31. Proper Forcing

Definition and Examples

Proper forcing was introduced by S. Shelah who isolated properness as the property of forcing that is common to many standard examples of forcing notions and that is preserved under countable support iteration.

Definition 31.1. A notion of forcing $(P, <)$ is *proper* if for every uncountable cardinal λ, every stationary subset of $[\lambda]^\omega$ remains stationary in the generic extension.

Properness is a generalization of both the countable chain condition and of being ω-closed. The following two lemmas are the analogs of Lemma 22.25 and Lemma 23.7 (for $\kappa = \aleph_1$):

Lemma 31.2. *If P satisfies the countable chain condition then for every uncountable λ, every closed unbounded set $C \subset [\lambda]^\omega$ in $V[G]$ has a subset $D \in V$ that is closed unbounded in V. Hence every stationary set $S \subset [\lambda]^\omega$ remains stationary in $V[G]$.*

Proof. Let $p \Vdash \dot{C}$ is closed unbounded; let $\dot{F}$ be a name for a function from $\lambda^{<\omega}$ into λ such that $p \Vdash C_{\dot{F}} \subset \dot{C}$ (where $C_{\dot{F}}$ is the set of all closure points of $\dot{F}$—see Theorem 8.28). Let $f : \lambda^{<\omega} \to [\lambda]^\omega$ be the function

$$f(e) = \{\alpha \in \lambda : \|\dot{F}(e) = \alpha\| \neq 0\}.$$

$f(e)$ is countable because P satisfies the countable chain condition. Let $D = C_f$.

Since $p \Vdash \dot{F}(e) \in f(e)$, if x is closed under f then $p \Vdash \dot{F}(e) \in x$, and so $p \Vdash D \subset \dot{C}$. □

Lemma 31.3. *If P is ω-closed then every stationary set $S \subset [\lambda]^\omega$ remains stationary in $V[G]$.*

Proof. Let $p \Vdash \dot{F} : \lambda^{<\omega} \to \lambda$. We shall find a condition $q \le p$ and some $x \in S$ such that $q \Vdash \dot{F}(x^{<\omega}) \subset x$.

Consider the model $(H_\kappa, \in, (P, <), p, \dot{F}, \Vdash)$ where $\kappa \ge \lambda$ is sufficiently large. Let C be the closed unbounded set in $[H_\lambda]^\omega$ of all countable elementary submodels of the model. By Theorem 8.27 there exists some $N \in C$ such that $N \cap \lambda \in S$. Let $x = N \cap \lambda$.

Enumerate $x^{<\omega} = \langle e_n : n < \omega \rangle$ and construct a sequence of conditions $p = p_0 \geq p_1 \geq \ldots \geq p_n \geq \ldots$ such that for each n there exists an $\alpha_n \in N \cap \lambda$ such that $p_n \Vdash \dot{F}(e_n) = \alpha_n$ (by elementarity). Let q be a lower bound for $\{p_n\}_n$. Then $q \Vdash \dot{F}(x^{<\omega}) \subset x$. ☐

Proper forcing does not collapse $\aleph_1$. In fact, an easy argument shows that a stronger property is true:

Lemma 31.4. *If P is proper then every countable set of ordinals in $V[G]$ is included in a set in V that is countable in V.*

Proof. Let X be a countable set of ordinals in $V[G]$ and let λ be uncountable in V such that $X \subset \lambda$. The set $([\lambda]^\omega)^V$ remains stationary in $V[G]$ and therefore meets the set $\{A \in [\lambda]^\omega : A \supset X\}$, which is a closed unbounded set in $V[G]$. Thus $X \subset A$ for some $A \in ([\lambda]^\omega)^V$. ☐

We shall now formulate a technical condition that is equivalent to properness of a forcing notion, and that will be used to prove that properness is preserved under countable support iteration. We refer the reader to the exercises for other equivalents of properness.

Let $(P, <)$ be a fixed notion of forcing. We say that λ is *sufficiently large* if λ is a cardinal and $\lambda > 2^{|P|}$. A *model* M is an elementary submodel of $(H_\lambda, \in, <, \ldots)$ where H_λ is the collection of all sets hereditarily of cardinality less than λ, $<$ is some unspecified well-ordering of H_λ (to allow for inductive constructions), and the structure of H_λ contains all the relevant parameters; in particular, M contains $(P, <)$.

Definition 31.5. A condition q is (M, P)-*generic* if for every maximal antichain $A \in M$, the set $A \cap M$ is predense below q.

The following lemma (the proof is a routine exercise) illuminates the concept of (M, P)-genericity:

Lemma 31.6. *Let λ be sufficiently large, let $M \prec H_\lambda$ be such that $P \in M$, and let $q \in P$. The following are equivalent*:

(i) *q is (M, P)-generic.*
(ii) *If $\dot{\alpha} \in M$ is an ordinal name then $q \Vdash \dot{\alpha} \in M$, i.e.,*

$$\forall r \leq q \; \exists s \leq r \; \exists \beta \in M \;\; s \Vdash \dot{\alpha} = \beta.$$

(iii) *$q \Vdash \dot{G} \cap M$ is a filter on P generic over M.* ☐

Theorem 31.7. *A forcing notion P is proper if and only if for all sufficiently large λ there is a closed unbounded set C of elementary submodels $M \prec (H_\lambda, \ldots)$ such that*

$$\forall p \in M \; \exists q \leq p \; (q \text{ is } (M, P)\text{-generic}). \tag{31.1}$$

Proof. First we show that the condition is necessary. Let P be proper and let λ be sufficiently large. Toward a contradiction assume that the set of all models $M \prec H_\lambda$ for which (31.1) fails is stationary. By normality there exist a stationary set $S \subset [H_\lambda]^\omega$ and a condition $p \in P$ such that for every $q \le p$ and every $M \in S$, q is not (M, P)-generic.

