B \cap \boldsymbol{C}_S \le_{\mathrm{reg}} B$. As A is dense in $B \cap \boldsymbol{C}_S$, it follows that $A \le_{\mathrm{reg}} B$. To see that C is unbounded, note that there are arbitrarily large countable sets S such that $B \cap \boldsymbol{C}_S$ is dense in $\boldsymbol{C}_S$ (because $\boldsymbol{C}_\kappa$ has the countable chain condition). Thus for any $a \in B$ we can find a countable S and a countable algebra $A \subset B$ such that $a \in A$, that A is dense in $B \cap \boldsymbol{C}_S$ and $B \cap \boldsymbol{C}_S$ is dense in $\boldsymbol{C}_S$.

To show that C is closed, let $\{A_n\}_{n=0}^\infty$ be an increasing chain in C and let $A = \bigcup_{n=0}^\infty A_n$; let $\{S_n\}_{n=0}^\infty$ be witnesses for $A_n \in C$. The sets S_n form a chain, and if we let $S = \bigcup_{n=0}^\infty S_n$, it follows that A is dense in $B \cap \boldsymbol{C}_S$ and $B \cap \boldsymbol{C}_S$ is dense in $\boldsymbol{C}_S$.

Now we verify (30.8); we shall show that if A_1 is dense in $\boldsymbol{C}_{S_1}$ and A_2 is dense in $\boldsymbol{C}_{S_2}$ then $A = \langle A_1 \cup A_2 \rangle$ is dense in $\boldsymbol{C}_S$ where $S = S_1 \cup S_2$. Let $b \in \boldsymbol{C}_S^+$; we shall find $a_1 \in A_1$ and $a_2 \in A_2$ such that $0 \ne a_1 \cdot a_2 \le b$.

Let $p \in P_S$ be such that $p \le b$, and let $p_1 = p{\restriction}S_1$, $p_2 = p{\restriction}S_2$. First we find some $a_1 \in A_1^+$ such that $a_1 \le p_1$ and then some $q_1 \in P_{S_1}$ such that $q_1 \le a_1$. Let $q_2 = p_2{\restriction}(S_2 - S_1) \cup (q_1{\restriction}S_2)$; we have $q_2 \in P_{S_2}$. Now we find some $a_2 \in A_2^+$ such that $a_2 \le q_2$. It remains to show that $a_1 \cdot a_2 \ne 0$: There exists some $r_2 \in P_{S_2}$ with $r_2 \le a_2$, and then $r_2 \cup (q_1{\restriction}(S_1 - S_2)) \in P_S$ is below both a_1 and a_2. $\square$

For the opposite direction, let B be an infinite Boolean algebra of uniform density κ and let C be a closed unbounded set of countable regular subalgebras of B that satisfies (30.8). First we note that B satisfies the countable chain condition: See Exercise 30.10. Let

$$\mathcal{S} = \{\langle \textstyle\bigcup X \rangle : X \subset C\}. \tag{30.10}$$

We claim that every $A \in \mathcal{S}$ is a regular subalgebra of B. Let $A = \langle \bigcup X \rangle$ and let W be a maximal antichain in A; we verify that W is maximal in B. As W is countable, we have $W \subset \langle \bigcup Y \rangle$ for some countable $Y \subset X$. Since C is closed unbounded and satisfies (30.8) it follows that $A_0 = \langle \bigcup Y \rangle \in C$ and hence $A_0 \le_{\mathrm{reg}} B$. Since W is a maximal antichain in A_0 and $A_0 \le_{\mathrm{reg}} B$, W is maximal in B.

A set $G \subset B$ is *independent* if

$$\pm x_1 \cdot \pm x_2 \cdot \ldots \cdot \pm x_n \ne 0$$

for all distinct $x_1, \ldots, x_n \in G$. If G is independent then $\langle G \rangle = \mathrm{Fr}_G$ is the unique *free* algebra over G; note that the completion of Fr_G is $\boldsymbol{C}_G$. Our goal is to find an independent $G \subset \overline{B}$ such that $\langle G \rangle$ is dense in $\overline{B}$.

Let A be a subalgebra of a Boolean algebra D. An element $u \in D$ is *independent over* A if $a \cdot u \ne 0 \ne a - u$ for all $a \in A^+$.

