32. More Descriptive Set Theory

$\mathbf{\Pi}^1_1$ Equivalence Relations

Theorem 32.1 (Silver). *If E is a $\mathbf{\Pi}^1_1$ equivalence relation on $\mathcal{N}$ then either E has at most $\aleph_0$ equivalence classes or there exits a perfect set of mutually inequivalent reals.*

Thus every $\mathbf{\Pi}^1_1$ (and in particular) Borel equivalence relation has either at most countably many or $2^{\aleph_0}$ equivalence classes. This can be viewed as a generalization of the perfect set property for analytic sets (Theorem 11.18); cf. Exercise 32.1. The theorem does not extend to $\mathbf{\Sigma}^1_1$, as there exists a Σ^1_1 equivalence relation with exactly $\aleph_1$ equivalence classes (Exercise 32.2).

We present a proof of Theorem 32.1 that is due to Leo Harrington. We start with an easy lemma.

Lemma 32.2. *Let E be a meager equivalence relation on $\mathcal{N}$. Then there exist a perfect set of inequivalent reals.*

Proof. Let $\{D_n\}_n$ be dense open sets in $\mathcal{N}\times\mathcal{N}$ such that $\mathcal{N}^2 - E \supset \bigcap_{n=0}^\infty D_n$. We construct a binary tree of finite sequences $\{u_s : s \in \mathit{Seq}(\{0,1\})\} \subset \mathit{Seq}$ such that for every n, if $|s| = |t| = n$ and $s \neq t$, then $O(u_s) \times O(u_t) \subset D_n$.

This is done by induction on the length of s. If the u_s have been defined for all $s \in \{0,1\}^n$, we consider successively all possible pairs $(s^\frown i, t^\frown j)$, and using the density of D_{n+1}, successively extend each $u_{s^\frown i}$ until $O(u_{s^\frown i}) \times O(u_{t^\frown j}) \subset D_{n+1}$ for all $s, t \in \{0,1\}^n$ and $i, j = 0, 1$.

For each $f \in \{0,1\}^\omega$ let a_f be the unique member of $\bigcap_{n=0}^\infty O(u_{f\restriction n})$. The set $\{a_f : f \in \{0,1\}^\omega\}$ is perfect, and if $f \neq g$ then $(a_f, a_g) \notin E$. □

We shall use a version of Lemma 32.2 for a different topology on $\mathcal{N}\times\mathcal{N}$. Toward the proof let us recall some basic facts about the property of Baire. In particular, Lemmas 11.16 and 11.17 as well as the fact that the σ-algebra of sets with the Baire property is closed under the Suslin operation $\mathcal{A}$, remain true in every second countable space (i.e., space that has a countable basis).

We shall prove Silver's Theorem for (lightface) Π^1_1 equivalence relations; the proof relativizes to $\Pi^1_1(a)$ for every real parameter a.

Definition 32.3. The Σ^1_1-*topology* on $\mathcal{N}$ is the topology with basic open sets being all the Σ_1 subsets of $\mathcal{N}$.

The Σ^1_1-topology has a countable base and is larger than the standard topology, as every basic open set $O(s)$ in $\mathcal{N}$ is Σ^0_1.

Lemma 32.4. *The Σ^1_1-topology satisfies the Baire Category Theorem.*

Proof. Exercise 32.3. □

Lemma 32.5. *If X is comeager in the Σ^1_1-topology then for every nonempty Σ^1_1 subset A of $\mathcal{N} \times \mathcal{N}$, $A \cap (X \times X) \neq \emptyset$.*

Proof. The lemma states that $X \times X$ is dense in the Σ^1_1-topology on $\mathcal{N} \times \mathcal{N}$ (which is larger than the product of the Σ^1_1-topology). If D is a dense open set in the Σ^1_1-topology then $D \times \mathcal{N}$ is dense open in the Σ^1_1-topology on $\mathcal{N}^2$: This is because if $A \neq \emptyset$ is a Σ^1_1 subset of $\mathcal{N}^2$ then its projection is Σ^1_1 and hence meets D.

Let $X \supset \bigcap_{n=0}^\infty D_n$ where each D_n is dense open. Then $X \times X \supset \bigcap_{n=0}^\infty (D_n \times \mathcal{N}) \cap \bigcap_{n=0}^\infty (\mathcal{N} \times D_n)$, and the latter set is dense, by the Baire Category Theorem applied to the Σ^1_1-topology on $\mathcal{N} \times \mathcal{N}$. □

Given a Π^1_1 equivalence relation E on $\mathcal{N}$, consider the set that is the complement of the union of all Σ^1_1 sets contained in some equivalence class:

(32.1) $H = \{a \in \mathcal{N} :$ for every Σ^1_1 set U, if $a \in U$ then there is a $b \in U$ with $(a, b) \notin E\}$.

Note that if H is empty then every equivalence class is the union of Σ^1_1 sets and therefore there are at most $\aleph_0$ equivalence classes. We shall prove that if $H \neq \emptyset$ then there exists a perfect set of inequivalent reals.