Now let $V[G]$ be a generic extension with $G \ni p$, and let us argue in $V[G]$. Every maximal antichain A below p (in V) meets G in a unique condition q_A. Let

$$C = \{M \prec (H_\lambda)^V : \text{if } A \in M \text{ then } q_A \in M\};$$

C is closed unbounded. Since S remains stationary in $V[G]$, there exists some $M \in S \cap C$.

For each $A \in M$ we have $\sum(A \cap M) \in G$ (because $q_A \in G$), and by genericity, $\prod_{A \in M} \sum(A \cap M) \in G$. Let $q \le \prod_{A \in M} \sum(A \cap M)$. Then $q \le p$ and q is (M, P)-generic, contradicting $M \in S$.

Now we prove that the condition is sufficient. Let P be a forcing notion that satisfies the condition of the theorem; we shall prove that P preserves stationary sets. Let λ be an uncountable cardinal and let $S \subset [\lambda]^\omega$ be stationary. Let $\dot{F}$ be a name for a function $\dot{F} : \lambda^{<\omega} \to \lambda$ and $p \in P$. We shall find a $q \le p$ and $x \in S$ such that $q \Vdash x$ is closed under $\dot{F}$.

Let $\mu \ge \lambda$ be sufficiently large. By the assumption there exists a closed unbounded set $C \subset [H_\mu]^\omega$ such that (31.1) holds for every $M \in C$. By Theorem 8.27, $\{M \cap \lambda : N \in C\}$ contains a closed unbounded set in $[\lambda]^\omega$ and hence there exists some $M \in C$ with $M \cap \lambda \in S$.

Let $q \le p$ be (M, P)-generic. We finish the proof by showing that $q \Vdash M \cap \lambda$ is closed under $\dot{F}$. Let $e \in (M \cap \lambda)^{<\omega}$; we shall show that $q \Vdash \dot{F}(e) \in M$. There is $A \in M$ such that A is a maximal antichain below p and every $w \in A$ decides $\dot{F}(e)$. Now if $r \le q$ forces $\dot{F}(e) = \alpha$, then because $A \cap M$ is predense below q, r is compatible with some $w \in A \cap M$ and so $w \Vdash \dot{F}(e) = \alpha$. Since α is definable from w, $\dot{F}$, and e, we have $\alpha \in M$. ☐

Another characterization of properness is formulated in terms of infinite games.

Definition 31.8. Let P be a forcing notion and let $p \in P$. The *proper game* (for P, below p) is played as follows: I plays P-names $\dot{\alpha}_n$ for ordinal numbers, and II plays ordinal numbers β_n. Player II wins if there exists a $q \le p$ such that

$$q \Vdash \forall n \, \exists k \, \dot{\alpha}_n = \beta_k. \tag{31.2}$$

Theorem 31.9. *A forcing notion P is proper if and only if for every $p \in P$, II has a winning strategy for the proper game.*

Proof. Exercise 31.3. ☐

We shall now present some examples of proper forcing. The following concept is due to J. Baumgartner:

Definition 31.10. A notion of forcing $(P, <)$ satisfies *Axiom A* if there is a collection $\{\leq_n\}_{n=0}^{\infty}$ of partial orderings of P such that $p \leq_0 q$ implies $p \leq q$ and for every n, $p \leq_{n+1} q$ implies $p \leq_n q$, and

(i) if $\langle p_n : n \in \omega \rangle$ is a sequence such that $p_0 \geq_0 p_1 \geq_1 \ldots \geq_{n-1} p_n \geq_n \ldots$ then there is a q such that $q \leq_n p_n$ for all n;
(ii) for every $p \in P$, for every n and for every ordinal name $\dot{\alpha}$ there exist a $q \leq_n p$ and a countable set B such that $q \Vdash \dot{\alpha} \in B$.

Lemma 31.11. *If P satisfies Axiom A then P is proper.*

Proof. Let P satisfy Axiom A and let $p \in P$. The following is a winning strategy for II in the game from Exercise 31.2: When I plays $\dot{\alpha}_n$, let II find a condition $p_n \leq_{n-1} p_{n-1}$ (with $p_0 \leq p$) and a countable set B_n such that $p_n \Vdash \dot{\alpha}_n \in B_n$. If q is a lower bound for $\{p_n\}_{n=0}^{\infty}$ then q witnesses that II wins the game. □

Example 31.12. Every ω-closed forcing satisfies Axiom A.
Let $p \leq_n q$ if and only if $p \leq q$, for all n. □

Example 31.13. Every c.c.c. forcing satisfies Axiom A.
Let $p \leq_n q$ if and only if $p = q$, for all $n > 0$. □

Example 31.14. The notions of forcing that add a Sacks real, a Mathias real or a Laver real satisfy Axiom A.
For Sacks reals, see (15.26). For Laver forcing, see (28.17); Mathias forcing is similar. □

In Exercises 31.5 and 31.6 we give Baumgartner's example of proper forcing that does not satisfy Axiom A.

Iteration of Proper Forcing

It is obvious that a two-step iteration of proper forcing is proper: If P preserves stationary sets and in V^P, $\dot{Q}$ preserves stationary sets then $P * \dot{Q}$ preserves stationary sets. What is more important however is that properness is preserved under countable support iteration. The present section is devoted to the proof of this.