Lemma 30.12. *Let D be a complete Boolean algebra of uniform density and let A be a complete subalgebra of D of smaller density. For every $v \in D$ there exists some $u \in D$ independent over A such that $v \in A(u)$.*

Proof. Exercises 30.11 and 30.12. $\square$

Let $\{d_\alpha : \alpha < \kappa\}$ be a dense subset of B. If A_1 and A_2 are subalgebras of $\overline{B}$ we say that A_1 and A_2 are *co-dense* if for every $a_1 \in A_1^+$ there exists some $a_2 \in A_2^+$ with $a_2 \le a_1$, and for every $a_2 \in A_2^+$ there exists some $a_1 \in A_1^+$ with $a_1 \le a_2$.

We construct, by induction on $\alpha < \kappa$, two continuous chains $G_0 \subset G_1 \subset \ldots \subset G_\alpha \subset \ldots$ and $B_0 \subset B_1 \subset \ldots \subset B_\alpha \subset \ldots$ such that

(30.11)
(i) $B_\alpha \in \mathcal{S}$,
(ii) $A_\alpha = \langle G_\alpha \rangle$ and B_α are co-dense,
(iii) $d_\alpha \in B_{\alpha+1}$,
(iv) $G_{\alpha+1} - G_\alpha$ is countable,
(v) G_α is an independent subset of $\overline{B}$.

This will prove that B is a Cohen algebra, because by (iii), $\bigcup_\alpha B_\alpha$ is dense in B, hence $\bigcup_\alpha A_\alpha$ is dense in $\overline{B}$, and by (v), $\bigcup_\alpha A_\alpha$ is the free algebra Fr_G (where $G = \bigcup_\alpha G_\alpha$).

At limit stages, we let $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$ and $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. To construct $G_{\alpha+1}$ and $B_{\alpha+1}$, we proceed as follows: Since A_α is dense in $\overline{B_\alpha}$, $\overline{A_\alpha} = \overline{B_\alpha}$ is a complete subalgebra of $\overline{B}$. Moreover, if $u_1, \ldots, u_n \in \overline{B}$ then $\overline{A_\alpha}(u_1, \ldots, u_n)$ is a complete subalgebra of $\overline{B}$.

Since $|A_\alpha| < \kappa$, we find, by Lemma 30.12, for every $b \in \overline{B}$ some $u \in \overline{B}$ independent over $\overline{A_\alpha}$ such that $b \in \overline{A_\alpha}(u)$. More generally, if $b, u_1, \ldots, u_n \in \overline{B}$, then there exists some u independent over $\overline{A_\alpha}(u_1, \ldots, u_n)$ such that $b \in \overline{A_\alpha}(u_1, \ldots, u_n, u)$.

Given $u \in \overline{B}$, there exists a countable set $\{b_n\}_{n=0}^\infty \subset B$ such that $\sum_{n=0}^\infty b_n = u$. Then there exists some $X \in C$ such that $\{b_n\}_n \subset X$ and so $\langle B_\alpha \cup X \rangle$ is dense in $\overline{A_\alpha}(u)$. Therefore there exist a countable set $\{u_n\}_{n=0}^\infty \subset \overline{B}$ and some $B_{\alpha+1} \in \mathcal{S}$ such that $d_\alpha \in B_{\alpha+1}$, that $G_{\alpha+1} = G_\alpha \cup \{u_n\}_{n=0}^\infty$ is independent and that $A_{\alpha+1} = \langle G_{\alpha+1} \rangle$ and $B_{\alpha+1}$ are co-dense. $\square$

The following property is a natural weakening of the characterization of Cohen algebras in Theorem 30.10:

Definition 30.13. An infinite Boolean algebra B of uniform density is *semi-Cohen* if $[B]^\omega$ has a closed unbounded set of countable regular subalgebras.

An immediate consequence of the definition is that if B is semi-Cohen and $|B| \le \aleph_1$ then B is a Cohen algebra. This is because $[B]^\omega$ has a closed unbounded subset that is a chain, and therefore satisfies (30.8).

The important feature of semi-Cohen algebras is that the property is hereditary:

Theorem 30.14. *If B is a semi-Cohen algebra and if A is a regular subalgebra of B of uniform density then A is semi-Cohen.*

Proof. $[B]^\omega$ has a closed unbounded subset of regular subalgebras of B. Since $A \le_{\text{reg}} B$, there exists for every $b \in B^+$ some $a \in A^+$ such that there is no $x \in A^+$ with $x \le a - b$. Let $F : B^+ \to A^+$ be a function that to each $b \in B^+$ assigns such an $a \in A^+$. Let $C \subset [B]^\omega$ be a closed unbounded set of regular subalgebras closed under F.

