"Bidrag til Theorien for Dissociationen," af C. M. Guldberg (10 Mai 1872).

"Bemærkninger om Formelen for Høidemaaling med Barometer," af C. M. Guldberg.

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On the Integration of Discontinuous Functions.

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1. Riemann, in his Memoir "Ueber die Darstellbarkeit einer Function durch eine Trigonometrische Reihe" (Abhandlungen der k. Gesellschaft der Wissenschaften zu Göttingen, vol. xiii., p. 87), has given an important theorem which serves to determine whether a function $f(x)$ which is discontinuous, but not infinite, between the finite limits a and b, does or does not admit of integration between those limits, the variable x, as well as the limits a and b, being supposed real. Some further discussion of this theorem would seem to be desirable, partly because, in one particular at least, Riemann's demonstration is wanting in formal accuracy, and partly because the theorem itself appears to have been misunderstood, and to have been made the basis of erroneous inferences.

2. Let d be any given positive quantity, and let the interval $b-a$ be divided into any *segments* whatever, $\delta_1 = x_1 - a$, $\delta_2 = x_2 - x_1$, ..., $\delta_n = b - x_{n-1}$, subject only to the condition that none of these segments surpasses d. We may term d the *norm* of the division; it is evident that there is an infinite number of different divisions having a given norm; and that a division appertaining to any given norm, appertains also to every greater norm. Let $\epsilon_1, \epsilon_2, \ldots\ldots \epsilon_n$ be positive proper fractions; if, when the norm d is diminished indefinitely, the sum

$$S = \delta_1 f(a+\epsilon_1\delta_1) + \delta_2 f(x_1+\epsilon_2\delta_2) + \ldots\ldots + \delta_n f(x_{n-1}+\epsilon_n\delta_n)$$

converges to a definite limit, whatever be the mode of division, and whatever be the fractions $\epsilon_1, \epsilon_2, \ldots\ldots \epsilon_n$, that limit is represented by the symbol $\int_a^b f(x)\,dx$, and the function $f(x)$ is said to admit of integration between the limits a and b. We shall call the values of $f(x)$ corresponding to the points of any segment the *ordinates* of that segment; by the *ordinate difference* of a segment we shall understand the difference between the greatest and least ordinates of the segment.

For any given division $\delta_1, \delta_2, \ldots\ldots \delta_n$, the greatest value of S is obtained by taking the maximum ordinate of each segment, and the least value of S by taking the minimum ordinate of each segment; if D_i is the ordinate difference of the segment δ_i, the difference θ between these two values of S is $\theta = \delta_1 D_1 + \delta_2 D_2 + \ldots\ldots + \delta_n D_n$.

But, for a given norm d, the greatest value of S, and the least value of S, will in general result, not from one and the same division, but from two different divisions, each of them having the given norm. Hence the difference Θ between the greatest and the least values that S can acquire for a given norm, is, in general, greater than the greatest of the differences θ. To satisfy ourselves, in any given case, that S converges to a definite limit, when d is diminished without limit, we must be sure that Θ diminishes without limit; and it is not enough to show (as the form of Riemann's proof would seem to imply) that θ diminishes without limit, even if this should be shown for every division having the norm d.

3. Let $A(d)$ be the greatest value of S appertaining to a given norm d, and let $B(d)$ be the least value of S appertaining to the same norm. If d_1 and d_2 are any two norms, of which d_1 is greater than d_2, it is evident that $A(d_1) \geqq A(d_2)$, $B(d_1) \leqq B(d_2)$, because every division appertaining to the norm d_2 also appertains to the norm d_1. And it may be proved (although, for brevity, we omit the demonstration here) that, given any norm d_1, we can always assign a norm d_2, less than d_1, which shall satisfy the inequalities $A(d_1) > A(d_2)$, $B(d_1) < B(d_2)$; except only when the function is such that the maximum (or minimum) ordinate is the same, throughout the whole interval, for all segments however small. In this excepted case, which is one by no means inconceivable, the value of $A(d)$, [or of $B(d)$,] is independent of d, and is simply $h(b-a)$, where h is the maximum (or minimum) ordinate common to all segments of the interval $b-a$. In all other cases, it is possible to assign a series of norms, decreasing without limit, and such that the corresponding maximum values of S form a decreasing series, while the corresponding minimum values of S form an increasing series.