Lemma 32.6. *H is a Σ^1_1 set.*

Proof. First note that if an equivalence class A of E contains a nonempty Σ^1_1 set U then A is Π^1_1:

$$x \in A \leftrightarrow \forall y\, (y \in U \to x \mathrel{E} y).$$

Then by the separation principle there exists a Δ^1_1 set V such that $U \subset V \subset A$. It follows that

(32.2) $H = \{a :$ for every Δ^1_1 set U, if $a \in U$ then $\exists b \in U$ with $(a, b) \notin E\}$.

The quantification "for every Δ^1_1 set" in (32.2) can be replaced by "for every Borel code for a Δ^1_1 set" and since we are dealing only with lightface Δ^1_1 sets, this can be replaced by a number quantifier $\forall n$. Similarly, "$a \in U$" and "$b \in U$" are Δ^1_1 properties, and it follows that H is Σ^1_1. □

Lemma 32.7. *For every $a \in \mathcal{N}$, $E_a \cap H$ is meager in the Σ^1_1-topology, where $E_a = \{b : (a, b) \in E\}$.*

Proof. If $H = \emptyset$ then there is nothing to prove; thus assume $H \neq \emptyset$. The set E_a is Π^1_1 and therefore has the Baire property in the Σ^1_1-topology. If $E_a \cap H$ is not meager then there exists a nonempty Σ^1_1 set U such that $E_a \cap U$ is comeager in U. As $U \subset H$, $U \times U$ is not contained in E and so $U^2 - E$ is nonempty; hence we have (by Lemma 32.5) $(U^2 - E) \cap (E_a \cap U)^2 \neq \emptyset$. In other words there exist $b, c \in U$ such that $a\ E\ b$, $a\ E\ c$ and $(b, c) \notin E$, a contradiction. □

Lemma 32.8. *$E \cap (H \times H)$ is meager (in the product of the Σ^1_1-topology).*

Proof. By Lemma 32.7 and Lemma 11.16. □

Proof of Theorem 32.1. If H is empty then E has at most $\aleph_0$ equivalence classes. If $H \neq \emptyset$ then H is Σ^1_1 and therefore a basic open set in the Σ^1_1-topology. By Lemma 32.8 $E \cap (H \times H)$ is meager in the product of the Σ^1_1-topology. The rest of the proof (which we omit) is a combination of the construction in the proof of Lemma 32.2 and the construction in Exercise 32.3: One can produce a perfect set $\{a_f : f \in \{0,1\}^\omega\} \subset H^2$ such that $(a_f, a_g) \notin E$ whenever $f \neq g$. □

$\mathbf{\Sigma}^1_1$ Equivalence Relations

Theorem 32.9. *If E is a $\mathbf{\Sigma}^1_1$ equivalence relation on $\mathcal{N}$ then either E has at most $\aleph_1$ equivalence classes or there exists a perfect set of mutually inequivalent reals.*

This theorem, due to J. Burgess, extends Silver's Theorem and uses it in the proof. Note that Exercise 32.2 makes it best possible.

Proof. Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation. There exists a tree T on $(\omega \times \omega) \times \omega$ such that for all $a, b \in \mathcal{N}$

$$a\ E\ b \leftrightarrow T(a, b) \text{ is ill-founded.} \tag{32.3}$$

We define, for each $\alpha < \omega_1$, a relation E^α on $\mathcal{N}$ as follows:

$$a\ E^\alpha\ b \leftrightarrow \text{not } (\|T(a, b)\| < \alpha). \tag{32.4}$$

It is clear that each E^α is a Borel relation, $E^\alpha \supset E^\beta$ if $\alpha < \beta$, $E^\alpha = \bigcap_{\beta<\alpha} E^\beta$ if α is limit, and $E = \bigcap_{\alpha<\omega_1} E^\alpha$. Moreover, each E^α is reflexive as $E^\alpha \supset E$.

Lemma 32.10. *There is a closed unbounded set $C \subset \omega_1$ such that for each $\alpha \in C$, E^α is an equivalence relation.*

Proof. If $T(x,y)$ is well-founded then so is $T(y,x)$ (by the symmetry of E) and so for every $\alpha<\omega_1$ the set $\{T(y,x) : \|T(x,y)\|<\alpha\}$ is a set of well-founded trees. The set is $\mathbf{\Sigma}^1_1$ and so, by the Boundedness Lemma there is a countable ordinal $f(\alpha)$ such that $\|T(y,x)\|<f(\alpha)$ whenever $\|T(x,y)\|<\alpha$. Let γ be a closure point of f, i.e., if $\alpha<\gamma$ then $f(\alpha)<\gamma$. Let $a,b\in\mathcal{N}$. If $(b,a)\notin E^\gamma$, or $\|T(b,a)\|<\gamma$, then $\|T(a,b)\|<\gamma$, or $(a,b)\notin E^\gamma$ and so E^γ is symmetric.

Similarly, there is a function $g:\omega_1\to\omega_1$ such that if γ is a closure point of g then for all $a,b,c\in\mathcal{N}$, if $(a,c)\notin E^\gamma$ then either $(a,b)\notin E^\gamma$ or $(b,c)\notin E^\gamma$. Let C be the set of all closure points of both f and g. □

Now assume that E has more than $\aleph_1$ equivalence classes. We shall prove that there exists a perfect set of E-inequivalent reals.