If $X \in C$ then $A \cap X \le_{\text{reg}} X$ because X is closed under F. Every maximal antichain in $A \cap X$ is maximal in X, hence in B (because $X \le_{\text{reg}} B$), hence in A. Therefore $A \cap X \le_{\text{reg}} A$.

There is a closed unbounded set $D \subset [A]^\omega$ such that $D \subset \{X \cap A : X \in C\}$; D witnesses that A is semi-Cohen. □

Corollary 30.15. *If B is semi-Cohen and has density $\aleph_1$ then B is a Cohen algebra.*

Proof. B has a dense subalgebra A of size $\aleph_1$. By Theorem 30.14, A is also semi-Cohen, and hence Cohen. But $\overline{A} = \overline{B}$, and hence B is Cohen. □

Corollary 30.16. *Every complete subalgebra of $\boldsymbol{C}_\kappa$ of uniform density $\aleph_1$ is isomorphic to $\boldsymbol{C}_{\omega_1}$.* □

The property of being semi-Cohen is also preserved by completion. This can be proved using the following lemma:

Lemma 30.17. *A Boolean algebra B of uniform density is semi-Cohen if and only if B is Cohen in V^P, where P is the collapse of $|B|$ onto $\aleph_1$ with countable conditions.*

Proof. As $|B| = \aleph_1$ in V^P, it suffices to show that B is semi-Cohen if and only if it is semi-Cohen in V^P.

As P does not add new countable sets, $[B]^\omega$ remains the same in V^P. By property (iii) of Lemma 30.9, the relation $\le_{\text{reg}}$ is absolute. Let S be the set of all regular subalgebras of B. If S contains a closed unbounded set C then C is closed unbounded in V^P. Conversely, if S does not contain a closed unbounded set, then it does not contain one in V^P. □

Corollary 30.18. *B is semi-Cohen if and only if its completion is semi-Cohen.*

Proof. Let B be semi-Cohen and let $A = \overline{B}$. Let P be the ω-closed collapse of $|A|$ to $\aleph_1$. In V^P, A has a dense subalgebra B that is a Cohen algebra, hence A itself is Cohen. Therefore A is semi-Cohen.

The converse follows from Theorem 30.14. □

Not every semi-Cohen algebra is a Cohen algebra, and Corollary 30.16 does not extend to density $\aleph_2$. Koppelberg and Shelah gave an example of a complete subalgebra of C_{ω_2} of (uniform density $\aleph_2$) that is not isomorphic to C_{ω_2}. Another example, due to Zapletal, is the forcing that adds $\aleph_2$ eventually different reals: Let

$$P = \{z : z \text{ is a finite function with } \mathrm{dom}(z) \subset \omega_2 \text{ and } \mathrm{ran}(z) \subset \omega^{<\omega}\},$$

and let $z \leq w$ if z is a coordinate-wise extension of w and for $\alpha \neq \beta$ in $\mathrm{dom}(w)$, if $n \in \mathrm{dom}(z(\alpha) - w(\alpha))$ and $n \in \mathrm{dom}(z(\beta))$, then $z(\alpha)(n) \neq z(\beta)(n)$.

If $B = B(P)$ then B can be embedded in $\boldsymbol{C}_{\omega_2}$ but is not isomorphic to $\boldsymbol{C}_{\omega_2}$. We omit the proof.

Suslin Algebras

Definition 30.19. A *Suslin algebra* is a complete atomless Boolean algebra that is ω-distributive and satisfies the countable chain condition.

If T is a normal Suslin tree, and P_T is the forcing with T upside down, then $B(P_T)$ is a Suslin algebra. Conversely, if B is a Suslin algebra of density $\aleph_1$ then $B = B(P_T)$ for some Suslin tree T; in general, if B is a Suslin algebra then B has a complete subalgebra B_T such that $B_T = B(P_T)$ for some Suslin tree T.

Theorem 30.20. *If B is a Suslin algebra then $|B| \leq 2^{\aleph_1}$.*

Proof. Let $\kappa = 2^{\aleph_1}$. Assume that there is a Suslin algebra B such that $|B| > \kappa$. We shall reach a contradiction.