Besides the maximum and minimum values of S corresponding to a given norm, we have also to consider the maximum and minimum values of S corresponding to a given division. Let $P(d)$ be the maximum value of S appertaining to a given division of norm d, and let $Q(d')$ be the minimum value of S appertaining to a different division, having the same norm or a different norm. It is important to observe that we shall always have $P(d) > Q(d')$, the sign of equality being inadmissible, except when the function is such as to be represented geometrically by a single segment, or a system of segments, parallel to the axis of x. Leaving out of consideration the excepted case, we may enunciate the theorem—"The least value of S that can be obtained by taking, in any division whatever, the greatest ordinate of each segment, is greater than the greatest value that can be obtained by taking, in any division whatever, the least ordinate of each segment." To prove this theorem, let the two divisions, which give the values $P(d)$ and $Q(d')$, be simultaneously applied to the interval $b-a$. To obtain $P(d)$, ach segment in the resulting division will have to be multiplied by its greatest ordinate, or by a still greater ordinate in some adjacent segment; whereas to obtain $Q(d')$ each segment will have to be multiplied by its least ordinate, or by a still less ordinate. It follows that we have, in general, $P(d) > Q(d')$. If, however, we regard the interval $b-a$ as composed of segments $l_1, l_2, \ldots$, each of which has for its extremities points which are also extremities of segments in each of the two given divisions, we shall find that the inequality $P(d) > Q(d')$ must be replaced by the equality $P(d) = Q(d')$, if it should so happen that the maximum ordinate of each segment l is the same as its minimum ordinate; *i.e.*, if the function $f(x)$ is represented geometrically by a series of segments parallel to the axis of x, and respectively equal to the segments $l_1, l_2, \ldots$

4. Again, let $B'(d)$ be the least value of S corresponding to the division which gives $A(d)$; and let $A'(d)$ be the greatest value of S corresponding to the division which gives $B(d)$; it is evident from what has been said that we shall have the inequalities

$$A(d) > A'(d) > B'(d) > B(d).$$

Now

$$A(d)-B(d) = [A(d)-B'(d)]+[A'(d)-B(d)]-[A'(d)-B'(d)];$$

and
$$A'(d) \geqq B'(d);$$

therefore $A(d)-B(d) \leqq [A(d)-B'(d)]+[A'(d)-B(d)]$.

Hence, to prove the evanescence of $A(d)-B(d)$ or Θ, it suffices to prove the evanescence of $A(d)-B'(d)$, and of $A'(d)-B(d)$, which are, in fact, the two values of θ corresponding to the two divisions which give the absolutely greatest and least values of S for the norm d.

5. The theorem of Riemann may be enunciated as follows:—

"Let σ be any given quantity, however small; if, in every division of norm d, the sum of the segments, of which the ordinate differences surpass σ, diminishes without limit, as d diminishes without limit, the function admits of integration; and, *vice versâ*, if the function admits of integration, the sum of these segments diminishes without limit with d."

The following (with a slight modification suggested by the preceding considerations) is Riemann's demonstration of the first part of the theorem:

Let s_1 be the sum of the segments which, in the division corresponding to $A(d)$ and $B'(d)$, have ordinate differences surpassing σ; and let Ω be the greatest ordinate difference in any division appertaining to the norm d; Ω is necessarily finite, because all the ordinates are finite. The contribution of the segments s_1 to the difference $A(d)-B'(d)$ cannot surpass $s_1 \times \Omega$, and the contribution of the remaining segments cannot surpass $\sigma \times (b-a-s_1)$; *i.e.*,

$$A(d)-B'(d) \leqq s_1 \times \Omega + \sigma(b-a-s_1).$$

Similarly, if s_2 is the sum of the segments which, in the division corresponding to $A'(d)$ and $B(d)$, have ordinate differences surpassing σ,

$$A'(d)-B(d) \leqq s_2 \times \Omega + \sigma(b-a-s_2).$$

Adding these two inequalities, we find

$$A(d)-B(d) \leqq (s_1+s_2)(\Omega-\sigma)+2\sigma(b-a).$$

But σ may be taken as small as we please, and, by hypothesis, however small σ may be, d can always be taken so small as to render s_1 and s_2 as small as we please; *i.e.*, the difference $A(d)-B(d)=\Theta$ diminishes without limit with d, and $f(x)$ admits of integration between the limits a and b.