Let $V[G]$ be a generic extension of V that collapses $\aleph_1$ and makes $\aleph_2^V = \aleph_1^{V[G]}$. Let $\tilde{E}$ denote the relation defined in $V[G]$ by (32.3), and for each $\alpha<\omega_1^{V[G]}$ let $\tilde{E}^\alpha$ be defined by (32.4). $\tilde{E}$ is $\mathbf{\Sigma}^1_1$, and each $\tilde{E}^\alpha$ is Borel. By absoluteness, $\tilde{E}\cap V = E$ and $\tilde{E}$ is an equivalence relation, $\tilde{E}^\alpha\cap V = E^\alpha$ for each $\alpha<\omega_1^V$, and if E^α is an equivalence relation then so is $\tilde{E}^\alpha$. Since $\tilde{E}^{\omega_1^V} = \bigcap_{\alpha<\omega_1^V}\tilde{E}^\alpha$, it is a Borel equivalence relation. We assume that E has, in V, a set X of size $\aleph_2$ of inequivalent reals. If $x,y\in X$ and $x,y\notin E$ then $(x,y)\notin E^\alpha$ for some $\alpha<\omega_1^V$. Hence X is a set of $\tilde{E}^{\omega_1^V}$-inequivalent reals, and X is uncountable in $V[G]$.

By Silver's Theorem, $\tilde{E}^{\omega_1^V}$ has a perfect set of inequivalent reals. These reals are $\tilde{E}$-inequivalent and so

(32.5) $\quad V[G]\vDash$ there is a perfect set of $\tilde{E}$-inequivalent reals.

However, the statement in (32.5) true in $V[G]$ is clearly $\mathbf{\Sigma}^1_2$ and so by Shoenfield's Absoluteness Theorem, it holds in V. Therefore in V, there exists a perfect set of E-inequivalent reals. □

Constructible Reals and Perfect Sets

We recall (Lemma 26.50) that if there exists a nonconstructible real then the set $\boldsymbol{R}\cap L$ is Lebesgue measurable only if it is null, and has the property of Baire only if it is meager. The following theorem proves a similar result for perfect sets.

Theorem 32.11. *If there exists a nonconstructible real then the set* $\boldsymbol{R}\cap L$ *does not have a perfect subset.*

Proof. As a first step we show that $\boldsymbol{R}\cap L$ does not have a *superperfect* subset. A tree $T\subset Seq$ is *superperfect* if for every $t\in T$ there exists an $s\supset t$ in T such that $s^\frown k\in T$ for infinitely many $k\in\omega$. (We call s an *ω-splitting node* of T.) A nonempty set $P\subset\mathcal{N}$ is *superperfect* if $P=[T]$ for some superperfect tree T.

Lemma 32.12. *If $\mathcal{N} \cap L$ has a superperfect subset then every real is constructible.*

Proof. Instead of $\mathcal{N}$, consider the space $[\omega]^\omega$ of increasing sequences of natural numbers. Let x, y, z be distinct elements of $[\omega]^\omega$ and let

$$(32.6) \quad O(x,y,z) = \{n \in \omega : z(n-1) \le x(n-1),\ z(n-1) \le y(n-1) \text{ and } z(n) > x(n),\ z(n) > y(n)\}.$$

If $O(x,y,z)$ is infinite, let $\langle n_k : k \in \omega\rangle$ be its increasing enumeration and let

$$(32.7) \qquad o(x,y,z) = \{k : x(n_k) \le y(n_k)\}.$$

Now assume that $P = [T]$ is a superperfect subset of $[\omega]^\omega$ such that every $x \in P$ is constructible. We shall prove that every real is constructible as follows: Let $A \subset \omega$ be arbitrary; we shall find $x, y, z \in [T]$ such that $o(x,y,z)$ (is defined and) is equal to A. Then A is constructible, as the definition (32.7) is absolute for L.

Thus let $A \subset \omega$ be arbitrary. We find $x, y, z \in [T]$ by constructing inductively their initial segments. We construct sequences $x_0 \subset x_1 \subset \ldots \subset x_k \subset \ldots$, $y_0 \subset y_1 \subset \ldots \subset y_k \subset \ldots$, and $z_0 \subset z_1 \subset \ldots \subset z_k \subset \ldots$ of ω-splitting nodes of T such that for each k, $n_k = |z_k|$ is the kth element of $O(x,y,z)$, and $k \in o(x,y,z)$ if and only if $k \in A$. Inductively, we arrange $l_k = |x_k| > n_k$ and $m_k = |y_k| > n_k$, as well as $z_k(n_k - 1) \le x_k(n_k - 1)$ and $z_k(n_k - 1) \le y_k(n_k - 1)$.

