36. Forcing and Large Cardinals

In this chapter we continue to develop the techniques introduced in Chapter 21. We shall describe several applications of forcing that use various large cardinal assumptions.

Violating GCH at a Measurable Cardinal

By Silver's Theorem 21.4 it is consistent, relative to a supercompact cardinal, that GCH can fail at a measurable cardinal. This, combined with Prikry forcing, shows further that the Singular Cardinal Hypothesis is unprovable. The consistency strength of both statements has been proved to be exactly $o(\kappa) = \kappa^{++}$:

Theorem 36.1 (Gitik). *The following are equiconsistent*:

(i) *There exists a measurable cardinal κ such that $2^\kappa > \kappa^+$.*
(ii) *There exists a strong limit singular cardinal κ such that $2^\kappa > \kappa^+$.*
(iii) *There exists a measurable cardinal κ of Mitchell order κ^{++}.*

As proved in Chapter 21 (Corollary 21.13), the consistency of (ii) follows from the consistency of (i) by Prikry forcing. The necessity of (iii) for the consistency of "not SCH" was proved by Gitik, by a combination of the pcf theory and Mitchell's inner model for sequences of measures. We omit the proof.

As for the consistency of (i) using $o(\kappa) = \kappa^{++}$, this improvement of Silver's Theorem 21.4 is a combination of an intermediate forcing result of Woodin which we outline below, and an additional forcing argument of Gitik that we also omit. $\square$

Theorem 36.2 (Woodin). *Assume* GCH *and assume that there exists an elementary embedding $j : V \to M$ with critical point κ such that $M^\kappa \subset M$ and that there exists a function $f : \kappa \to \kappa$ with $j(f)(\kappa) = \kappa^{++}$. Then there is a generic extension in which κ is a measurable cardinal and $2^\kappa > \kappa^+$.*

The assumption of Theorem 36.2 is easily seen to follow from κ being $(\kappa + 2)$-strong (Exercise 36.1). By Gitik, the statement holds in some generic extension of the canonical inner model for $o(\kappa) = \kappa^{++}$.

Proof. We outline the proof, which follows loosely the proof of Silver's Theorem 21.4. However, since the assumption is considerably weaker than supercompactness, more delicate arguments are needed.

We may assume that $j = j_E$ and $M = \mathrm{Ult}_E$, where E is a (κ, κ^{++})-extender (Exercise 36.2). Let U be the ultrafilter $U = \{X \subset \kappa : \kappa \in j(X)\}$ and consider the commutative diagram

$$\begin{tikzcd} V \arrow[rr, "j"] \arrow[dr, "i"'] & & M \\ & N \arrow[ur, "k"'] & \end{tikzcd} \tag{36.1}$$

where $N = \mathrm{Ult}_U(V)$ and $i : V \to N$ is the corresponding elementary embedding. Let

$$\lambda = i(f)(\kappa) \quad \text{and} \quad \mu = \mathrm{crit}(k). \tag{36.2}$$

We have the following inequalities:

$$\kappa^+ = (\kappa^+)^N = (\kappa^+)^M < \mu \le \lambda < i(\kappa) < \kappa^{++} < j(\kappa) < \kappa^{+++}. \tag{36.3}$$

Moreover,

$$M = \{k(t)(a) : a \subset \kappa^{++} \text{ finite, } t \in N \text{ and } t : [\lambda]^{|a|} \to N\} \tag{36.4}$$

(see Exercise 36.3).

For cardinals α and β, let $\mathrm{Add}(\alpha, \beta)$ denote the notion of forcing that adds β subsets of α, cf. (15.3). The model for $2^\kappa > \kappa^+$ is constructed in two stages: The first stage is (as in Silver's proof) an iteration of length $\kappa + 1$, with Easton support. The final model is then obtained by a forcing extension of this model.

Let $P = P_{\kappa+1}$ be the Easton support iteration of $\dot{Q}_\alpha$, where for each $\alpha < \kappa$, $\dot{Q}_\alpha = \{1\}$ unless α is inaccessible and a closure point of the function f, in which case $\dot{Q}_\alpha = \mathrm{Add}(\alpha, f(\alpha))$ (in V^{P_α} where P_α is the αth iterate). For $\alpha = \kappa$, $\dot{Q}_\kappa = (\mathrm{Add}(\kappa, \kappa^{++}))^{V^{P_\kappa}}$. Let G be a generic filter on P; we have $V[G] = V[G_\kappa][H_\kappa]$ where G_κ is V-generic on P_κ and H_κ is $V[G_\kappa]$-generic on $Q_\kappa = \mathrm{Add}(\kappa, \kappa^{++})^{V[G_\kappa]}$.