Theorem 31.15 (Shelah). *If P_α is a countable support iteration of $\{\dot{Q}_\beta : \beta < \alpha\}$ such that every $\dot{Q}_\beta$ is a proper forcing notion in $V^{P_\alpha \restriction \beta}$, then P_α is proper.*

Toward the proof of Theorem 31.15 we first observe that the properness condition in Theorem 31.7 can be somewhat simplified:

Lemma 31.16. *P is proper if and only if for every $p \in P$, every sufficiently large λ and every countable $M \prec (H_\lambda, \in, <)$ containing P and p, there exists a $q \le p$ that is (M, P)-generic.*

Proof. Let P be proper and $p \in P$. Let $\mu = 2^{|P|}$ and $\lambda > \mu$; we recall that $<$ is a well-ordering of H_λ. By Theorem 31.7 the set of all countable elementary submodels of H_μ with property (31.1) contains a closed unbounded set, and so it contains C_F for some function $F : H_\mu^{<\omega} \to H_\mu$. If F is the least such function in H_λ then every $M \prec (H_\lambda, \in, <)$ is closed under F and so $M \cap H_\mu$ satisfies (31.1). Hence every such M with $P, p \in M$ satisfies the condition of the lemma. □

In order to prove that an iteration P_α is proper, we wish to show that if λ is sufficiently large and $M \prec H_\lambda$ contains P_α then for every $p \in P_\alpha \cap M$ there is some (M, P_α)-generic $q \in P_\alpha$ such that $q \Vdash_\alpha p \in \dot{G}$. We prove this by induction; the main point is that the inductive condition is somewhat stronger:

Lemma 31.17. *Let P_α be a countable support iteration of proper forcing notions. Let λ be sufficiently large and let $M \prec (H_\lambda, \in, <)$ be countable, with $P_\alpha \in M$. For every $\gamma \in \alpha \cap M$, every $q_0 \in P_\gamma = P_\alpha \restriction \gamma$ that is (M, P_γ)-generic, and every $\dot{p} \in V^{P_\gamma}$ such that*

$$q_0 \Vdash_\gamma \dot{p} \in (P_\alpha \cap M) \text{ and } \dot{p} \restriction \gamma \in \dot{G}_\gamma \tag{31.3}$$

there exists an (M, P_α)-generic condition $q \in P_\alpha$ such that $q \restriction \gamma = q_0$ and $q \Vdash_\alpha \dot{p} \in \dot{G}_\alpha$.

$\dot{G}_\alpha$ and $\dot{G}_\gamma$ are the canonical names for generic filters on P_α and P_γ respectively. Letting $\gamma = 0$ (and q_0 the trivial condition 1 in $P_0 = \{1\}$), we get the desired result.

Lemma 31.17 is proved by induction on α. In order to handle the successor stages we need first to prove the special case $\alpha = 2$, $\gamma = 1$; then the inductive step from α to $\alpha + 1$ is a routine modification of the special case:

Lemma 31.18. *Let P be proper, let $\dot{Q} \in V^P$ be such that $\Vdash_P \dot{Q}$ is proper and let $R = P * \dot{Q}$. Let $M \prec H_\lambda$ be countable, with $R \in M$. For every (M, P)-generic $q_0 \in P$ and every $\dot{p} \in V^P$ such that*

$$q_0 \Vdash_P \dot{p} \in (M \cap R) \text{ and } \dot{p}_0 \in \dot{G}_P$$

(where $\dot{p}$ is a name for $(\dot{p}_0, \dot{p}_1)$ and $\dot{G}_P$ is generic on P) there is some $\dot{q}_1 \in V^P$ such that $(q_0, \dot{q}_1)$ is (M, R)-generic and $(q_0, \dot{q}_1) \Vdash_R \dot{p} \in \dot{G}_R$.

Proof. To find the name $\dot{q}_1$, let G be a generic filter on P containing q_0. Let $p = \dot{p}^G$ and $q = \dot{Q}^G$; then $p \in M \cap R$ and $p = (p_0, \dot{p}_1)$ with $p_0 \in G$. Since $\dot{p}_1 \in M$, we have $p_1 \in M[G] \cap Q$, and since Q is proper, there exists (in $V[G]$)

a stronger condition q_1 that is $(M[G], Q)$-generic. (Here we use the fact that $M[G] \prec H_\lambda^{V[G]}$ which we leave as an exercise.) This describes $\dot{q}_1$.

That $(q_0, \dot{q}_1)$ is (M, R)-generic follows from q_0 being (M, P)-generic and $q_0 \Vdash \dot{q}_1$ is $(M[\dot{G}_P], \dot{Q})$-generic (this is routine). Also, since $q_0 \Vdash \dot{p}_0 \in \dot{G}_P$ and $q_0 \Vdash \dot{q}_1 \le \dot{p}_1$, we conclude that $(q_0, \dot{q}_1) \Vdash_R \dot{p} \in \dot{G}_R$. □

Proof of Lemma 31.17. We assume that α is a limit ordinal; hence $\alpha \cap M$ is a countable set of ordinals without a maximal element. Let $\langle \gamma_n : n \in \omega \rangle$ be an increasing set of ordinals in M with $\gamma_0 = \gamma$, cofinal in $\alpha \cap M$. Let $\{D_n : n \in \omega\}$ be an enumeration of al dense subsets of P_α that are in M. Let $q_0 \in P_{\gamma_0}$ be (M, P_{γ_0})-generic and let $\dot{p}$ be a $V^{P_{\gamma_0}}$-name such that (31.3) holds. We shall find a (M, P_α)-generic condition $q \in P_\alpha$ such that $q \restriction \gamma_0 = q_0$ and $q \Vdash_\alpha \dot{p} \in \dot{G}_\alpha$.

We construct q as the limit of conditions $q_n \in P_{\gamma_n}$ such that $q_{n+1} \restriction \gamma_n = q_n$, and such that each q_n is (M, P_{γ_n})-generic.

Along with the q_n we construct P_{γ_n}-names $\dot{p}_n$ such that $\dot{p}_0 = \dot{p}$ and that for each n, q_n forces

(31.4) (i) $\dot{p}_n \in (\dot{P}_\alpha \cap M)$,
(ii) $\dot{p}_n \le \dot{p}_{n-1}$,
(iii) $\dot{p}_n \in D_{n-1}$,
(iv) $\dot{p}_n \restriction \gamma_n \in \dot{G}_{\gamma_n}$.