Without loss of generality we assume that $|B{\restriction}u| > \kappa$ for all $u \in B^+$. We shall construct a κ-sequence

$$B_0 \subset B_1 \subset \ldots \subset B_\alpha \subset \ldots \qquad (\alpha < \kappa) \tag{30.12}$$

of complete subalgebras of B, each of size $\leq \kappa$. If $D \subset B$ and $|D| \leq \kappa$, then there are $\kappa^{\aleph_1} = \kappa$ D-valued names for relations on ω_1; thus for every such D let $\dot{R}^D_\gamma$, $\gamma < \kappa$, be a fixed enumeration of all such names. Let $\alpha \mapsto (\beta_\alpha, \gamma_\alpha)$ be the canonical mapping of κ onto $\kappa \times \kappa$; we recall $\beta_\alpha \leq \alpha$ for all α.

The sequence (30.12) is constructed as follows: We let $B_0 = \{0, 1\}$; if α is a limit ordinal, then B_α is the complete subalgebra of B generated by $\bigcup_{\nu<\alpha} B_\nu$. If $|B_\nu| \leq \kappa$ for each $\nu < \alpha$, then $|B_\alpha| < \kappa$. At successor steps, we construct $B_{\alpha+1}$ as follows: Let $D = B_{\beta_\alpha}$ and let $\dot{R} = \dot{R}^D_{\gamma_\alpha}$. If

$$\|(\omega_1, \dot{R}) \text{ is a Suslin tree}\|_{B_\alpha} = 1, \tag{30.13}$$

if $\dot{C} \in V^{B_\alpha}$ is the Suslin algebra (in V^{B_α}) corresponding to the Suslin tree and if $B_\alpha * \dot{C}$ is (isomorphic to) a complete subalgebra of B, then we let $B_{\alpha+1} = B_\alpha * \dot{C}$. Otherwise, we let $B_{\alpha+1} = B_\alpha$. In either case, if $|B_\alpha| \leq \kappa$, then $|B_{\alpha+1}| \leq \kappa$.

Now let B_κ be the complete subalgebra of B generated by $\bigcup_{\alpha<\kappa} B_\alpha$. Clearly, $|B_\kappa| \le \kappa$. Let $\dot{A} \in V^{B_\kappa}$ be the complete Boolean algebra $B : B_\kappa$ (in V^{B_κ}). Since both B and B_κ satisfy the c.c.c., we have

$$\|\dot{A} \text{ satisfies the c.c.c.}\|_{B_\kappa} = 1.$$

Similarly, since both B and B_κ are ω-distributive, we have

$$\|\dot{A} \text{ is } \omega\text{-distributive}\|_{B_\kappa} = 1.$$

We have assumed that $|B{\restriction}u| > \kappa$ for all $u \ne 0$, and we also have $|B_\kappa| \le \kappa$. Thus

$$\| \, |\dot{A}| > \kappa\|_{B_\kappa} = 1$$

and consequently

$$\|\dot{A} \text{ is not atomic}\|_{B_\kappa} = 1.$$

Now we work inside V^{B_κ}; There exists a $\dot{T} \subset \dot{A}$ such that $(\dot{T}, \ge_{\dot{A}})$ is a normal Suslin tree; let $\dot{B}_T \subset \dot{A}$ be the Suslin algebra, subalgebra of $\dot{A}$, generated by $\dot{T}$. Let $\dot{R}$ be a binary relation on ω_1 isomorphic to $\dot{T}$.

The name $\dot{R}$ is B_κ-valued; and since B_κ satisfies the countable chain condition, $\dot{R}$ involves at most $\aleph_1$ elements of B_κ. Since $\operatorname{cf}\kappa > \aleph_1$, there exists a $\beta < \kappa$ such that $\dot{R} \in V^{B_\beta}$; furthermore, let $\gamma < \kappa$ be such that $\dot{R}$ is the γth B_β-valued binary relation on ω_1, $\dot{R} = \dot{R}_\gamma^{B_\beta}$.