We recall some of the facts established in the proof of Theorem 21.4: P_κ is κ-c.c. forcing notion of cardinality κ, κ remains inaccessible in $V[G]$, and $V[G]$ satisfies $2^\kappa = \kappa^{++}$. Since κ is in M a closure point of $j(f)$, $\dot{Q}_\kappa = (\dot{Q}_\kappa)^M$, and so $P = (j(P))_{\kappa+1}$.

As for $i : V \to N$, we use the fact that $N^\kappa \subset N$ and that P_κ is κ-c.c. to conclude (as in Lemma 21.9) that in $V[G_\kappa]$, $(N[G_\kappa])^\kappa \subset N[G_\kappa]$. Also, $(Q_\kappa)^{N[G_\kappa]} = \mathrm{Add}(\kappa, \lambda)^{V[G_\kappa]}$.

Using $k(P_\kappa) = P_\kappa$ and the fact that $k(p) = p$ for every $p \in P_\kappa$, we extend $k : N \to M$ to an embedding $k : N[G_\kappa] \to M[G_\kappa]$. Now consider the forcing Q_κ in $N[G_\kappa]$, and let

$$h_\kappa = k_{-1}(H_\kappa) = \{p \in \mathrm{Add}(\kappa,\lambda)^{V[G_\kappa]} : k(p) \in H_\kappa\}. \tag{36.5}$$

As $Q_\kappa^{N[G_\kappa]}$ has, in $N[G_\kappa]$, the κ^+-chain condition, and $\mathrm{crit}(k) > \kappa^+$, h_κ is $Q_\kappa^{N[G_\kappa]}$-generic over $N[G_\kappa]$ (see Exercise 36.4). Moreover, $G_\kappa * h_\kappa$ is V-generic on $(i(P))_{\kappa+1}$ and in $V[G_\kappa][h_\kappa]$, $(N[G_\kappa][h_\kappa])^\kappa \subset N[G_\kappa][h_\kappa]$ (Exercise 36.5). Note also that (because $\lambda < \kappa^{++}$), $V[G_\kappa][h_\kappa]$ satisfies $2^\kappa = \kappa^+$. It follows that, in $V[G_\kappa][h_\kappa]$, k can be extended to an embedding $k : N[G_\kappa][h_\kappa] \to M[G_\kappa][H_\kappa]$.

Let $\dot{R} \in N$ be the name for the iteration after stage $\kappa + 1$:

$$P_\kappa * \dot{Q}_\kappa^N * \dot{R} = i(P_\kappa) \tag{36.6}$$

and let $R \in N[G_\kappa][h_\kappa]$ be the interpretation of $\dot{R}$ by $G_\kappa * h_\kappa$. In $N[G_\kappa][h_\kappa]$, R is an $i(\kappa)$-c.c. forcing of cardinality $i(\kappa)$, and because the least α for which the αth iterate is nontrivial is above λ, R is λ-closed.

Using the fact that R is λ-closed and that the number of antichains of R in $N[G_\kappa][h_\kappa]$ is small in $V[G_\kappa][h_\kappa]$ we conclude that there exists in $V[G_\kappa][h_\kappa]$ an R-generic filter H over $N[G_\kappa][h_\kappa]$ (Exercise 36.6).

Now define $k(H)$ as follows (in $V[G_\kappa][h_\kappa]$):

$$k(H) = \{q \in k(R) : \exists p \in H\, k(p) \le q\}. \tag{36.7}$$

We claim that $k(H)$ is an $M[G_\kappa][h_\kappa]$-generic filter on $k(R)$. We omit the proof (but see Exercise 36.7).

As $p \in H$ implies $k(p) \in k(H)$ for every $p \in R$, k can be extended, in $V[G_\kappa][h_\kappa]$, to an embedding $k : N[G_\kappa][h_\kappa][H] \to M[G_\kappa][h_\kappa][k(H)]$. It follows that i and j can be extended (in $V[G]$), so that we have the following commutative diagram:

$$\begin{tikzcd} V[G_\kappa] \arrow[rr, "j"] \arrow[dr, "i"'] & & M[G_\kappa][h_\kappa][k(H)] \\ & N[G_\kappa][h_\kappa][H] \arrow[ur, "k"'] & \end{tikzcd} \tag{36.8}$$

Now we describe the second stage of the construction, namely a generic extension of $V[G] = V[G_\kappa * H_\kappa]$, and in this extension, an elementary embedding that extends j. We force over $V[G]$ with the partial order $Q = i(Q_\kappa)$. Since $i(Q_\kappa)$ is $<i(\kappa)$-closed in $N[G_\kappa][h_\kappa][H]$, it follows from Exercise 36.5 that Q is κ-closed in $V[G_\kappa][h_\kappa]$. However, as the model $V[G] = V[G_\kappa][H_\kappa]$ is a generic extension of $V[G_\kappa][h_\kappa]$ by a κ^+-c.c. forcing $\mathrm{Add}(\kappa, \lambda)$, Q is κ-distributive in $V[G]$ (Exercise 36.8).