We omit the initial stage of the induction as it is similar to the induction step: At stage $k+1$ we find an integer i greater than $x_k(l_k - 1)$ and $y_k(m_k - 1)$ such that $z_k{}^\frown i \in T$. Then we let $z_{k+1} \supset z_k$ be an ω-splitting node above such that $n_{k+1} = |z_{k+1}|$ is greater than l_k and m_k. Now if $k+1 \in A$, let $j > z_{k+1}(n_{k+1} - 1)$ be such that $x_k{}^\frown j \in T$, and let $x_{k+1} \supset x_k{}^\frown j$ be an ω-splitting node such that $l_{k+1} = |x_{k+1}| > n_{k+1}$. Then let $h > x_{k+1}(l_{k+1} - 1)$ be such that $y_k{}^\frown h \in T$ and let $y_{k+1} \supset y_k{}^\frown h$ be an ω-splitting node such that $m_{k+1} = |y_{k+1}| \ge l_{k+1}$. If $k+1 \notin A$, we reverse the construction of x_{k+1} and y_{k+1}. Since x_{k+1}, y_{k+1}, and z_{k+1} are all increasing it follows that n_{k+1} is the least $n > n_k$ that belongs to $O(x,y,z)$, and the construction guarantees that $x(n_{k+1}) \le y(n_{k+1})$ if and only if $k+1 \in A$. □

Now we complete the proof of the theorem. If $\boldsymbol{R} \cap L$ is countable then the theorem is true trivially, so assume that $\aleph_1^L = \aleph_1$. If X is a countable subset of $\boldsymbol{R} \cap L$, then given a constructible enumeration $\langle a_\alpha : \alpha < \omega_1\rangle$ of $\boldsymbol{R} \cap L$, we have $X \subset \{a_\alpha : \alpha < \gamma\}$ for some $\gamma < \omega_1$, and so there exists a constructible $Y \subset \boldsymbol{R} \cap L$ such that $X \subset Y$ and $|Y|^L = \aleph_0$.

Let P be a perfect subset of the Cantor space and assume that $P \subset \{0,1\}^\omega \cap L$. Applying the preceding argument to a countable dense subset $X \subset P$, we obtain a constructible countable set $D \in L$ that is a dense subset of $\boldsymbol{C} = \{0,1\}^\omega$ and that $D \cap P$ is dense in P. Let $\boldsymbol{C} - D = \mathcal{X}$.

The space $\mathcal{X}$ is homeomorphic to the irrationals which in turn is homeomorphic to $\mathcal{N}$ and $\mathcal{N}$ is homeomorphic to $[\omega]^\omega$. Thus there exists a homeomorphism h between $\mathcal{X}$ and $[\omega]^\omega$; moreover h is coded in L because $D \in L$. The set $P - D$, a closed subset of $\mathcal{X}$, contains no compact subset with nonempty interior, and therefore the set $h(P - D)$ has the same property in $[\omega]^\omega$; it follows that $h(P - D)$ is superperfect. Hence $h(P - D)$ is a superperfect subset of $[\omega]^\omega \cap L$, contradicting Lemma 32.12. □

Projective Sets and Large Cardinals

One of the successes of modern set theory has been the discovery of the close relationship between the hierarchy of definable sets of reals and the hierarchy of large cardinals. We shall elaborate on this relationship in subsequent chapters. In the present section we apply the large cardinal theory to $\mathbf{\Sigma}^1_3$ sets.

By Theorem 25.38, the perfect set property for $\mathbf{\Sigma}^1_2$ sets is equivalent to the large cardinal assumption

$$\aleph_1 \text{ is inaccessible in } L[a], \text{ for every } a \in \boldsymbol{R} \tag{32.8}$$

(see Exercise 32.4). The statement (32.8) also implies that every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable and has the Baire property.

By Solovay's Theorem 26.14, inaccessibility is sufficient for the consistency of Lebesgue measurability and the Baire property of all projective sets. The following theorem shows that the assumption is necessary for Lebesgue measurability, while by another result of Shelah, the Baire property for all projective sets is consistent relative to ZFC only:

Theorem 32.13 (Shelah [1984]). *If every $\mathbf{\Sigma}^1_3$ set of reals is Lebesgue measurable then $\aleph_1$ is an inaccessible cardinal in L.* □

We shall outline a result that shows that under a suitable strengthening of (32.8), every $\mathbf{\Sigma}^1_3$ set is Lebesgue measurable, has the Baire property, and has the perfect set property. The key is a tree representation of $\mathbf{\Sigma}^1_3$ sets in the presence of a measurable cardinal.

Theorem 32.14 (Martin and Solovay [1969], Mansfield [1971]). *If there exists a measurable cardinal then for every $\mathbf{\Sigma}^1_3$ set A there exists a tree T on $\omega \times \lambda$ (for some λ) such that $A = p[T]$.*

Proof. Let κ be a measurable cardinal and let U be a normal measure on κ. For each n, let U_n be the ultrafilter $\{X \subset \kappa^n : X \supset [Z]^n \text{ for some } Z \in U\}$, and let $j_n = i_{n,n+1}$ be the canonical elementary embedding $i_{n,n+1} : \mathrm{Ult}_{U_n}(V) \to \mathrm{Ult}_{U_{n+1}}(V)$.