Assume that q_n and $\dot{p}_n$ have been constructed. To find $\dot{p}_{n+1}$, let G be a P_{γ_n}-generic filter such that $q_n \in G$, and let $p_n = \dot{p}_n^G$. We have $p_n \in P_\alpha \cap M$ and $p_n \restriction \gamma_n \in G$. Since q_n is (M, P_{γ_n})-generic and $D_n \in M$, we can find a condition $p_{n+1} \le p_n$ in $D_n \cap M$ such that $p_{n+1} \restriction \gamma_n \in G$. This describes the $P_{\gamma_{n+1}}$-name $\dot{p}_{n+1}$. Now we apply the inductive condition to γ_{n+1} (in place of α) and γ_n (in place of γ), for q_n and $\dot{p}_{n+1} \restriction \gamma_{n+1}$; we obtain a $q_{n+1} \in P_{\gamma_n+1}$ that forces (31.4) (with n replaced by $n + 1$).

Now we let q be the limit of the q_n. Clearly, $q \in P_\alpha$ and $q \restriction \gamma_0 = q_0$. We complete the proof by showing that for every n, $q \Vdash_\alpha \dot{p}_n \in \dot{G}_\alpha$. This implies not only that $q \Vdash_\alpha \dot{p} \in \dot{G}_\alpha$, but also that q is (M, P_α)-generic, because $q \Vdash \dot{p}_n \in (D_{n-1} \cap M)$.

To verify that $q \Vdash_\alpha \dot{p}_n \in \dot{G}_\alpha$, let G be a generic filter on P_α and let $p_n = \dot{p}_n^G$. We have $p_n \in M$ and $p_n \restriction \gamma_k \in G_{\gamma_k} \cap M$ for all $k \ge n$. Thus if we let $\delta = \sup(\alpha \cap M)$, we have $p_n \restriction \delta \in G_\delta$. Since $p_n \in M$, its support is included in M and therefore $p_n \restriction \delta = p_n$. It follows that $p_n \in G$. □

A significant consequence of Theorem 31.15 is that countable support iteration of proper forcing preserves $\aleph_1$. As for cardinals above $\aleph_1$, one often needs additional assumptions on the iterates $\dot{Q}_\beta$ to calculate the chain condition. The easiest case was already stated in Exercise 16.20: If P is a countable support iteration of length $\kappa \ge \aleph_2$ such that each $P \restriction \beta$, $\beta < \kappa$, has a dense subset of size $< \kappa$, then P satisfies the κ-chain condition. In particular, iteration of length ω_2 with each $P \restriction \beta$ having a dense set of size $\aleph_1$, satisfies the $\aleph_2$-chain condition, and all cardinals are preserved.

A somewhat better result is the following which we state without a proof. For a proof, see Abraham's paper [∞] in the Handbook of Set Theory. [Shelah's book [1998] contains more general chain condition theorems.]

Theorem 31.19. *Assume* CH. *If P is a countable support iteration of length $\kappa \le \omega_2$ of proper forcings $\dot{Q}_\beta$ of size $\aleph_1$, then P satisfies the $\aleph_2$-chain condition.* □

The Proper Forcing Axiom

When we replace the countable chain condition in Martin's Axiom $\mathrm{MA}_{\aleph_1}$ by properness we obtain a more powerful statement, the *Proper Forcing Axiom* (PFA):

Definition 31.20 (Proper Forcing Axiom (PFA)). If $(P, <)$ is a proper notion of forcing and if $\mathcal{D}$ is a collection of $\aleph_1$ dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.

It turns out that PFA implies that $2^{\aleph_0} = \aleph_2$, and therefore PFA is a generalization of Martin's Axiom MA. Unlike MA, consistency of PFA requires large cardinals: It follows from the results stated later in this chapter that at least a Woodin cardinal is necessary. The consistency proof given below uses a supercompact cardinal.

Theorem 31.21. *If there exists a supercompact cardinal then there is a generic model that satisfies* PFA.

Proof. The proof follows loosely the proof of the consistency of MA. Let κ be a supercompact cardinal. The model is obtained by countable support iteration of length κ. Each notion of forcing used in the iteration is proper and has size $< \kappa$, thus both $\aleph_1$ and all cardinals $\ge \kappa$ are preserved. Cardinals between $\aleph_1$ and κ are collapsed and so κ becomes $\aleph_2$, and the model satisfies $2^{\aleph_0} = \aleph_2$.

In order to show that the resulting model satisfies PFA, we use a Laver function (see Theorem 20.21); this makes it possible to handle all potential proper forcing notions in κ steps.

Let $f : \kappa \to V_\kappa$ be a Laver function. We construct a countable support iteration P_κ of $\{\dot{Q}_\alpha : \alpha < \kappa\}$ as follows: At stage α, if $f(\alpha)$ is a pair $(\dot{P}, \dot{\mathcal{D}})$ of P_α-names such that $\dot{P}$ is a proper forcing notion and $\mathcal{D}$ is a γ-sequence of dense subsets of $\dot{P}$ for some $\gamma < \kappa$, we let $\dot{Q}_\alpha = \dot{P}$; otherwise, $\dot{Q}_\alpha$ is the trivial forcing.

Let G be a generic filter on P_κ, the countable support iteration of $\{\dot{Q}_\alpha : \alpha < \kappa\}$. Since each $\dot{Q}_\alpha$ is proper, P_κ is proper and therefore $\aleph_1$ is preserved. Each P_α (the iteration of $\{\dot{Q}_\beta : \beta < \alpha\}$) has size less than κ (because $f(\alpha) \in V_\alpha$) and so P_κ has the κ-chain condition; hence all cardinals $\ge \kappa$ are preserved.

Lemma 31.22. *In $V[G]$, if P is proper and $\mathcal{D} = \{D_\alpha : \alpha < \gamma\}$, with $\gamma < \kappa$, is a family of dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.*

This lemma will complete the proof of the theorem: For every $\gamma < \kappa$, let P be the forcing that collapses γ onto ω_1 with countable conditions, and for $\alpha < \gamma$ let $D_\alpha = \{p \in P : \alpha \in \operatorname{ran}(p)\}$. By Lemma 31.22, there exists a collapsing map of γ onto ω_1. Thus $\kappa = \aleph_2$ in $V[G]$. Now Lemma 31.22 implies that $V[G]$ satisfies PFA. Moreover, $2^{\aleph_0} = \aleph_2$ in $V[G]$: On the one hand, PFA implies $\mathrm{MA}_{\aleph_1}$ and so $2^{\aleph_0} > \aleph_1$, and on the other hand, $2^{\aleph_0} \le \kappa$ because $|P_\kappa| = \kappa$.