We also claim that in $V[G]$, Q is κ^{++}-c.c. To prove the claim, let $\tilde{Q}$ be, in $V[G_\kappa]$, the full support product of κ copies of Q_κ. Note that $\tilde{Q}$ is κ^{++}-c.c. in $V[G_\kappa]$, and since $\tilde{Q} \simeq Q_\kappa \times \tilde{Q}$, $\tilde{Q}$ is κ^{++}-c.c. in $V[G_\kappa][H_\kappa] = V[G]$. Since conditions in $Q = i(Q_\kappa)$ have the form $i(g)(\kappa)$ where $g : \kappa \to Q_\kappa$, an antichain in Q yields an antichain in $\tilde{Q}$, proving the claim.

Hence forcing with Q preserves κ^+ and κ^{++}, and so $2^\kappa = \kappa^{++}$ holds in the extension. Let K be a Q-generic filter over $V[G]$. The final step is to find in $V[G][K]$ a generic $j(K)$ over $M[G][k(H)]$ such that the embedding j from (36.8) extends to an embedding $j : V[G][K] \to M[G][k(H)][j(K)]$. This step, which we omit, first applies k to K and produces a generic X such that j extends to $j : V[G] \to M[G][k(H)][X]$, and then applies j to K and produces a generic Y such that j extends to $j : V[G][K] \to M[G][k(H)][X][Y]$. Details can be found in Gitik's paper [1989].

As this final step is performed inside $V[G][K]$, it follows that in $V[G][K]$, κ is measurable. □

The Singular Cardinal Problem

By Corollary 21.13, the negation of the Singular Cardinal Hypothesis is consistent relative to large cardinals, and its consistency strength is determined by Theorem 36.1. These results belong to a wide area of theorems and conjectures known collectively as the *Singular Cardinal Problem.* Unlike the behaviour of the continuum function on regular cardinals, which by Easton's Theorem can be quite arbitrary, the values of the continuum function at singular cardinals are subject to three kinds of constraint:

(1) By Silver's Theorems 8.12 and 8.13, the value of 2^κ for a singular cardinal κ of uncountable cofinality depends on the continuum function below κ.
(2) The Galvin-Hajnal Theorem 24.1 and Shelah's results in the pcf theory give upper bounds for the value of 2^κ when κ is a strong limit singular cardinal such that $\kappa < \aleph_\kappa$.
(3) Jensen's Covering Theorem 18.30 and the subsequent theory of core models shows that the consistency of the failure of SCH requires large cardinal assumptions.

There is a large body of forcing constructions that, using large cardinals, yield models with various behaviour of the continuum function subject to the above mentioned constraints. There is, however, no comprehensive solution of the Singular Cardinal Problem analogous to Easton's Theorem.

Below we list some of the advances in this area:

Theorem 36.3 (Magidor [1977a], [1977b]).

(i) *If there exists a supercompact cardinal then there is a generic extension in which $2^{\aleph_n} < \aleph_\omega$ for all $n < \omega$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$.*

(ii) *If there exist $\kappa < \lambda$ with κ supercompact and λ huge then there exists a generic extension in which $2^{\aleph_n} = \aleph_{n+1}$ for all $n < \omega$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$.* □

Theorem 36.4 (Woodin, Gitik [1989]). *If there exists a measurable cardinal κ of Mitchell order κ^{++}, then there exists a generic extension in which* GCH *holds below $\aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$.* □

Theorem 36.5 (Magidor [1977a], Shelah [1983], Gitik [∞]). *Assume that there exists a supercompact cardinal.*

(i) *There is a generic extension in which* GCH *holds below $\aleph_\omega$, and $2^{\aleph_\omega} = \aleph_{\omega+\alpha+1}$, where α is any prescribed countable ordinal.*
(ii) *There is a generic extension in which $\aleph_{\omega_1}$ is strong limit and $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+\alpha+1}$ for any prescribed ordinal $\alpha < \omega_2$.*
(iii) *There is a generic extension in which* GCH *holds below the least fixed point of the aleph function $\kappa = \aleph_\kappa$ while 2^κ is any arbitrarily large prescribed successor cardinal.* □

Theorem 36.6 (Woodin, Cummings [1992]).

(i) *If there exists a supercompact cardinal, then there is a generic extension in which $2^\kappa = \kappa^{++}$ for each cardinal κ.*
(ii) *If there exists a strong cardinal, then there is a generic extension in which $2^\kappa = \kappa^+$ for each successor cardinal and $2^\kappa = \kappa^{++}$ for each limit cardinal.* □

There are additional results on the failure of SCH by Gitik, Shelah and others. The main open problem in this area is the following:

Problem 36.7. *Is it consistent that $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} > \aleph_{\omega_1}$?*

Compare this with Shelah's Theorem 24.33.