Let $\alpha < \kappa$ be such that $\beta = \beta_\alpha$ and $\gamma = \gamma_\alpha$. Since

$$\|(\omega_1, \dot{R}) \text{ is a Suslin tree}\|_{B_\kappa} = 1,$$

it follows that

$$\|(\omega_1, \dot{R}) \text{ is a Suslin tree}\|_{B_\alpha} = 1.$$

If $\dot{C}$ denotes the corresponding Suslin algebra in V^{B_α}, we have

$$B_\alpha * \dot{C} \subset B_\kappa * \dot{B}_T \subset B_\kappa * \dot{A} = B$$

and it follows that $B_{\alpha+1} = B_\alpha * \dot{C}$. However, forcing with a Suslin tree destroys its Suslinity, and we have

$$\|(\omega_1, \dot{R}) \text{ is not a Suslin tree}\|_{B_{\alpha+1}} = 1,$$

a contradiction. $\square$

Suslin algebras of size $2^{\aleph_1}$ can be constructed by forcing (cf. Jech [1973b]), or in L (an unpublished result of Laver).

Simple Algebras

Definition 30.21. A complete Boolean algebra B is *simple* if it is atomless and if it has no proper atomless complete subalgebra.

The problem of existence of simple algebras originated in forcing and was first discussed by McAloon in [1971]. It is clear that a simple algebra is *minimal*, i.e., when forcing with it, there is no intermediate model between the ground model and the generic extension. Minimality, when formulated in Boolean-algebraic terms, is the following property:

(30.14) If A is a complete atomless subalgebra of B then there exists a partition W such that $A{\restriction}w = B{\restriction}w$ for all $w \in W$.

(An example of a minimal algebra is $B(P)$ where P is the Sacks forcing.)

Simple algebras, in addition to being minimal, are *rigid*, i.e., have no nontrivial automorphisms (Exercise 30.13). It turns out that the conjunction of these two properties also implies that the algebra is simple (Exercise 30.14). Thus we have:

Theorem 30.22. *An atomless complete Boolean algebra is simple if and only if it is rigid and minimal.* □

An example of a rigid and minimal algebra is B_P where P is Jensen's forcing from Theorem 28.1 that produces a minimal Δ^1_3 real. B_P is minimal because the generic real has minimal degree of constructibility, and rigid because it is definable. It follows that if $V = L$ then a simple complete Boolean algebra exists.

In L, one can also construct Suslin algebras that are simple (see Exercises 30.15 and 30.16 for the construction of a rigid Suslin algebra).

Simple complete Boolean algebras have been constructed in ZFC; we refer the reader to Jech-Shelah's papers [1996] and [2001]. The former constructs a countably generated simple algebra and uses a modification of the Sacks forcing to produce a minimal definable real. The latter construction is somewhat less complicated and yields forcing that produces a minimal definable uncountable set.

Infinite Games on Boolean Algebras

Infinite games have many applications in set theory, particularly in descriptive set theory, and we shall investigate these methods in some detail in the chapter on Axiom of Determinacy. In the present section we look into some properties of complete Boolean algebras, and of forcing, that are formulated in terms of infinite games.

Let B be a Boolean algebra, and let $\mathcal{G}$ be the following infinite game between two players I and II: I chooses a nonzero element $a_0 \in B$ and then II chooses some $b_0 \in B^+$ such that $b_0 \le a_0$. Then I plays (chooses) $a_1 \le b_0$ and II plays $b_1 \le a_1$ (both $\ne 0$). The game continues, with I's moves $a_n \in B^+$, $n < \omega$, and II's moves $b_n \in B^+$, $n < \omega$, such that

$$a_0 \ge b_0 \ge a_1 \ge b_1 \ge \ldots \ge a_n \ge b_n \ge \ldots . \tag{30.15}$$

Player I wins the game if $\prod_{n=0}^{\infty} a_n = 0$; player II wins otherwise: if the chain (30.15) has a nonzero lower bound. A *strategy* for player I is a function $\sigma : B^{<\omega} \to B$; it is a *winning strategy* if I wins every play (30.15) in which I follows σ, i.e., for each n, $a_n = \sigma(\langle b_0, \ldots, b_{n-1}\rangle)$. A (winning) strategy for II is defined similarly. If player I has a winning strategy then II does not, and vice versa, and in general, neither player need have a winning strategy.

Lemma 30.23. *Player I has a winning strategy in $\mathcal{G}$ if and only if B is not ω-distributive.*

Proof. Let σ be a winning strategy for I. Let $a_0 = \sigma(\langle\rangle)$; we shall find partitions W_n of a_0 without a common refinement. Let $W_0 = \{a_0\}$. Having constructed $W_0, \ldots, W_n$, consider all finite sequences $a_0 \ge b_0 \ge \ldots \ge a_n \ge b_n$ where the a_n's are chosen by σ and $a_k \in W_k$ for all $k \le n$. Let W_n be a maximal antichain whose members are elements $a_{n+1} = \sigma(\langle b_0, \ldots, b_n\rangle)$ where $a_0 \ge b_0 \ge \ldots \ge a_n \ge b_n$ is as described. The W_n's are partitions of a_0 and do not have a common refinement.