Proof of Lemma 31.22. Let $\dot{P}$ and $\dot{\mathcal{D}}$ be P_κ-names for P and $\mathcal{D}$. Let $\lambda > 2^{2^{|P|}}$ be sufficiently large; we may also assume that $P \subset \lambda$. Since f is a Laver function, there exists an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$, $M^\lambda \subset M$, and $(jf)(\kappa) = (\dot{P}, \dot{\mathcal{D}})$.

P is a proper forcing in $V[G]$. This is witnessed by some closed unbounded set $C \subset [H_\eta]^\omega$ of countable models for some η with $2^{|P|} < \eta < \lambda$. Since $M^\lambda \subset M$ and P_κ has the κ-chain condition, $V[G]$ satisfies that $M[G]^\lambda \subset M[G]$, and therefore C is closed unbounded in $M[G]$. Therefore P is proper in the model $M[G]$.

Now consider the forcing notion $j(P_\kappa)$ in M. It is a countable support iteration of length $j(\kappa)$ using the Laver function $j(f)$. Since $j{\restriction}V_\kappa$ is the identity, we have $j(P_\kappa){\restriction}\kappa = P_\kappa$. As $(jf)(\kappa) = (\dot{P}, \dot{\mathcal{D}})$ and P is proper in $M[G]$, it follows that $(j\dot{Q})_\kappa = \dot{P}$. Hence

$$j(P_\kappa) = P_\kappa * \dot{P} * \dot{R}$$

for some $\dot{R}$.

Let $H * K$ be a $V[G]$-generic ultrafilter on $\dot{P} * \dot{R}$. In $V[G * H * K]$ we extend the elementary embedding $j : V \to M$ to an elementary embedding $j^* : V[G] \to M[G * H * K]$ as follows: For every P_κ-name $\dot{x}$, let

$$j^*(\dot{x}^G) = j(\dot{x})^{G*H*K}.$$

The definition of j^* does not depend on the choice of the name $\dot{x}$, since $\|\dot{x} = \dot{y}\| \in G$ implies $\|j(\dot{x}) = j(\dot{y})\| \in G * H * K$ (because $j(p) = p$ for every $p \in P_\kappa$). Similarly, $\|\varphi(\dot{x})\| \in G$ implies $\|\varphi(j(\dot{x}))\| \in G * H * K$, and so j^* is elementary. Clearly, j^* extends j.

The filter H on P is $V[G]$-generic and thus meets every D_α, $\alpha < \gamma$. Let $E = \{j(p) : p \in H\}$. Since $j{\restriction}\lambda \in M$, the set E is in $M[G * H * K]$, and generates a filter on $j^*(P)$ that is $j^*(\mathcal{D})$-generic. Thus

$$M[G * H * K] \vDash \text{there exists a } j^*(\mathcal{D})\text{-generic filter on } j^*(P)$$

and since $j^* : V[G] \to M[G * H * K]$ is elementary, there exists in $V[G]$ a $\mathcal{D}$-generic filter on P. ∎

Applications of PFA

Our first goal is to outline the proof of the following theorem:

Theorem 31.23 (Todorčević). PFA *implies* $2^{\aleph_0} = \aleph_2$.

As the first step we show that the Open Coloring Axiom (29.6) is a consequence of PFA. If $[X]^2 = K_0 \cup K_1$ with K_0 open, let us call $Z \subset X$ 0-*homogeneous* if $[Z]^2 \subset K_0$ and 1-*homogeneous* if $[Z]^2 \subset K_1$. It is clear that the closure of a 1-homogeneous set is also 1-homogeneous, and so in (29.6) we can further assume that the sets H_n are closed.

The proof of OCA from PFA uses the following technical lemma that we state without proof:

Lemma 31.24 (Todorčević). *Assume* $2^{\aleph_0} = \aleph_1$. *Let* $X \subset \boldsymbol{R}$ *and* $[X]^2 = K_0 \cup K_1$ *with* K_0 *open, and assume that* X *is not the union of countably many closed* 1-*homogeneous sets. Then there exists an uncountable* $Y \subset X$ *such that in any uncountable set* $W \subset \{p \in [Y]^{<\omega} : p$ *is* 0-*homogeneous*$\}$ *there exist* $p \neq q$ *such that* $p \cup q$ *is* 0-*homogeneous.*

Proof. See Theorem 4.4 of Todorčević [1989]. (To apply the theorem, let $F(x)$ be the closure of $\{y \in X : x < y$ and $\{x, y\} \in K_1\}$.) □

Theorem 31.25. PFA *implies* OCA.

Proof. Let $X \subset \boldsymbol{R}$ and let $[X]^2 = K_0 \cup K_1$ with K_0 open, and assume that X is not the union of countably many closed 1-homogeneous sets. We shall use PFA to find an uncountable 0-homogeneous set.

Let P be the forcing (15.2) that adds a subset of ω_1 with countable conditions. By Exercise 15.14, V^P satisfies $2^{\aleph_0} = \aleph_1$. Since P does not add new reals, it does not add new closed sets of reals and so in V^P, X is not the union of countably many closed 1-homogeneous sets.

By Lemma 31.24 there exists an uncountable $\dot{Y} \in V^P$ such that if we let $\dot{Q} = \{p \in [\dot{Y}]^{<\omega} : p$ is 0-homogeneous$\}$ (and p is stronger than q if $p \supset q$) then the forcing notion $\dot{Q}$ satisfies the countable chain condition. Hence $P * \dot{Q}$ is proper.