Violating SCH at $\aleph_\omega$

We shall now outline the Woodin-Gitik modification of Magidor's technique for getting a model in which $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} = \aleph_{\omega+2}$. First we describe the preparation forcing (which replaces Magidor's use of a supercompact cardinal):

Lemma 36.8. *Assume that there exists a measurable cardinal κ with $o(\kappa) = \kappa^{++}$.*

(i) *There is a model V that satisfies* GCH *and*

$$\exists j : V \to M,\ \mathrm{crit}(j) = \kappa,\ M^\kappa \subset M \text{ and } (\kappa^{++})^M = \kappa^{++}. \tag{36.9}$$

(ii) *There exists a model V with a measurable cardinal κ such that $2^\kappa = \kappa^{++}$ and a normal measure U on κ with $N = \mathrm{Ult}_U(V)$, and there exists a set $G \in V$ which is an N-generic filter on $\mathrm{Col}^N((\kappa^{+++})^N, {<}j_U(\kappa))$ (a Lévy collapse in N).*

Proof. (i) We outline how to get a model that satisfies (36.9). First one uses Gitik's forcing from [1989] to get a model of GCH that has a (κ, κ^{++})-extender E and a function $f : \kappa \to \kappa$ such that $j(f)(\kappa) = \kappa^{++}$ (the assumption of Theorem 36.2). Note that in M (where $j : V \to M$), κ^{++} is an inaccessible cardinal. To obtain (36.9), one uses the iteration of length κ, with Easton support, of Lévy collapses $\mathrm{Col}(\alpha^+, {<}f(\alpha))$, followed by the Lévy collapse $\mathrm{Col}(\kappa^+, {<}\kappa^{++})$. In this generic extension, the embedding $j : V \to M$ can be extended to an embedding that satisfies (36.9). (For details, see Gitik [1989].)

(ii) Starting with $j : V \to M$ that satisfies (36.9), we do Woodin's construction described in the proof of Theorem 36.2, except that by (36.9), we may assume that $f(\alpha) = \alpha^{++}$ for all α. In the resulting model, κ is measurable, $2^\kappa = \kappa^{++}$, and if U is the measure $\{X : \kappa \in j(X)\}$ given by the extended embedding j, then U has the required property. (Again, details are in Gitik's [1989].) $\square$

For the rest of this section we assume that κ is a measurable cardinal with $2^\kappa = \kappa^{++}$, U is a normal measure on κ, $N = \mathrm{Ult}_U(V)$ and $j = j_U : V \to N$, and $G \in V$ is an N-generic filter on $\mathrm{Col}^N((\kappa^{+++})^N, {<}j(\kappa))$, the Lévy collapse in N. Magidor's forcing conditions are as follows:

(36.10) A forcing condition has the form $p = (\kappa_0, f_0, \kappa_1, f_1, \ldots, \kappa_{n-1}, f_{n-1}, A, F)$ where

- (i) $\kappa_0 < \kappa_1 < \ldots < \kappa_{n-1}$ are inaccessible cardinals $< \kappa$,
- (ii) $f_i \in \mathrm{Col}(\kappa_i^{+++}, {<}\kappa_i)$, for $i < n-1$ and $f_{n-1} \in \mathrm{Col}(\kappa_{n-1}^{+++}, {<}\kappa)$,
- (iii) $A \in U$,
- (iv) F is a function on A and $F(\alpha) \in \mathrm{Col}(\alpha^{+++}, {<}\kappa)$ for all $\alpha \in A$,
- (v) $[F]_U$, the element of $\mathrm{Col}((\kappa^{+++})^N, {<}j(\kappa))$ represented by F, belongs to G.

(36.11) A condition $p' = (\kappa'_0, f'_0, \ldots, \kappa'_{m-1}, f'_{m-1}, A', F')$ is stronger than p if

- (i) $m \geq n$,
- (ii) $\kappa'_i = \kappa_i$ for all $i < n$ and $\kappa'_i \in A$ for all i, $n \leq i < m$,
- (iii) $f'_i \supset f_i$ for all $i < n$ and $f'_i \supset F(\kappa'_i)$ for all i, $n \leq i < m$,
- (iv) $A' \subset A$,
- (v) $F'(\alpha) \supset F(\alpha)$ for all $\alpha \in A'$.

This forcing produces a Prikry sequence $\langle \kappa_n : n < \omega \rangle$ cofinal in κ. A consequence of (36.10)(v) is that the forcing satisfies the κ^+-chain condition (Exercise 36.9) and so (if κ is preserved) $2^\kappa = \kappa^{++}$ in the generic extension. The crucial property of this forcing is that the cardinals κ_n are preserved, and $2^{\kappa_n} < \kappa_{n+1}$. Since all but finitely many cardinals between κ_n and κ_{n+1} are collapsed, there remain exactly ω cardinals between κ_0 and κ. Thus we can follow this generic extension by a Lévy collapse $\mathrm{Col}(\aleph_0, {<}\kappa_0)$ and in the resulting model we have $\kappa = \aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$ with $\aleph_\omega$ strong limit.

