Mastermath Exam Set Theory
04-06-2013; 14:00–17:00.

This exam consists of multiple-choice questions, 1–12, and open questions, 13–17.
Record your answers to the multiple-choice questions in a readable table on the exam paper.

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(2) 1. Given our definitions of ordered pairs ($\langle x, y\rangle = \{\{x\}, \{x, y\}\}$) and natural numbers ($n = \{0, \ldots, n-1\}$), which of the following is true:

A. $\langle 0, 1\rangle \subseteq 2$

B. $2 \subseteq \langle 0, 1\rangle$

C. $\langle 0, 1\rangle = \{1, 2\}$

D. $0 \in \langle 0, 1\rangle$

(2) 2. Consider the structure $\langle \mathbb{N}, <\rangle$, for the language of set theory. Which of the following axioms of ZF holds in this structure.

A. Pairing

B. Separation

C. Union

D. Infinity

(2) 3. Which of the following is not a filter on ω_1:

A. $\{A \subseteq \omega_1 : \omega_1 \setminus A \text{ is finite}\}$

B. $\{A \subseteq \omega_1 : \omega_1 \setminus A \text{ is countable}\}$

C. $\{A \subseteq \omega_1 : \omega_1 \setminus A \text{ is not stationary}\}$

D. $\{A \subseteq \omega_1 : A \text{ is cub}\}$

(2) 4. Assume ZFC. The set V_{ω_1}, viewed as a structure for the language of Set Theory, does *not* satisfy which axiom:

A. Choice

B. Replacement

C. Separation

D. Pairing

(2) 5. Let κ be a regular and uncountable cardinal; which statement about subsets of κ is true:

A. Every unbounded set is stationary

B. Every stationary set is unbounded

C. Every stationary set is cub

D. Every bounded set is stationary

(2) 6. Which of the following *ordinal* inequalities does not hold:

A. $2^\omega < 3^\omega$

B. $\omega^2 < \omega^3$

C. $\omega \cdot 2 < \omega \cdot 3$

D. $\omega + 2 < \omega + 3$

More problems on the next page.

(2) 7. Which of the following *cardinal* inequalities does hold:
A. $2^{\aleph_0} < 3^{\aleph_0}$

B. $\aleph_0^2 < \aleph_0^3$

C. $\aleph_0 \cdot 2 < \aleph_0 \cdot 3$

D. $2^{\aleph_0} < 2^{2^{\aleph_0}}$

(2) 8. Let M be a countable elementary substructure of $H(\aleph_{25})$, with respect to $\in$. Which of the following is not provable about M:
A. $\omega_{20} \in M$

B. $\varepsilon_0 \subseteq M$

C. $\mathcal{P}(\omega) \in M$

D. $\omega_1 \times \omega_1 \in M$

(2) 9. Which of the following partition relations is provable in ZFC:
A. $\aleph_2 \to (\aleph_2, \aleph_1)^2$

B. $\aleph_2 \to (\aleph_1)^2_2$

C. $2^{\aleph_0} \to (\aleph_1, \aleph_0)^2$

D. $7 \to (4,3)^2$

(2) 10. "The GCH holds below $\aleph_\eta$" means that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha < \eta$. Which of the following implications is provable in ZFC:
A. If the GCH holds below $\aleph_\omega$ then $\aleph_\omega^{\aleph_0} = \aleph_{\omega+2}$

B. If the GCH holds below $\aleph_{1000}$ then $\aleph_{1000}^{\aleph_{999}} = \aleph_{1000}$

C. If the GCH holds below $\aleph_{\omega_2}$ then $2^{\aleph_{\omega_3}} = \aleph_{\omega_4}$

D. If the GCH holds below $\aleph_{\omega_1}$ then $\aleph_{\omega_1}^{\aleph_0} = \aleph_{\omega_1+1}$

(2) 11. The weakest assumption needed to prove the statement "If X is infinite then there is an injection from $\mathbb{N}$ into X" is
A. ZF

B. ZF plus the Countable Axiom of Choice

C. ZF plus the Principle of Dependent Choices

D. ZFC

(2) 12. Which of the following statements is *not* provable in ZFC (κ, λ and μ denote *infinite* cardinals):
A. $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}$

B. If $\kappa \leqslant \lambda$ then $\kappa^\lambda = 3^\lambda$

C. If $\kappa \leqslant \lambda$ then $\kappa^\mu \leqslant \lambda^\mu$

D. If $\kappa < \lambda$ then $\mu^\kappa < \mu^\lambda$

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13. Recall that a set A is *finite* if there are $n \in \mathbb{N}$ and a bijection $f : n \to A$. Define A to be *T-finite* if every $X \subseteq \mathcal{P}(A)$ has a maximal element with respect to $\subset$. In this problem we do not assume the Axiom of Choice. Prove:

(7) a. (by induction) Every $n \in \mathbb{N}$ set is T-finite (hence every finite set is T-finite).

(4) b. $\mathbb{N}$ is not T-finite.

(7) c. Every T-finite set is finite.

More problems on the next page.

(8) 14. Let X be an infinite set and let $\prec$ and $\lhd$ be two well-orders of X. Prove that there is a subset A of X such that $|A| = |X|$ and the well-orders agree on A, i.e., for all $x, y \in A$ we have $x \prec y$ if and only if $x \lhd y$. *Hint*: Try a suitable partition of $[X]^2$.

15. For every countable ordinal $\alpha \geqslant \omega$ let $f_\alpha : \alpha \to \omega$ be a bijection. For α and n define

$$U(\alpha, n) = \{\beta \in \omega_1 : \beta > \alpha \text{ and } f_\beta(\alpha) = n\}$$

Prove:

(3) a. For every $n \in \omega$ the family $\{U(\alpha, n) : \alpha \geqslant \omega\}$ is pairwise disjoint.

(3) b. For every $\alpha \geqslant \omega$ there is an n such that $U(\alpha, n)$ is stationary in ω_1.

(4) c. There is an n such that $\{\alpha \geqslant \omega : U(\alpha, n) \text{ is stationary}\}$ is uncountable.

(4) d. Every stationary subset of ω_1 can be decomposed into $\aleph_1$ many pairwise disjoint stationary sets.

(8) 16. a. Give a self-contained proof of $\kappa < \operatorname{cf} 2^\kappa$ for infinite cardinal numbers κ, one that does not rely on Kőnig's inequality.

(8) b. Give a direct proof of the statement "If κ is a strong limit cardinal then $2^\kappa = \kappa^{\operatorname{cf}\kappa}$" by constructing appropriate injections and invoking the Cantor-Bernstein theorem.

(10) 17. Let $\langle A_\alpha : \alpha \in \omega_2 \rangle$ be a sequence of countable subsets of ω_2 such that $\alpha \notin A_\alpha$ for all α. Prove that there is a subset X of ω_2 of cardinality $\aleph_2$ and such that $\alpha \notin A_\beta$ whenever α and β are distinct elements of X. *Hint*: Think of the set $\{\alpha \in \omega_2 : \operatorname{cf} \alpha = \aleph_1\}$ and, possibly, the pressing-down lemma.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END