Let $\langle y_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of $\dot{Y}$ in V^P. For each $\alpha < \omega_1$, the set $D_\alpha = \{(p, q) \in P \times \dot{Q} : p \Vdash y_\alpha \in q\}$ is a dense set in $P * \dot{Q}$. Let $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\}$. By PFA there exists a $\mathcal{D}$-generic filter G on $P * \dot{Q}$, and then the set $Y = \bigcup\{q : (p, q) \in G\}$ is an uncountable 0-homogeneous set. □

Theorem 31.25 appears in Todorčević [1989]. Its proof does not require the full force of PFA. What we used is a weaker statement that is obtained by replacing "proper notion of forcing" in Definition 31.20 by "Axiom A forcing of cardinality $\leq 2^{\aleph_0}$." This axiom is weaker than PFA (and stronger than $\mathrm{MA}_{\aleph_1}$) and is consistent relative to ZFC+ "there exists a weakly compact cardinal" (see Baumgartner [1984]).

The consistency of related partition axioms was first established in Abraham, Rubin and Shelah [1985].

By Theorems 31.25 and 29.8, PFA implies OCA which implies $\mathfrak{b} = \aleph_2$. Thus to complete the proof of Theorem 31.23 it is enough to show that PFA implies $\mathfrak{b} = 2^{\aleph_0}$. We shall use another technical lemma of Todorčević that we state without a proof.

Let $\kappa \le 2^{\aleph_0}$ be a regular uncountable cardinal and let $F : [\kappa]^2 \to \omega$ be a partition. Let P be the forcing with countable conditions that adds a subset of ω_1. In V^P, $|\kappa| = \aleph_1$ and $\operatorname{cf} \kappa = \omega_1$; let $\dot{C} \in V^P$ be a closed unbounded subset of κ of order-type ω_1, consisting of limit ordinals. For every n and k let $\dot{R}^k_n$ be the forcing where conditions are finite k-homogeneous (for F) subsets of $\dot{C} + n = \{\alpha + n : \alpha \in \dot{C}\}$. $\dot{R}^k_n$ adds a k-homogeneous subset $\dot{G}^k_n$ of $\dot{C} + n$. (In general, $\dot{R}^k_n$ need not satisfy the countable chain condition.) Let $\dot{Q}^k_n$ be the product of ω copies of $\dot{R}^k_n$, and for every real $r \in \omega^\omega$, let $\dot{Q}_r = \dot{Q}_r(\dot{C})$ be the product of $\dot{Q}^{r(n)}_n$, $n < \omega$.

Lemma 31.26. *There exists a partition $F : [\mathfrak{b}]^2 \to \omega$ such that in V^P, for every $\dot{C}$ as above and every $r \in \omega^\omega$, $\dot{Q}_r(\dot{C})$ satisfies the countable chain condition.*

Proof. See Bekkali [1991], page 49. The partition F is obtained by using oscillating real numbers, cf. Chapter 1 of Todorčević [1989]. □

Lemma 31.27. PFA *implies* $\mathfrak{b} = 2^{\aleph_0}$.

Proof. Let $F : [\mathfrak{b}]^2 \to \omega$ be as in Lemma 31.26. Let P be the ω-closed forcing that adds a subset of ω_1, and let $\dot{C} \in V^P$ be a closed unbounded subset of $\mathfrak{b}$, of order-type ω_1.

Let $r \in \omega^\omega$. The forcing $P * \dot{Q}_r(\dot{C})$ is proper and we apply PFA to obtain a sufficiently generic filter $G \times \prod_n \prod_i G_{n,i}$. Let $C(r) = C = \dot{C}^G$; C is a closed unbounded subset of some $\delta(r) = \delta < \mathfrak{b}$, $\operatorname{cf} \delta = \omega_1$, and for each n, each $G_{n,i}$ is an $r(n)$-homogeneous subset of $C + n$. Let $C_{n,i} = G_{n,i} - n$; by genericity, we have $C = \bigcup_{i<\omega} C_{n,i}$ for each n, and

$$(31.5) \qquad r(n) = k \quad \text{if and only if} \quad \forall i \; \forall \alpha, \beta \in C_{n,i} \; F(\alpha + n, \beta + n) = k.$$

We claim that if $r \ne s$ then $\delta(r) \ne \delta(s)$.

Let n be such that $r(n) \ne s(n)$. Assuming that $\delta(r) = \delta(s) = \delta$, the set $C(r) \cap C(s)$ is closed unbounded in δ, and we can find i and j such that $C_{n,i}(r) \cap C_{n,j}(s)$ is unbounded. Let $\alpha < \beta$ be in this unbounded set; then by (31.5), $F(\alpha + \beta, \beta + n) = r(n) = r(s)$, a contradiction.

Thus we have produced a one-to-one mapping of ω^ω into $\mathfrak{b}$. □

The next theorem establishes the consistency strength of PFA (see the discussion following the proof):

Theorem 31.28 (Todorčević). PFA *implies that $\square_\kappa$ fails for every uncountable cardinal κ.*

Proof. Let κ be an uncountable cardinal, assume that $\square_\kappa$ holds, and let $\langle C_\alpha : \alpha \in \text{Lim}(\kappa^+)\rangle$ be a square-sequence (cf. (23.4)).

Let T be the tree whose nodes are limit ordinals below κ^+, and $\beta \prec \alpha$ if $\beta \in \text{Lim}(C_\alpha)$. Since $\langle C_\alpha\rangle_\alpha$ is a square-sequence, T has no κ^+-branch.

Let λ be sufficiently large, and consider countable elementary submodels M of H_λ such that $\langle C_\alpha : \alpha < \kappa^+\rangle \in M$; let $\delta_M = \sum(M \cap \kappa^+)$. An *elementary chain* is a sequence $\langle M_\alpha : \alpha < \omega_1\rangle$ of elementary submodels of H_λ such that $M_\alpha \subset M_\beta$ and $M_\alpha \in M_\beta$ whenever $\alpha < \beta$, and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ if α is a limit ordinal. If E is a finite subset of ω_1 then an E-chain is $\langle M_\alpha : \alpha \in E\rangle$ such that each M_α is an elementary submodel of H_λ, and $M_\alpha \cup \{M_\alpha\} \subset M_\beta$ for $\alpha < \beta$ in E.