The key to preservation of the cardinals κ_n is an analog of Prikry's Lemma 21.12:

Lemma 36.9. *Let σ be a sentence of the forcing language and let $p = (\kappa_0, f_0, \ldots, \kappa_{n-1}, f_{n-1}, A, F)$ be a condition. Then there exists a stronger condition $p' = (\kappa_0, f_0', \ldots, \kappa_{n-1}, f_{n-1}', A', F')$ (with the same n) that satisfies σ.* $\square$

We omit the proof of Lemma 36.9 as well as its application and refer the reader to Magidor [1977a] and Gitik [1989] for the details.

Radin Forcing

As a consequence of Jensen's Covering Theorem, and more generally of the inner model theory, large cardinals are necessary for any nontrivial change of cofinality (with the exception of Namba forcing, Theorem 28.10). Prikry forcing is the prime example of forcing that changes cofinality. In this section we describe its generalizations.

The first example, due to Magidor, generalizes Prikry forcing to change the cofinality of a large cardinal κ to a given regular cardinal $\lambda < \kappa$ while preserving κ as a cardinal:

Let λ be a regular cardinal and let $\kappa > \lambda$ be a measurable cardinal such that $o(\kappa) = \lambda$. Using an inner model for $o(\kappa) = \lambda$, we may assume that there exists a sequence

$$U_0 < U_1 < \ldots < U_\alpha < \ldots \qquad (\alpha < \lambda) \tag{36.12}$$

of normal measures on κ, ordered by the Mitchell order. For every $\alpha < \beta < \lambda$, let $f_\alpha^\beta : \kappa \to V_\kappa$ be the function that represents U_α in Ult_{U_β}, i.e., $[f_\alpha^\beta]_{U_\beta} = U_\alpha$.

(36.13) A forcing condition is a pair (g, G) such that
- (i) g is an increasing function from a finite subset of λ into κ,
- (ii) G is a function on λ such that $G(\alpha) \subset \kappa$ for all $\alpha < \lambda$,
- (iii) if $\alpha > \max(\mathrm{dom}\, g)$ then $G(\alpha) \in U_\alpha$,
- (iv) if $\alpha < \max(\mathrm{dom}\, g)$ and β is the least $\beta \in \mathrm{dom}(g)$ above α, then $G(\alpha) \in f_\alpha^\beta(g(\beta))$.

The finite function g plays the role of the finite sequence in the Prikry forcing. The function G plays the role of the measure one set: Clause (iii) is an obvious generalization while (iv) states that $G(\alpha)$ has measure one with respect to the measure $f_\alpha^\beta(g(\beta))$ on $g(\beta)$, which reflects the properties of the measure U_α in the ultrapower by U_β.

(36.14) A condition (h, H) is stronger than (g, G) if
(i) $h \supset g$,
(ii) for every α, $H(\alpha) \subset G(\alpha)$,
(iii) for every $\alpha \in \operatorname{dom}(h) - \operatorname{dom}(g)$, $h(\alpha) \in G(\alpha)$.

Magidor's forcing (36.13) is a generalization of Prikry forcing. A generic filter yields a cofinal λ-sequence in κ (Exercise 36.10). Similarly to the Prikry forcing, (36.13) has a κ^+-chain condition (Exercise 36.11). The crucial property of Magidor's forcing is that it preserves cardinals. The proof is a generalization of the proof for the Prikry forcing and the following is the key lemma:

Lemma 36.10. *Let σ be a sentence of the forcing language and let (g, G) be a condition. Then there exists a stronger condition (g, H) (with the same g) that decides σ.* □

Magidor's forcing changes the cofinality of κ to $\operatorname{cf} \kappa = \lambda$, under the assumption that $o(\kappa) = \lambda$. If λ is uncountable then by Mitchell [1984] this assumption is necessary.

Radin forcing generalizes Prikry forcing further and uses objects called *measure sequences.* Let $j : V \to M$ be an elementary embedding with critical point κ. Let us define a sequence $\langle u(\alpha) : \alpha < \vartheta \rangle$ as follows:

$$\begin{aligned} u(0) &= \kappa, \\ u(\alpha) &= \{X \subset V_\kappa : u{\restriction}\alpha \in j(X)\} \qquad (\alpha > 0). \end{aligned} \tag{36.15}$$

The sequence $\langle u(\alpha) \rangle_\alpha$ is defined for all α for which $u{\restriction}\alpha \in M$. Thus the length ϑ depends on the strength of the embedding j. For example, if $j = j_U$ is the ultrapower embedding by a normal measure U on κ then $\lambda = 2$,

$$u(1) = \{X \in V_\kappa : \{\alpha : \langle \alpha \rangle \in X\} \in U\}$$

is a measure on V_κ concentrating on 1-sequences $\langle \alpha \rangle$, $\alpha < \kappa$, and $\langle u(0), u(1) \rangle \notin M = \mathrm{Ult}_U$. As long as $u(\alpha)$ is defined, $u(\alpha)$ is a measure on V_κ concentrating on α-sequences.