Let $A \subset \mathcal{N}$ be a Σ^1_3 set. A can be expressed as

$$x \in A \leftrightarrow \exists y \forall z\, R(x,y,z) \text{ is ill-founded} \tag{32.9}$$

where R is a recursive function, $R(x,y,z)$ is, for each x, y, z, a linear order of ω and $R(x,y,z)$ restricted to $n = \{0,\ldots,n-1\}$ depends only on $x{\restriction}n$, $y{\restriction}n$, $z{\restriction}n$. Let $\pi : Seq_3 \to \omega$ be defined so that $\pi(x{\restriction}n, y{\restriction}n, z{\restriction}n)$ is the position of $n-1$ in the order $R(x,y,z){\restriction}n$. Let $\{s_k\}_{k=0}^{\infty}$ be an enumeration of Seq. We let $\alpha = i_{0,\omega}(\kappa)$, and define

$$(x{\restriction}n, y{\restriction}n, \langle\beta_0,\ldots,\beta_{n-1}\rangle) \in T \leftrightarrow j_{\pi(x\restriction l, y\restriction l, s_k)}(\beta_k) > \beta_i \tag{32.10}$$

for every $i = 0, \ldots, n-2$, where $l = \text{length}(s_i)$ and $s_k = s_i{\restriction}l$.

We leave to the reader to verify that $x \in A$ if and only if $T(x)$, a tree on $\omega \times \alpha$, is ill-founded. For details we refer to Kanamori's book [1994], Chapter 15. □

A careful analysis of the tree representation in Theorem 32.14 shows that the assumption can be weakened to

$$\text{for every } a \in \boldsymbol{R},\ a^\sharp \text{ exists} \tag{32.11}$$

and the tree T can be constructed on $\omega \times \omega_2$ (see Kanamori [1994] for details). Thus one obtains:

Theorem 32.15 (Martin). *If for every $a \in \boldsymbol{R}$, $a^\sharp$ exists, then every $\boldsymbol{\Sigma}^1_3$ set is ω_2-Suslin, and hence a union of $\aleph_2$ Borel sets.* □

The following theorem establishes good behavior of $\boldsymbol{\Sigma}^1_3$ sets under a large cardinal assumption:

Theorem 32.16 (Magidor [1980]). *Let us assume that there exists a measurable cardinal, and that ω_1 carries a precipitous ideal. Then every $\boldsymbol{\Sigma}^1_3$ set is Lebesgue measurable, has the Baire property, and is either countable or contains a perfect subset.*

Proof. Let A be a $\boldsymbol{\Sigma}^1_3$ set and let $A = p[T]$ where T is the tree defined in the proof of Theorem 32.14. We shall prove that under the given assumptions,

$$\boldsymbol{R} \cap L[T] \text{ is countable.} \tag{32.12}$$

Then the statements on Lebesgue measurability and the Baire property can be derived as the corresponding result (Theorem 26.20 and Corollary 26.21) for $\boldsymbol{\Sigma}^1_2$ sets: Using absoluteness, one can show that

$$A = \{x \in \boldsymbol{R} : L[T] \vDash \varphi(x)\}$$

for some formula φ, and apply Corollary 26.6. The perfect set property is derived by using Lemma 25.24.

Let I be a precipitous ideal on ω_1, and let $M = \text{Ult}_G(V)$ be the generic ultrapower by I, that is by a generic ultrafilter G obtained by forcing P consisting of I-positive sets. As I is precipitous, M is well-founded and we identify it with a transitive class $M \subset V[G]$. Let $i : V \to M$ be the corresponding elementary embedding. We shall prove

$$i(T) = T. \tag{32.13}$$

This will suffice, as (32.13) implies (32.12), as follows: Assume that a_ξ, $\xi < \omega_1$, are uncountably many (distinct) reals in $L[T]$. The function $\langle a_\xi : \xi < \omega_1 \rangle$ represents a real $a \in \text{Ult}_G$, and since each $a_\xi \in L[T]$, we have $a \in L[i(T)] = L[T]$; hence $a \in V$. But then $i(a) = a$, and so $a = a_\xi$ for G-almost all ξ. This is a contradiction since G is nonprincipal.

Toward the proof of (32.13), let κ, U, U_n, and j_n be as in the proof of Theorem 32.14.

If γ is an inaccessible cardinal then γ is still inaccessible in $V[G]$ and it follows that $i(\gamma) = \gamma$. In particular $i(\kappa) = \kappa$. Let $\overline{U}$ be the filter in $V[G]$ generated by U; similarly $\overline{U_n}$.

Lemma 32.17. $i(U) = \overline{U} \cap M$, $i(U_n) = \overline{U_n} \cap M$.