We now define a forcing notion P as follows: A condition $p \in P$ is a pair $(\langle N_\alpha : \alpha \in E\rangle, f)$ where

(31.6)
(i) E is a finite subset of ω_1 and $\langle N_\alpha : \alpha \in E\rangle$ is an E-chain such that there exists an elementary chain $\langle M_\alpha : \alpha < \omega_1\rangle$ such that $N_\alpha = M_\alpha$ for all $\alpha \in E$,
(ii) f is a function from $\{\delta_{N_\alpha} : \alpha \in E\}$ into ω such that $f(\gamma) \neq f(\delta)$ whenever $\gamma \prec \delta$.

A condition q is stronger than p if $p = q{\restriction}E$.

Note that (i) resembles the forcing that adds a closed unbounded set with finite conditions, and (ii) resembles the forcing that specializes an Aronszajn tree.

Lemma 31.29. *P is proper.*

Proof. We omit the proof, as it is similar to the proof of properness in Exercise 31.5 (and using the fact that $(T, \prec)$ has no κ^+-branch). □

Now we use PFA to reach a contradiction. Let G be a sufficiently generic filter on P. The filter G yields an elementary chain $\langle N_\alpha : \alpha < \omega_1\rangle$ and a closed unbounded set $\{\delta_\alpha : \alpha < \omega_1\}$ (where $\delta_\alpha = \delta_{N_\alpha}$) with supremum γ. There is a closed unbounded set $C \subset \omega_1$ such that for all $\alpha \in C$, δ_α is a limit point of C_γ. Since $\delta_\alpha \prec \delta_\beta$ whenever $\alpha < \beta \in C$, it follows that $\{\delta_\alpha : \alpha \in C\}$ is an ω_1-chain in T.

On the other hand, the filter G yields a specializing function on $\{\delta_\alpha : \alpha < \omega_1\}$, that is a function F with values in ω such that $F(\delta_\alpha) \neq F(\delta_\beta)$ whenever $\delta_\alpha \prec \delta_\beta$. A contradiction. □

The proof of Theorem 31.28 has been modified by Magidor to show that under PFA, even a weak version of $\square$ fails (we shall discuss these versions of $\square$ in Chapter 38). It has been proved by Schimmerling that the failure of those principles imply an inner model for a Woodin cardinal. Thus we have:

Theorem 31.30 (Schimmerling). *If* PFA *holds then there exists an inner model of* "*there exists a Woodin cardinal.*" □

Martin's Axiom $\mathrm{MA}_{\aleph_1}$ implies that there are no Suslin trees, and moreover, that every Aronszajn tree is special. PFA implies a stronger result. If T is a normal ω_1-tree and $C \subset \omega_1$ a closed unbounded set, then $T \restriction C$ is the tree $\{t \in T : o(t) \in C\}$. Two trees T_1 and T_2 are *club-isomorphic* if there exists a closed unbounded C such that $T_1 \restriction C$ and $T_2 \restriction C$ are isomorphic.

Theorem 31.31. *If* PFA *holds then any two normal Aronszajn trees are club-isomorphic.*

Proof. Let T_1 and T_2 be two normal Aronszajn trees. Consider the forcing with finite conditions (E, f) such that

(31.7) (i) E is a finite subset of ω_1,
(ii) $\mathrm{dom}(f)$ is a subtree of $T_1 \restriction E$ in which every branch has size $|E|$; similarly for $\mathrm{ran}(f) \subset T_2 \restriction E$,
(iii) f us an isomorphism.

We omit the proof that P is proper and refer the reader to Todorčević [1984], Theorem 5.10.

A sufficiently generic filter on P yields an uncountable set A and an isomorphism between $T_1 \restriction A$ and $T_2 \restriction A$, which easily extends to $T_1 \restriction C$ where C is the closure of A. □

We present one more consequence of PFA, due to J. Baumgartner (compare with Theorem 28.24):

Theorem 31.32. *If* PFA *holds then there are no $\aleph_2$-Aronszajn trees.*

Proof. Assume that T is an $\aleph_2$-Aronszajn tree. Let P be the forcing that adds a subset of ω_1 with countable conditions. Since $2^{\aleph_0} = \aleph_2$, P collapses ω_2 and so there is in V^P a closed unbounded subset $\dot{C}$ of ω_2, of order-type ω_1. The tree T has no new branches (this is proved as in Lemma 27.10, because the levels of T have size $\aleph_1 < 2^{\aleph_0}$). Thus $\dot{U} = T \restriction \dot{C}$ is in V^P an ω_1-tree with no ω_1-branches.

Now let $\dot{Q} \in V^P$ be the specializing forcing for $\dot{U}$, as in Theorem 16.17. $\dot{Q}$ satisfies the countable chain condition, and so $P * \dot{Q}$ is proper.

Let G be a sufficiently generic filter on $P * \dot{Q}$. It yields a closed unbounded subset C of some $\gamma < \omega_2$, a tree $U = T \restriction C$, and a specializing function $f : U \to \omega$. This is a contradiction, since a special tree has no ω_1-branches, while every $t \in T$ at level γ produces an ω_1-branch in U. □

Exercises

31.1. If P is strategically ω-closed then P is proper.

The following two exercises present equivalent versions of the proper game:

31.2. Let $p \in P$. Player II has a winning strategy in the proper game if and only if II has a winning strategy in the game where I plays ordinal names $\dot\alpha_n$ and II plays countable sets of ordinals B_n, and II wins if some $q \le p$ forces $\forall n\ \exists k\ \dot\alpha_n \in B_k$.

31.3. P is proper if and only if for every $p \in P$, II has a winning strategy in the following game: At move n, I plays a maximal antichain A_n and II responds by playing countable sets $B_0^n \subset A_0, \ldots, B_n^n \subset A_n$. II wins if for some $q \le p$, $\forall n$ $\bigcup_{k=n}^\infty B_n^k$ is predense below q.