We define measure sequences as sequences obtained by (36.15) from elementary embeddings, but since we want the measures in measure sequences to concentrate on measure sequences, the definition is as follows.

Definition 36.11 (Measure Sequences). Let

$$\begin{aligned} \mathrm{MS}_0 &= \text{the class of all } u{\restriction}\alpha \text{ where } u \text{ is as in (36.15) for some} \\ &\qquad \text{elementary } j : V \to M, \\ \mathrm{MS}_{n+1} &= \{u \in \mathrm{MS}_n : (\forall \alpha > 0)\, \mathrm{MS}_n \cap V_{u(0)} \in u(\alpha)\}, \\ \mathrm{MS} &= \textstyle\bigcap_{n=0}^\infty \mathrm{MS}_n = \text{the class of all measure sequences.} \end{aligned}$$

Since all the measures are σ-complete it follows that for every measure sequence u and every $0 < \alpha < \text{length}(u)$, $u{\restriction}\alpha \in \text{MS}$ and $\text{MS} \cap V_{u(0)} \in u(\alpha)$. Clearly, some large cardinal assumption is necessary for the existence of nontrivial measure sequences. Under the assumption of a strong cardinal, there exist long measure sequences (Exercise 36.12).

Let U be a measure sequence of length at least 2, and let $\kappa = U(0)$. We associate with U the *Radin forcing* for U, R_U:

(36.16) A forcing condition $p \in R_U$ is a finite sequence

$$\langle (u_0, A_0), \ldots, (u_n, A_n) \rangle$$

such that $u_n = U$, and that, letting $\kappa_i = u_i(0)$ for $i = 0, \ldots, n$,

(i) for each $i = 0, \ldots, n$, $u_i \in \text{MS}$, $A_i \subset \text{MS}$ and $A_i \in u_i(\alpha)$ for every $0 < \alpha < \text{length}(u_i)$,

(ii) for each $i = 0, \ldots, n-1$, $(u_i, A_i) \in V_{\kappa_{i+1}}$.

(Thus $\kappa_0 < \kappa_1 < \ldots < \kappa_{n-1} < \kappa_n = \kappa$. These ordinals will produce a cofinal sequence in κ, a generalization of a Prikry sequence.)

(36.17) A forcing condition $p = \langle (u_0, A_0), \ldots, (u_n, A_n) \rangle$ is stronger than $q = \langle (v_0, B_0), \ldots, (v_m, B_m) \rangle$ if

(i) $n \geq m$,

(ii) $\{u_0, \ldots, u_n\} \supset \{v_0, \ldots, v_m\}$,

(iii) for each $j = 0, \ldots, m$, if $u_i = v_j$ then $A_i \subset B_j$,

(iv) for each i such that $u_i \notin \{v_0, \ldots, v_n\}$ if v_j is the first v_j such that $u_i(0) < v_j(0)$, then $u_i \in B_j$ and $A_i \subset B_j$.

If U is a measure sequence of length 2, then R_U is more or less the Prikry forcing (Exercise 36.13); if U has length 3, R_U produces a cofinal sequence of order type ω^2 (Exercise 36.14).

A generic filter G on R_U produces a set

(36.18) $D_G = \{u : \exists p \in G\ p = \langle (u_i, A_i) : i \leq n \rangle \text{ and } u = u_i \text{ for some } i < n\}$.

As in Prikry forcing, one proves that $V[D_G] = V[G]$. Let

(36.19) $$C_G = \{u(0) : u \in D_G\}.$$

It is not difficult to show:

Lemma 36.12. *C_G is a closed unbounded subset of κ.*

Proof. Exercise 36.15. □

When $\text{length}(U) < \kappa$, Radin forcing is similar to Magidor's forcing (36.13), see Exercise 36.16.

The analog of Prikry's Lemma 21.12 holds for Radin's forcing as well:

Lemma 36.13. *Let σ be a sentence of the forcing language and let $p = \langle (u_i, A_i) : i \leq n\rangle$ be a condition. Then there exists a stronger condition $q = \langle (u_i, B_i) : i \leq n\rangle$ (with the same $\{u_i : i \leq n\}$) that decides σ.* □

As a consequence, all cardinals are preserved in the generic extension.

Radin's forcing is more flexible than Magidor's forcing, and under suitable large cardinal assumptions, κ retains its regularity, or even its large cardinal properties:

Lemma 36.14. *If* $\mathrm{cf}(\mathrm{length}(U)) > \kappa$ *then κ remains regular in the forcing extension by R_U.* □

Lemma 36.15. *Let $j : V \to M$ and let $U \in \mathrm{MS}$ be defined from j as in* (36.15).