Proof. It suffices to show that $i(U) \subset \overline{U} \cap M$; if $X \in i(U)$ we want a $W \in U$ such that $X \supset W$. X is represented by $\langle X_\xi : \xi < \omega_1 \rangle$, so let $Y = \bigcap_{\xi < \omega_1} X_\xi$; we have $Y \in U$ and $i(Y) \subset X$. Now if $W = \{\gamma \in Y : \gamma \text{ is inaccessible}\}$ we have $W \in U$ and $W = i``Y \subset i(Y) \subset X$. □

Lemma 32.18. *Let $h \in V[G]$ be a function $h : \kappa \to V$. Then there exists a function $H \in V$ such that $h(\alpha) = H(\alpha)$ a.e.* mod U. *Similarly for $h : \kappa^n \to V$ (and U_n).*

Proof. For each $\alpha < \kappa$ there is a maximal antichain W_α in P and a set $\{x_p^\alpha : p \in W_\alpha\}$ such that $p \Vdash \dot{h}(\alpha) = x_p^\alpha$. Let W be such that $W_\alpha = W$ for U-almost all α, and let p be the unique $p \in G \cap W$. Now let $H(\alpha) = x_p^\alpha$, for all $\alpha < \kappa$. □

Lemma 32.19. *Let $f \in V$ be a function $f : \kappa \to Ord$. Then there exists a function $g \in M$ such that $f(\alpha) = g(\alpha)$ a.e.* mod U. *Similarly for $f : \kappa^n \to Ord$.*

Proof. Every ordinal β is represented in M by some $h_\beta : \omega_1 \to Ord$, $h_\beta \in V$. For each $\alpha < \kappa$, pick (in $V[G]$) some $h_{f(\alpha)} : \omega_1 \to Ord$ that represents $f(\alpha)$ and let $h(\alpha) = h_{f(\alpha)}$. By Lemma 32.18 there is some $H \in V$ such that $H(\alpha) = h_{f(\alpha)}$ a.e.; let $A \in U$ be a set of inaccessibles such that $H(\alpha) = h_{f(\alpha)}$ for all $\alpha \in A$.

For each $\xi < \omega_1$, let g_ξ (a function on κ, in V) be defined by $g_\xi(\alpha) = (H(\alpha))(\xi)$, and let $G(\xi) = g_\xi$. G is in V and represents in M some function g.

For each $\alpha \in A$, $i(\alpha) = \alpha$, and $g(\alpha)$ is represented by the function that sends ξ to $(H(\alpha))(\xi)$, but since $(H(\alpha))(\xi) = h_{f(\alpha)}(\xi)$ for all ξ, $g(\alpha)$ is represented by $h_{f(\alpha)}$. It follows that $g(\alpha) = f(\alpha)$. □

As a consequence of Lemmas 32.17, 32.18, and 32.19, if a function $f \in V$ represents an ordinal in $\mathrm{Ult}_{U_n}(V)$ then there is an U_n-equivalent function $g \in M$ that represents the same ordinal in $\mathrm{Ult}_{i(U_n)}(M)$. Consequently,

$$(i(j_n))(\alpha) = j_n(\alpha) \text{ for all } \alpha \tag{32.14}$$

(where $i(j_n) = \bigcup_{\gamma \in Ord} i(j \restriction V_\gamma)$). Using (32.14) and the definition of T, one can now verify that $i(T) = T$. □

The existence of a measurable cardinal alone is not sufficient in Theorem 32.16. If $V = L[U]$ then there exists an uncountable Σ^1_3 set that is not Lebesgue measurable, does not have the Baire property, and does not contain a perfect subset:

Theorem 32.20 (Silver [1971a]). *$\boldsymbol{R} \cap L[U]$ is a Σ^1_3 set. The ordering $<_{L[U]}$ of $\boldsymbol{R}$ is a Σ^1_3 relation.* □

Also, the analog of Lemma 25.27 holds for $<_{L[U]}$, and the arguments for Σ^1_2 sets and L can be adopted for Σ^1_3 and $L[U]$.

Universally Baire sets

Definition 32.21. A set $A \subset \boldsymbol{R}$ is *universally Baire* if for any compact Hausdorff space X and any continuous function $f : X \to \boldsymbol{R}$, the set $f_{-1}(A)$ has the property of Baire in X.

The set of all universally Baire sets is a σ-algebra and is closed under operation $\mathcal{A}$. Thus every $\boldsymbol{\Sigma}^1_1$ set is universally Baire. We show below that every universally Baire set is Lebesgue measurable and that the statement that every $\boldsymbol{\Delta}^1_2$ set is universally Baire has consistency strength between inaccessible and Mahlo cardinals. The assumption that every projective (or even every $\boldsymbol{\Sigma}^1_2$ set) is universally Baire is considerably stronger; we refer to Feng, Magidor and Woodin [1992] for details.

Theorem 32.22. *A set $A \subset \boldsymbol{R}$ is universally Baire if and only if for every notion of forcing P there exist trees T and S on $\omega \times \lambda$ (where $\lambda = 2^{|P|}$) such that*

$$A = p[T], \qquad \boldsymbol{R} - A = p[S] \tag{32.15}$$

and for every generic filter G on P,

$$V[G] \vDash p[T] \cup p[S] = \boldsymbol{R} \text{ and } p[T] \cap p[S] = \emptyset. \tag{32.16}$$ □

We omit the proof of this equivalence. We remark that in the definition the space X can be replaced by the generalized Cantor space λ^ω (for all λ), and in the theorem, the forcing notion P can be replaced by $\mathrm{Col}(\lambda) = \mathrm{Col}(\omega, \lambda)$, the collapse of λ with finite conditions.

