Mastermath Exam Set Theory
25-06-2013; 16:00–19:00.

This exam consists of multiple-choice questions, 1–12, and open questions, 13–16.
Record your answers to the multiple-choice questions in a readable table on the exam paper.

(2) 1. Given our definitions of ordered pairs ($\langle x, y\rangle = \{\{x\}, \{x, y\}\}$) and natural numbers ($n = \{0, \ldots, n-1\}$), which of the following is true:

A. $\langle 1, 2\rangle \subseteq 3$

B. $\{1, 2\} \subseteq 3$

C. $3 \subseteq \langle 1, 2\rangle$

D. $3 \subseteq \{1, 2\}$

(2) 2. Consider the structure $\langle \mathbb{N}, <\rangle$, for the language of set theory. Which of the following axioms of ZF *does not hold* in this structure.

A. Power Set

B. Pairing

C. Regularity

D. Extensionality

(2) 3. Which of the following is a filter on $\mathbb{R}$:

A. $\{A \subseteq \mathbb{R} : A \text{ is uncountable}\}$

B. $\{A \subseteq \mathbb{R} : A \text{ has cardinality } |\mathbb{R}|\}$

C. $\{A \subseteq \mathbb{R} : A \text{ is open}\}$

D. $\{A \subseteq \mathbb{R} : |\mathbb{R} \setminus A| < |\mathbb{R}|\}$

(2) 4. Assume ZFC. The set $H(\aleph_2)$, viewed as a structure for the language of Set Theory, does *not* satisfy which axiom:

A. Choice

B. Replacement

C. Power set

D. Pairing

(2) 5. Let κ be a regular and uncountable cardinal; which statement about subsets of κ is *not* true:

A. Every stationary set is unbounded

B. Every cub set is stationary

C. Every unbounded set has cardinality κ

D. Every unbounded set is stationary

(2) 6. Which of the following *ordinal* inequalities does hold:

A. $2^\omega < 3^\omega$

B. $\omega^2 < \omega^3$

C. $2 \cdot \omega < 3 \cdot \omega$

D. $2 + \omega < 3 + \omega$

More problems on the next page.

(2) 7. Which of the following *cardinal* inequalities *does not hold*:
A. $2^{\aleph_0} < 3^{\aleph_0}$

B. $\aleph_0^{2013} < \aleph_{2013}$

C. $\aleph_\omega < \aleph_\omega^{\aleph_0}$

D. $2^{\aleph_0} < 2^{2^{\aleph_0}}$

(2) 8. Let $\kappa > 2^{\aleph_1}$ be regular. Which of the following is *not* provable about countable elementary substructures of $H(\kappa)$:
A. if S is stationary in ω_1 then for every M: if $S \in M$ then $M \cap \omega_1 \in S$

B. if S is stationary in ω_1 then there is an M such that $S \in M$ and $M \cap \omega_1 \in S$

C. if C is cub in ω_1 then for every M: if $C \in M$ then $M \cap \omega_1 \in C$

D. if C is cub in ω_1 then there is an M such that $C \in M$ and $M \cap \omega_1 \in C$

(2) 9. Which of the following partition relations is *not* provable in ZFC:
A. $\aleph_2 \to (\aleph_0, \aleph_1)^2$

B. $\aleph_2 \to (2013, \aleph_2)^2$

C. $\aleph_2 \to (\aleph_2, \aleph_2)^2$

D. $9 \to (4, 3)^2$

(2) 10. "The GCH holds below $\aleph_\eta$" means that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha < \eta$. Which of the following implications is *not* provable in ZFC:
A. If the GCH holds below $\aleph_\omega$ then $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4}$

B. If the GCH holds below $\aleph_{2013}$ then $\aleph_{2013}^{\aleph_{2013}} = \aleph_{2014}$

C. If the GCH holds below $\aleph_{\omega_2}$ then $2^{\aleph_{\omega_2}} = \aleph_{\omega_2+1}$

D. If the GCH holds below $\aleph_{\omega_1}$ then $\aleph_{\omega_1}^{\aleph_0} = \aleph_{\omega_1}$

(2) 11. The weakest assumption needed to prove the statement "$|X| \leqslant |Y|$ if there is a surjection $f : Y \to X$" is
A. ZF

B. ZF plus the Countable Axiom of Choice

C. ZF plus the Principle of Dependent Choices

D. ZFC

(2) 12. Which of the following statements is provable in ZFC (κ, λ and μ denote *infinite* cardinals):
A. $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+2}$

B. If $\mu < \kappa$ and $\mu^\lambda > \kappa$ then $\kappa^\lambda = \mu^\lambda$

C. If $\kappa < \lambda$ then $\kappa^\mu < \lambda^\mu$

D. If $\kappa < \lambda$ then $\mu^\kappa < \mu^\lambda$

More problems on the next page.

13. Recall that a set A is *finite* if there are $n \in \mathbb{N}$ and a bijection $f : n \to A$. Define A to be *D-finite* if every injective map $f : A \to A$ is surjective. In this problem we do not assume the Axiom of Choice. Prove:

(7) a. (by induction) Every $n \in \mathbb{N}$ set is D-finite (hence every finite set is D-finite).

(4) b. $\mathbb{N}$ is not D-finite.

(7) c. For a set A the following are equivalent

(1) A is *not* D-finite

(2) $|A| + 1 = |A|$, i.e., there is a bijection $f : A \to A \cup \{p\}$, where $p \notin A$

(2) $|\mathbb{N}| \leqslant |A|$, i.e., there is an injection $f : \mathbb{N} \to A$

(8) 14. Prove the first non-trivial version of Ramsey's theorem:

$$\aleph_0 \to (\aleph_0)^2_2$$

(8) 15. a. Prove the following version of the Δ-system lemma: assume $2^{\aleph_0} = \aleph_1$ and let $\langle A_\alpha : \alpha \in \omega_2 \rangle$ be a sequence of countable sets. Then there are a countable set R and a subset T of ω_2 such that $|T| = \aleph_2$ and $\{A_\alpha : \alpha \in T\}$ is a Δ-system with root R. *Hint*: Think of the set $\{\alpha \in \omega_2 : \operatorname{cf} \alpha = \aleph_1\}$ and, possibly, the pressing-down lemma.

(8) b. Let A be the set of functions from ω to 2. Prove that if $B \subseteq A$ is a Δ-system then $|B| \leqslant 2$.
Hint: Functions are *sets* of ordered pairs.

16. For ordinal-valued functions f and g with domain ω_1 we define $f \prec g$ to mean that $\{\alpha : f(\alpha) \geqslant g(\alpha)\}$ is not stationary.

(8) a. Prove that $\prec$ is a well-founded partial order.

We denote the associated rank function by $\|\varphi\|$, so that $\|\varphi\| = \sup\{\|\psi\| + 1 : \psi \prec \varphi\}$.

(8) b. Prove $\|\varphi\| < \omega_1$ if and only if there is $\beta \in \omega_1$ such that $\{\alpha \in \omega_1 : \varphi(\alpha) = \beta\}$ is stationary, and in that case $\|\varphi\|$ is equal to the smallest such β.

(8) c. Determine the rank of the identity function on ω_1.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END