Conversely, if B is not ω-distributive, there exist some a_0 and open dense sets D_n below a_0 such that $\bigcap_{n=0}^{\infty} D_n = \emptyset$. We define $\sigma(\langle\rangle) = a_0$, and if $a_0 \ge b_0 \ge \ldots \ge a_n \ge b_n$ is such that the a_n's are chosen by σ, let $\sigma(\langle b_0, \ldots, b_n\rangle)$ be some element of D_n below b_n. The function σ is a winning strategy for I. $\square$

Let P be a separative notion of forcing, and consider the infinite game $\mathcal{G}$ in which players I and II take turns to play a descending chain $a_0 \ge b_0 \ge \ldots \ge a_n \ge b_n \ge \ldots$ in P. I wins if and only if the chain does not have a lower bound. It is easy to see that either player has a winning strategy in this game if and only if the same player has a winning strategy in $\mathcal{G}$ played on $B(P)$ (Exercise 30.17).

Definition 30.24. A separative notion of forcing P (a Boolean algebra B) is *strategically ω-closed* if player II has a winning strategy in the game $\mathcal{G}$.

Being strategically ω-closed is a hereditary property. If B is strategically ω-closed and if A is a regular subalgebra of B then also A is strategically ω-closed (Exercise 30.18).

It is obvious that if P is ω-closed then player II has a winning strategy in $\mathcal{G}$. Hence if B has a dense ω-closed subset, then B is strategically ω-closed. The converse is true for small algebras:

Theorem 30.25 (Foreman). *If B has density $\aleph_1$ and is strategically ω-closed, then it has a dense ω-closed subset.*

Proof. Let $\{d_\alpha : \alpha < \omega_1\}$ be a dense set in B. By induction on α, we find partitions W_α of 1 such that W_β refines W_α if $\alpha < \beta$, and every W_α has some $w \le d_\alpha$; at limit stages, we use ω-distributivity of B. Let $T = \bigcup_{\alpha<\omega_1} W_\alpha$; T is dense in B and is a tree. Let σ be a winning strategy for II in the game $\mathcal{G}$ on T. We shall find a dense subset P of T that is ω-closed.

If $t \in T$, we call $p = \langle a_0, b_0, \ldots, a_n, b_n\rangle$ a *partial play above* t if the b_k's are played by σ and $b_n > t$. We claim:

(30.16) $(\forall t \in T)\,(\exists t^* < t)$ if p is a partial play above t^* and if $u > t^*$ then there is a partial play $q \supset p$ above t^* with last move b such that $u > b > t^*$.

To prove the claim, we construct $t_0 > t_1 > \ldots > t_n > \ldots$ such that $t_0 = t$ and that for every n and every partial play p above t_n, if $u > t_n$ then there exist some $q \supset p$ with last move b and some m such that $u > b > t_m$. This is possible because there are only countably many such p's and u's. Therefore there exists a play $\langle a_0, b_0, \ldots, a_n, b_n, \ldots\rangle$ that is played according to σ and that is cofinal in $\langle t_n\rangle_{n=0}^\infty$. As σ is a winning strategy, $\langle t_n\rangle_{n=0}^\infty$ has a lower bound, let t^* be a maximal lower bound (it exists because T is a tree). This proves (30.16). Now let

$$P = \{s \in T : \text{for some descending chain } \{t_n\}_{n=0}^\infty,\ s \text{ is a maximal lower bound of } \{t_n^*\}_{n=0}^\infty\}.$$

The set P is ω-closed: Given $s_0 > s_1 > \ldots$ in P, find $\{t_n\}_{n=0}^\infty$ such that $t_0^* > s_0 > t_1^* > \ldots$. The chain $\{t_n^*\}_{n=0}^\infty$ has a lower bound (because there exists a cofinal play by σ) and its maximal lower bound is in P.