Corollary 32.23. *Every universally Baire set is Lebesgue measurable.*

Proof. Let $A \subset \boldsymbol{R}$ be universally Baire. Let B be the measure algebra, and let T and S be trees on $\omega \times \lambda$ such that $A = p[T]$, $\boldsymbol{R} - A = p[S]$, and (32.16) holds for every generic ultrafilter G on B.

Let $\dot{a}$ be the canonical name for a random real and let B be a Borel set such that $\|\dot{a} \in p[T]\| = [B]$. We will show that $A \triangle B$ has measure 0, and thus A is measurable.

Let M be a countable elementary submodel of H_κ where H_κ is sufficiently large. We claim that for every random real x over M,

$$x \in p[T] \leftrightarrow x \in B. \tag{32.17}$$

If $x \in B$ then $B \Vdash \dot{a} \in p[T]$ and hence $M \vDash (B \Vdash \dot{a} \in p[T])$. Thus $M[x] \vDash x \in p[T]$, and so $x \in p[T]$. If $x \notin B$ then $-B \Vdash \dot{a} \in p[S]$ and $M[x] \vDash x \in p[S]$, and hence $x \notin p[T]$, proving (32.17).

Since M is countable, almost all reals are random over M, and therefore $A \triangle B$ is null. □

Theorem 32.24. *The following are equivalent*:

(i) *Every $\boldsymbol{\Delta}^1_2$ set is universally Baire.*
(ii) *V is $\boldsymbol{\Sigma}^1_3$-absolute with respect to every generic extension.*

The statement (ii) states precisely: If P is a forcing notion and $\varphi(x_1, \ldots, x_n)$ a Π^1_3 formula then for all reals $a_1, \ldots, a_n$,

$$\varphi(a_1, \ldots, a_n) \quad \text{if and only if} \quad \Vdash_P \varphi(a_1, \ldots, a_n).$$

Its consistency strength is between inaccessible and Mahlo: It is the existence of an inaccessible cardinal κ such that $V_\kappa \prec_{\Sigma_2} V$; see Exercises 32.6, 32.7, 32.8.

Proof. First assume $\boldsymbol{\Sigma}^1_3$-absoluteness for generic extensions and let A be a $\boldsymbol{\Delta}^1_2$ set. We have

$$x \in A \leftrightarrow \exists y\, \varphi(x, y) \leftrightarrow x \in p[T] \tag{32.18}$$

and

$$x \in A \leftrightarrow \exists y\, \psi(x, y) \leftrightarrow x \in p[S] \tag{32.19}$$

where φ and ψ are $\boldsymbol{\Pi}^1_1$ and T and S are trees on $\omega \times \omega_1$.

If $V[G]$ is a generic extension then the second equivalences in (32.18) and (32.19) hold in $V[G]$, by $\boldsymbol{\Sigma}^1_2$-absoluteness. Since $p[T] \cup p[S] = \boldsymbol{R}$ is a $\boldsymbol{\Pi}^1_3$ statement (namely $\forall x\, (\exists y\, \varphi \vee \exists y\, \psi)$), and $p[T] \cap p[S] = \emptyset$ is $\boldsymbol{\Pi}^1_2$, $\boldsymbol{\Sigma}^1_3$-absoluteness gives (32.16).

Now assume that every $\mathbf{\Delta}^1_2$ set is universally Baire and prove the generic $\mathbf{\Sigma}^1_3$-absoluteness. It is enough to prove it for $P = \text{Col}(\lambda)$, as every V^P embeds in $V^{\text{Col}(\lambda)}$ for sufficiently large λ.

Let φ be a $\mathbf{\Sigma}^1_1$ formula (with a parameter in $\boldsymbol{R}^V$) and assume that $V[G] \vDash \exists x \, \forall y \, \varphi(x, y)$. Let $\dot{x} \in V^{\text{Col}(\lambda)}$ be such that $\Vdash \forall y \, \varphi(\dot{x}, y)$. There is a function $f : \lambda^\omega \to \omega^\omega$ such that f is continuous on a comeager G_δ set such that for every generic collapse $G \in \lambda^\omega$, $f(G) = \dot{x}^G$.

Toward a contradiction, assume that $V \vDash \forall x \, \exists y \, \neg\varphi(x, y)$. By Kondô's Uniformization Theorem, there exists a $\mathbf{\Pi}^1_1$ function g such that $\forall x \, \neg\varphi(x, g(x))$. We claim that the function $g \circ f$ is continuous on a comeager set in λ^ω. For each $s \in Seq$, $g_{-1}(O(s))$ is a $\mathbf{\Delta}^1_2$ set, therefore universally Baire, and so there exists an open set D_s such that $B_s = D_s \bigtriangleup f_{-1}(g_{-1}(O(s)))$ is meager. Let $A = \lambda^\omega - \bigcup\{B_s : s \in Seq\}$; A is comeager and $g \circ f$ is continuous on A.

We may assume that $A = \bigcap_{n=0}^\infty D_n$, with each D_n dense open. For $x \in A$, let $F(x) = (f(x), g(f(x)))$; F is continuous on A.