(i) *If j witnesses that κ is $(\kappa + 2)$-strong then κ remains measurable in V^{R_U}.*

(ii) *If j witnesses that κ is λ-supercompact then κ remains λ-supercompact.* □

Among applications of Radin forcing is the following theorem:

Theorem 36.16 (Mitchell). *If $\exists \kappa\, o(\kappa) = \kappa^{++}$ is consistent then so is* $\mathrm{ZF} + \mathrm{DC} +$ "*the closed unbounded forcing on $\aleph_1$ is an ultrafilter.*" □

Stationary Tower Forcing

We now describe the general version of the stationary tower forcing (Definition 34.10) which can be used, among others, to change cofinalities in a way that is not possible without very large cardinals.

Let A be an uncountable set. A set $S \subset P(A)$ is *stationary* in $P(A)$ if for every $F : [A]^{<\omega} \to A$, S contains a closure point of F, i.e., a set $X \subset A$ such that $F(e) \in X$ for all $e \in [X]^{<\omega}$. As in Theorem 8.27, projections and liftings of stationary sets are stationary. Also, the analog of Theorem 8.24 holds. For the relation to stationary sets in $P_\kappa(\lambda)$ see Exercise 36.17.

Definition 36.17 (Stationary Tower Forcing). Let δ be a Woodin cardinal. The forcing notion $P = P_{<\delta}$ consists of conditions (A, S) where $A \in V_\delta$ is uncountable and S is stationary in $P(A)$. (B, T) is stronger than (A, S) if $B \supset A$ and $T{\restriction}A \subset S$.

If G is a generic filter on $P_{<\delta}$ then we form the generic ultrapower $\mathrm{Ult}_G(V)$ as in (34.10). The general form of Theorem 34.14 is as follows:

Theorem 36.18 (Woodin [1988]). *Let δ be a Woodin cardinal. If G is generic on $P_{<\delta}$ then the generic ultrapower* $\mathrm{Ult}_G(V)$ *is well-founded and the model* $\mathrm{Ult}_G(V)$ *is closed under $< \delta$-sequences.* □

Forcing with $P_{<\delta}$ gives more flexibility than forcing with $Q_{<\delta}$ (from Definition 34.10). In a typical application, one can collapse a successor of a singular cardinal and give it any prescribed cofinality, see Example 36.19 below. In fact, the consistency strength of this is exactly that of a Woodin cardinal.

Example 36.19 (Woodin). Assume that $\aleph_\omega$ be strong limit, and let S be the following stationary set in $P(V_{\aleph_{\omega+1}})$:

$$S = \{X \in [V_{\aleph_{\omega+1}}]^{\aleph_\omega} : X \cap \aleph_{\omega+1} \in \aleph_{\omega+1} \text{ and } \operatorname{cf}(X \cap \aleph_{\omega+1}) = \aleph_3\}.$$

Let G be a generic filter on $P_{<\delta}$ such that $(V_{\aleph_{\omega+1}}, S) \in G$, let $M = \operatorname{Ult}_G(V)$ and let $j : V \to M$ be the generic ultrapower embedding. Then $\operatorname{crit}(j) = \aleph_{\omega+1}$ and $\operatorname{cf}^M \aleph_{\omega+1} = \aleph_3$ (Exercise 36.20). As $P^{V[G]}(\omega_n) = P^M(\omega_n) = P(\omega_n)$ for all n, the forcing $P_{<\delta}$ below $(V_{\aleph_{\omega+1}}, S)$ changes the cofinality of $\aleph_{\omega+1}$ to $\aleph_3$ while preserving $\aleph_\omega$. $\square$

Exercises

36.1. Assume that κ is $(\kappa+2)$-strong, and $j : V \to M$ with critical point κ be such that $V_{\kappa+2} \subset M$. Then the function $f(\alpha) = \alpha^{++}$ $(\alpha < \kappa)$ satisfies $j(f)(\kappa) = \kappa^{++}$.

36.2. Let $j : V \to M$ and $f : \kappa \to \kappa$ be as in Theorem 36.2, and let E be the (κ, κ^{++})-extender derived from j. Then $j_E(f)(\kappa) = \kappa^{++}$ and $(\operatorname{Ult}_E)^\kappa \subset \operatorname{Ult}_E$.

36.3. Prove (36.4).

[Use the fact that $j = j_E$.]

36.4. The filter h_κ is $Q_\kappa^{N[G_\kappa]}$-generic over $N[G_\kappa]$.

[Use the $\operatorname{crit}(k)$-chain condition.]