The set P is dense in T: Given $t \in T$, let $\{t_n\}_{n=0}^\infty$ be the chain t, t^*, t^{**}, ... There exists a cofinal σ-play, and so $\{t_n\}_{n=0}^\infty$ has a lower bound. The maximal lower bound is in P. □

As a corollary, we get the following characterization of strategically ω-closed forcings:

Corollary 30.26. *B is strategically ω-closed if and only if B is a regular subalgebra of some algebra that has an ω-closed dense subset.*

Proof. Sufficiency follows from Exercise 30.18. Thus assume that B is strategically ω-closed and let σ be a winning strategy for II. Let P be the collapse with countable conditions of $|B|$ to $\aleph_1$. In V^P, σ is still a winning strategy, and by Theorem 30.25, B has an ω-closed dense subset $\dot{E}$. Let $A = B(P \times B^+)$; B is a regular subalgebra of A. Let $D = \{(p, b) : p \Vdash b \in \dot{E}\}$; D is dense in A. D is ω-closed: Let $\{(p_n, b_n)\}_n$ be descending and let $p = \bigcup_n p_n$. Then p forces that $\{b_n\}_n$ is descending, and there is a $b \in B^+$ such that $p \Vdash (b \in \dot{E}$ and $b \le b_n$ for all $n)$. Hence (p, b) is a lower bound of $\{(p_n, b_n)\}_n$. □

Foreman's Theorem does not extend to $\aleph_2$: It is consistent that there is a strategically ω-closed complete Boolean algebra of density $\aleph_2$ that does not have an ω-closed dense subset (Jech and Shelah [1996]).

There are many other infinite games that can be used to define properties of forcing and Boolean algebras, see Jech [1984]. We'll show in Chapter 31 that proper forcing admits such characterization. See also Exercise 30.19.

Exercises

Let B be a σ-complete Boolean algebra. If $\{a_n\}_{n<\omega}$ is a sequence in B, let $\limsup_n a_n = \prod_{n=0}^{\infty} \sum_{k\geq n} a_n$ and $\liminf_n a_n = \sum_{n=0}^{\infty} \prod_{k\geq n} a_n$. If $\limsup_n a_n = \liminf_n a_n = a$, we say that $\{a_n\}_{n<\omega}$ *converges*, and let $\lim_n a_n = a$.

30.1. If A is a subalgebra a measure algebra B then the complete subalgebra of B σ-generated by A consists of all limits of convergent seequences in A.

30.2. If μ is a measure on a measure algebra B and if $a = \lim_n a_n$, then $\mu(a) = \lim_n \mu(a_n)$.

30.3. If A is a finite subalgebra of B then $A \leq_{\text{reg}} B$.

30.4. If $A \leq_{\text{reg}} B$ and $B \leq_{\text{reg}} C$ then $A \leq_{\text{reg}} C$.

30.5. If A is a subalgebra of B, B is a subalgebra of C, and $A \leq_{\text{reg}} C$ then $A \leq_{\text{reg}} B$.

30.6. If A is a dense subalgebra of B then $A \leq_{\text{reg}} B$.

30.7. $A \leq_{\text{reg}} B$ if and and only if $\overline{A} \leq_{\text{reg}} \overline{B}$.

30.8. If A and B are complete then $A \leq_{\text{reg}} B$ if and only if A is a complete subalgebra of B.

30.9. If $\mathrm{pr}^A(b)$ exists for all $b \in B$, then $A \leq_{\text{reg}} B$.

30.10. If $\{A \in [B]^\omega : A \leq_{\text{reg}} B\}$ is stationary, then B has the countable chain condition.

[Let W be a maximal antichain and consider the model $M = (B, \leq, W)$. There exists an elementary submodel A of M such that $A \leq_{\text{reg}} B$. $W \cap A$ is maximal in A, therefore in B, and hence $W = W \cap A$.]

30.11 (Vladimirov's Lemma). Let D be a complete Boolean algebra of uniform density and A a complete subalgebra of smaller density. Then there exists an element $u \in D$ independent over A.

[Let $X = \{x \in D^+ :$ there is no $a \in A^+$ with no $a \in A^+$ with $a \leq x\}$; X is dense. Let $Y = \{\mathrm{pr}^A(x) : x \in X\}$; Y is dense. Let $W \subset Y$ be a maximal antichain, and let $Z \subset X$ be such that $W = \{\mathrm{pr}^A(z) : z \in Z\}$. Let $u = \sum Z$. If $a \in A^+$, let $z \in Z$ be such that $a \cdot \mathrm{pr}^A(z) \neq 0$; we also have $a \cdot (\mathrm{pr}^A(z) - z) \neq 0$. Hence u is independent over A.]