Let T be a tree on $\omega \times \omega \times \omega$ such that $p[T]$ is the $\mathbf{\Sigma}^1_1$ set $\{(x, y) : \varphi(x, y)\}$. Since $V[G] \vDash \varphi(f(G), g(f(G)))$ and $G \in \bigcap_{n=0}^\infty D_n$, we have that $T(F(G))$ is ill-founded. In other words, $V[G]$ satisfies

$$\exists x \, T(F(x)) \text{ is ill-founded.} \tag{32.20}$$

The statement (32.20) can be expressed as "T^* is ill-founded" where T^* is the tree

$$\begin{aligned}(\sigma, s, t, u) \in T^* \leftrightarrow (s, t, u) \in T, \ &\text{length}(\sigma) = \text{length}(s) = n \\ &\text{and } \exists \tau \, O(\sigma^\frown\tau) \subset \textstyle\bigcap_{i \le n} D_i \text{ and} \\ &\quad F``(O(\sigma^\frown\tau) - \textstyle\bigcup_{n=0}^\infty(\lambda^\omega - D_n)) \subset O(s, t)\end{aligned}$$

(where $O(\sigma^\frown\tau)$ and $O(s, t)$ are basic open sets in λ^ω and $\omega^\omega \times \omega^\omega$).

By absoluteness, T^* is ill-founded in V, and so (32.20) holds in V. In other words, for some $x \in A$ we have $\varphi(f(x), g(f(x)))$, a contradiction. $\square$

Exercises

32.1. Let $A \subset \mathcal{N}$ be $\mathbf{\Sigma}^1_1$. The equivalence relation on $\mathcal{N}$ whose equivalence classes are the singletons $\{a\}$ where $a \in A$, and the complement of A, is $\mathbf{\Pi}^1_1$. If A is uncountable then it has a perfect subset.

32.2. The relation "$\|a\| = \|b\|$ or $a, b \notin \text{WO}$" is $\mathbf{\Sigma}^1_1$ and has $\aleph_1$ equivalence classes.

32.3. Let $\{D_n\}_{n=1}^\infty$ be dense open sets in the Σ^1_1-topology and let B be a nonempty Σ^1_1 set. Then $B \cap \bigcap_{n=1}^\infty D_n \neq \emptyset$.

[Let $B = p[T_0]$ for some recursive tree T on $\omega \times \omega$. By induction on n, construct recursive trees T_n, a finite sequence s_n of length n, finite sequences t^i_n $(0 \le i \le n)$ of length n such that $\emptyset = s_0 \subset \ldots \subset s_n$ and $t^i_0 \subset \ldots \subset t^i_n$, and $\bigcap_{i=0}^n \{x : x \supset s_n \text{ and } (\exists y \supset t^i_n)\,(x, y) \in [T_i]\} \neq \emptyset$, and for all $1 \le i \le n$, $\{x : x \supset s_n \text{ and } (\exists y \supset t^i_n)\,(x, y) \in [T_i]\} \subset D_i$.]

32.4. If $a \in \boldsymbol{R}$ and $\aleph_1$ is a successor cardinal in $L[a]$, then for some $b \in \boldsymbol{R}$, $\aleph_1^{L[a,b]} = \aleph_1$.

[If $\aleph_1 = (\kappa^+)^{L[a]}$, let $b \subset \omega$ code the countable ordinal κ.]

32.5. Every universally Baire set is Ramsey.

[Use Mathias forcing.]

32.6. $\boldsymbol{\Sigma}^1_3$-absoluteness for generic extensions implies that $\aleph_1$ is inaccessible in each $L[a]$, $a \in \boldsymbol{R}$.

["$\omega_1^{L[a]}$ is countable" is $\Sigma^1_3(a)$.]

32.7. Generic $\boldsymbol{\Sigma}^1_3$-absoluteness implies that $L_\kappa \prec_{\Sigma_2} L$ where $\kappa = \aleph_1$.

32.8. If $V_\kappa \prec_{\Sigma_2} V$ and κ is inaccessible, let P be the Lévy collapse bellow κ. Show that V^P satisfies the generic $\boldsymbol{\Sigma}^1_3$-absoluteness.

[If $\dot{Q} \in V^P$, and φ is Σ^1_3, then $V^P \vDash \varphi$ if and only if $V^{P*\dot{Q}} \vDash \varphi$.]

Historical Notes

Theorem 32.1 on $\boldsymbol{\Pi}^1_1$ equivalence relations is due to Silver [1980]. The present proof is Harrington's as presented in Kechris and Martin [1980]. Lemma 32.5 is due to Louveau. Silver's Theorem was extended to Theorem 32.9 by J. Burgess in [1978].

Theorem 32.11 is due to Groszek and Slaman [1998]. The proof presented here is from Veličković and Woodin [1998].

Theorem 32.13: Shelah [1984]. The tree representation of $\boldsymbol{\Sigma}^1_3$ sets is implicit in Martin and Solovay [1969] and described in Mansfield [1971]. Theorem 32.16 is due to Magidor [1980].

Universally Baire sets are investigated in Feng, Magidor, and Woodin [1992].