36.5. The filter $G_\kappa * h_\kappa$ is $(i(P))_{\kappa+1}$-generic over V, and in $V[G_\kappa * h_\kappa]$,

$$(N[G_\kappa * h_\kappa])^\kappa \subset N[G_\kappa * h_\kappa].$$

[Use that $(i(P))_{\kappa+1}$ is κ^+-c.c.]

36.6. In $V[G_\kappa * h_\kappa]$ there exists an $N[G_\kappa * h_\kappa]$-generic filter on R.

[Use the fact that the number of antichains to meet is small, to build R.]

36.7. $k(H)$ is generic over $M[G_\kappa][h_\kappa]$.

[Use (36.4), or rather the corresponding description of $M[G_\kappa][h_\kappa]$. If D is an open dense set in $k(R)$, let $D = k(t)(a)$, where for each $x \in [\lambda]^{|a|}$, $t(x)$ is an open dense subset of R. Then use the fact that $\bigcap_x t(x)$ is open dense to show that $k(H)$ meets D.]

36.8. If Q is κ-closed then it remains κ-distributive in every κ^+-c.c. forcing extension.

[Let P be κ^+-c.c. Show that $\Vdash_Q P$ is κ^+-c.c., that the generics for P and Q are mutually generic, and that V^P and $V^{P*Q} = V^{Q*P}$ have the same κ-sequences of ordinals.]

36.9. The forcing (36.10) satisfies the κ^+-chain condition.
[Use (iii) and (v).]

36.10. If G is generic for the Magidor forcing (36.13) then in $V[G]$, κ has a cofinal subset of order-type λ.

36.11. The forcing (36.13) has the κ^+-chain condition.

36.12. Let κ be a $(\kappa+2)$-strong cardinal. Then there exists a measure sequence u of length $\vartheta \geq \kappa^{++}$.

36.13. Let $U \in \mathrm{MS}$ have length 2, $U = \langle \kappa, u(1) \rangle$. A condition $p \in R_U$ has the form $\langle (\alpha_0, \emptyset), \ldots, (\alpha_{n-1}, \emptyset), (u(1), A) \rangle$. Compare with the Prikry forcing.

36.14. Analyze R_U when $\mathrm{length}(U) = 3$.

36.15. Prove that C_G is a closed unbounded subset of κ.

36.16. Assume that $\mathrm{length}(U) = \lambda$ is a regular uncountable cardinal, $\lambda < \kappa$. Show that the order-type of C_G is λ.

36.17. Stationary sets in $P_\kappa(\lambda)$ are exactly the sets of the form $S \restriction \{X \in P_\kappa(\lambda) : X \cap \kappa \in \kappa\}$ where S is stationary in $P(\lambda)$.

36.18. Let $M = \mathrm{Ult}_G(V)$ where G is generic on $P_{<\delta}$, and let $j : V \to M$ be the generic ultrapower embedding. For each $(A, S) \in P_{<\delta}$, $(A, S) \Vdash j``A \in j(S)$.

36.19. Each $\alpha \leq \delta$ is represented in $\mathrm{Ult}_G(V)$ by the function $f_\alpha(x) = x \cap \alpha$.

36.20. In Example 36.19, $\mathrm{crit}(j) = \aleph_{\omega+1}$ and $\mathrm{cf}^M \aleph_{\omega+1} = \aleph_3$.

Historical Notes

Following the (unpublished) Theorem 36.2 of Woodin, Gitik proved in [1989] that $o(\kappa) = \kappa^{++}$ suffices for the consistency of a measurable cardinal κ with $2^\kappa = \kappa^{++}$, as well as for a model of "$\aleph_\omega$ is strong limit and $2^{\aleph_\omega} = \aleph_{\omega+2}$." In [1991], Gitik showed that the assumption $o(\kappa) = \kappa^{++}$ is necessary for the negation of SCH.

Methods for violating GCH at $\aleph_\omega$ were originated by Magidor in [1977a, 1977b]. Woodin (unpublished) improved the method by using a $(\kappa+2)$-strong cardinal, and Gitik [1989] obtained the result from $o(\kappa) = \kappa^{++}$.

In [1983], Shelah improved Magidor's method in the direction of getting an arbitrary countable gap between $\aleph_\omega$ and $2^{\aleph_\omega}$. In [1992], Gitik and Magidor introduced a novel method for blowing up the power of 2^κ for singular cardinals, leading to results that give the precise consistency strength (e.g., the large cardinal assumptions for Theorem 36.5 are considerably weaker than supercompactness).

In [1991], Foreman and Woodin constructed a model in which GCH fails everywhere; this was then improved by Woodin to Theorem 36.6(i), and Cummings followed with 36.6(ii).

Magidor's forcing for changing cofinality appeared in [1978]. For Radin's forcing, see Radin [1982]. Our presentation is based on improvements by Mitchell [1982], Woodin (unpublished) and Cummings [1992]. Theorem 36.16 is due to Mitchell [1982].

Stationary tower forcing is due to Woodin.