30.12. Under same assumptions, for every $v \in D - A$ there exists some $u \in D$ independent over A such that $v \in A(u)$.

[Let $z = \mathrm{pr}_A(v) + -\mathrm{pr}^A(v)$. If $z = 0$, let $u = v$. Otherwise, apply Exercise 30.11 to $D{\restriction}z$, to get some $w \le z$ independent over $A{\restriction}z$. Then let $u = w + (v - z)$. We have $v \in A(u)$ since $v = \mathrm{pr}_A(v) + u \cdot \mathrm{pr}^A(v)$; also, u is independent over A.]

30.13. Every simple complete Boolean algebra is rigid.

[Let π be a nontrivial automorphism. There exist disjoint a and b such that $\pi(a) = b$. Each x has a decomposition $x = a \cdot x + b \cdot x + y$; let A be the complete subalgebra $\{x : b \cdot x = \pi(a \cdot x)\}$. A is atomless and $a \notin A$.]

30.14. Every rigid minimal complete Boolean algebra is simple.

[Let B be minimal and A a complete atomless subalgebra such that $A \ne B$. There exists a $z \notin A$ such that $A{\restriction}z = B{\restriction}z$. Let $u_1 = z - \mathrm{pr}_A(z)$, $v_1 = \mathrm{pr}^A(z) - z$. Let $0 \ne v \le v_1$ be such that $A{\restriction}v = B{\restriction}v$, and let $u = u_1 \cdot \mathrm{pr}^A(v)$. For all $a \in A$ let $\pi(a \cdot u) = a \cdot v$; show that π is an automorphism between $B{\restriction}u$ and $B{\restriction}v$. Then π extends to a nontrivial automorphism of B.]

30.15. Let T be a normal Suslin tree and let B_T be the corresponding Suslin algebra. If π is an automorphism of B_T then there is a closed unbounded set $C \subset \omega_1$ such that $\pi{\restriction}T^C$ is an automorphism of T^C, where $T^C = \{t \in T : o(t) \in C\}$.

30.16. If $V = L$ then there exists a Suslin tree T such that B_T is rigid.

[Use $\diamondsuit$ and Exercise 30.15 to destroy all potential automorphisms of B_T.]

30.17. Player I (player II) has a winning strategy in $\mathcal{G}$ played on P if and only if the same player has one in $\mathcal{G}$ on $B(P)$.

30.18. If a complete Boolean algebra B is strategically ω-closed and if A is a complete subalgebra of B then A is strategically ω-closed.

[Let σ be a winning strategy on B; then the following σ_A is a winning strategy on A: When I plays a_0, let $b_0 = \sigma(a_0)$ and let $\sigma_A(a_0) = \mathrm{pr}^A(b_0)$. When I plays $a_1 \le \sigma_A(a_0)$, let $b_1 = \sigma(\langle a_0, a_1 \cdot b_0\rangle)$ and $\sigma_A(\langle a_0, a_1\rangle) = \mathrm{pr}^A(b_1)$. And so on.]

30.19. Let B be a Boolean algebra of uniform density. Consider the infinite game on B in which two players select elements a_0, b_0, ..., a_n, b_n, ... and II wins if and only if the set $\{a_n, b_n\}_{n=0}^{\infty}$ generates a regular subalgebra of B. Show that II has a winning strategy if and only if B is semi-Cohen.

[If σ is a winning strategy then the set C of all countable subalgebras closed under σ is a closed unbounded set of regular subalgebras; the converse is similar.]

Historical Notes

Maharam's Theorem 30.1 appeared in [1942]; the present proof is based on Fremlin's article [1989]. Theorem 30.10 appeared in Balcar, Jech and Zapletal [1997] improving a similar earlier result of Koppelberg [1993]. The [1997] investigates semi-Cohen algebra, the concept introduced by Fuchino and Jech. Corollary 30.16: Koppelberg. Theorem 30.20 is due to Solovay.

Rigid minimal algebras were studied by McAloon in [1971]. Constructions of a simple complete Boolean algebra in ZFC appeared in Jech and Shelah [1996] and [2001].

The game $\mathcal{G}$ on a Boolean algebra was introduced in Jech [1978]; this and similar games were studied in Jech [1984]. Foreman's Theorem 30.25 appeared in [1983].

Exercises 30.11 and 30.12: Vladimirov [1969].

Exercises 30.13 and 30.14: McAloon [1971].