[In the forward direction, let λ be sufficiently large and let C be a closed unbounded set of models $M \prec H_\lambda$ that satisfy (31.1) and $p \in M$. The following is a winning strategy for II: When I plays A_n, let II choose some $M_n \in C$ such that $M_n \supset M_{n-1}$ and $A_n \in M_n$, and let $B_k^n = A_k \cap M_n$, $k = 0, \ldots, n$. Let $M = \bigcup_{n=0}^\infty M_n$ and let $q \le p$ be (M, P)-generic. Since $A_n \cap M = \bigcup_{k=0}^\infty B_n^k$, II wins.

Conversely, let σ be a winning strategy for II, and let λ be sufficiently large with $\sigma \in H_\lambda$. Show that for every $M \prec H_\lambda$ such that $P, p, \sigma \in M$ there is some (M, P)-generic $q \le p$ (by playing a game in which I plays successively all maximal antichains $A \in M$). Let C_p be the closed unbounded set of all such M; the diagonal intersection $\triangle_p C_p$ witnesses that P is proper.]

31.4. If P satisfies Axiom A and $p \in P$ then II has a winning strategy in the following game (more difficult for player II than the proper game): I plays ordinal names $\dot\alpha_n$ and II plays countable sets of ordinals B_n; II wins if some $q \le p$ forces $\forall n\ \dot\alpha_n \in B_n$.

Adding a closed unbounded set with finite conditions: A condition $p \in P$ is a finite function with $\mathrm{dom}(p) \subset \omega_1$, $\mathrm{ran}(p) \subset \omega_1$ such that there exists a normal function $f : \omega_1 \to \omega_1$ with $f \supset p$. A condition q is stronger than p if $q \supset p$. If G is generic then $f_G = \bigcup\{p : p \in G\}$ is a normal function. Note that if $\alpha = \omega^\beta$ (an *indecomposable* ordinal) and $p \subset \alpha \times \omega$ is a condition then $p \cup \{(\alpha, \alpha)\}$ is also a condition.

31.5. Let P be as above and let $p \in P$. Then II has a winning strategy in the game from Exercise 31.3. Hence P is proper.

[When I plays A_n, II finds some indecomposable $\alpha_n > \alpha_{n-1}$ such that for all $k \le n$

$$(\forall\beta < \alpha)(\exists\gamma < \alpha)(\forall p \subset \gamma \times \gamma)(\exists q \subset \gamma \times \gamma)\ q \in A_k$$

(and q is compatible with p), and plays $B_k^n = \{p \in A_k : p \subset \alpha_n \times \alpha_n\}$.]

31.6. Let P be as above. Then I has a winning strategy in the game from Exercise 31.4. Hence P does not satisfy Axiom A.

[Let $\dot f$ be the name for f_G. At move n, player I chooses an indecomposable ordinal α_n greater than B_{n-1} and plays $\dot f(\alpha_n)$.]

31.7. Let P be the ω-closed forcing for collapsing ω_2 to ω_1 with countable conditions. There exists a set $\mathcal{D}$ of $\aleph_2$ dense sets for which there is no $\mathcal{D}$-generic filter.

[For $\alpha < \omega_2$, let $D_\alpha = \{p \in P : \alpha \in \mathrm{ran}(p)\}$.]

PFA$^+$ is the following statement: If P is proper, if $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\}$ are dense sets and if $\Vdash \dot S \subset \omega_1$ is stationary, then there exists a $\mathcal{D}$-generic filter G such that $\dot S^G$ is stationary (where $\dot S^G = \{\alpha : \exists p \in G\ p \Vdash \alpha \in \dot S\}$).

31.8. PFA$^+$ is consistent relative to a supercompact cardinal.

[Modify the proof of Theorem 31.21.]

31.9. PFA$^+$ implies that for every regular $\kappa \geq \omega_2$, every stationary set $A \subset E^\kappa_\omega$ reflects at some γ of cofinality ω_1.

[Let $A \subset E^\kappa_\omega$ be stationary. Let P consist of closed countable subsets of κ, ordered by end-extension. P is ω-closed and adds a closed unbounded subset $\dot{C} \subset \kappa$ of order-type ω_1. A remains stationary and so $A \cap \dot{C}$ is a stationary subset of $\dot{C}$; let $\dot{S} = f_{-1}(A \cap \dot{C})$ where f is the isomorphism between ω_1 and $\dot{C}$. If G is sufficiently generic such that $\dot{S}^G$ is stationary, then $A \cap \dot{C}^G$ is stationary in $\gamma = \sup \dot{C}^G$.]

PFA$^-$ is the statement: If P is proper such that $|P| \leq \aleph_1$ and if $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\}$ are dense then there exists a $\mathcal{D}$-generic filter. In [1982] Shelah proves that PFA$^-$ is consistent relative to ZFC only.

31.10. PFA$^-$ implies that any two normal Aronszajn trees are club-isomorphic.

[The forcing in (31.7) has size $\aleph_1$.]

Historical Notes

Proper forcing was introduced by Shelah, cf. [1982] and [1998]. The iteration Theorem 31.15 is due to Shelah; our treatment follows Abraham's article [∞]. The proper game was formulated independently by Shelah and C. Gray.

Proper Forcing Axiom was introduced by Baumgartner [1984]; earlier (Baumgartner [1983]) he introduced Axiom A. Theorem 31.21 is due to Baumgartner.

Theorem 31.23: Todorčević [1989], see also Bekkali [1991]. (The claim in Veličković [1992] to this result cannot be substantiated.)

Theorem 31.25: Todorčević [1989]; Abraham, Rubin and Shelah [1985].

Theorem 31.28: Todorčević [1984].

Theorem 31.31: Abraham and Shelah [1985].

Theorem 31.32: Baumgartner [1984].

The forcing for adding a closed unbounded set with finite conditions is due to Baumgartner [1983].

Exercises 31.8, 31.9: Baumgartner [1984].

Exercise 31.10: Abraham and Shelah [1985].