6. Riemann's demonstration of the second part of the theorem requires no modification. For, if S converges to a definite limit, Θ must be comminuent with d, and, *à fortiori*, each of the quantities θ must be comminuent with d. But, evidently, in any given division in which s is the sum of the segments having ordinate differences which surpass σ, $\sigma s \leqq \theta$. Hence, however small the given quantity σ may be, we can always, by taking d small enough, make $\frac{\theta}{\sigma}$ less than any assigned quantity; *i.e.*, if S converges to a definite limit, s must diminish without limit at the same time with d.

7. It will be observed that, in order to establish the convergence of S to a definite limit, it is sufficient to know that the sum of the segments, having ordinate differences surpassing σ, is comminuent with d in each of two specified divisions [viz., in the division which gives $A(d)$ the maximum value of S, and in that which gives $B(d)$ the minimum value of S]. Hence, if these two sums are comminuent with d, the corresponding sum in any other division of norm d is also comminuent with d.

8. Let us suppose that the function $f(x)$ has any number of dis-

continuities between a and b; and let there be $\psi(\sigma)$ points at which there are discontinuities surpassing σ. (We say that a discontinuity surpassing σ exists at a given point, when any segment, however small, being taken which includes that point, the ordinate difference of the segment surpasses σ.) If $\psi(\sigma)$ has a finite and assignable value for every value of σ, however small, the condition of integrability is certainly satisfied, even if $\psi(\sigma)$ increase without limit, when σ diminishes without limit. For, in any division of norm d, the sum of the segments having ordinate differences which surpass σ, cannot surpass $2d \times \psi(\sigma)$; and, however small σ may be, d can be taken so small that $2d \times \psi(\sigma)$ shall be less than any quantity that can be assigned. As an example, we may take the function considered by Riemann, viz.,

$$f(x) = \frac{(x)}{1} + \frac{(2x)}{4} + \frac{(3x)}{9} + \dots\dots,$$

where, by (x) we are to understand the (positive or negative) excess of x above the whole number nearest to x; or, if x lies half-way between two whole numbers, the arithmetical mean between the two differences $\frac{1}{2}$ and $-\frac{1}{2}$, *i. e.*, zero. In this function, if $x = \frac{m}{2n}$, where m and $2n$ are relatively prime, we have

$$f\left(\frac{m}{2n} + 0\right) = f\left(\frac{m}{2n}\right) - \frac{\pi^2}{16n^2},$$

$$f\left(\frac{m}{2} - 0\right) = f\left(\frac{m}{2n}\right) + \frac{\pi^2}{16n^2}.$$

Thus the number of discontinuities in any given interval is infinitely great. But the number of discontinuities which in any given interval surpass a given quantity σ, is always finite. For example, the number of discontinuities between 0 and 1 which surpass σ, is equal to the number of irreducible proper fractions, having even denominators $2n$, which verify the inequality $\frac{\pi^2}{8n^2} > \sigma$; or, if $\phi(m)$ be the number of numbers not surpassing m and prime to m, and if h be the greatest integer not surpassing $\frac{\pi}{2\sqrt{2\sigma}}$, the number of discontinuities in question is

$$\phi(1) + \phi(2) + \dots\dots\phi(h) = \psi(\sigma),$$

which is evidently finite for any given value of σ, although it increases without limit when σ diminishes without limit.

9. Next, let us suppose that $f(x)$ in the interval $b-a$ has an infinite number of discontinuities surpassing a given quantity σ. The points at which these discontinuities occur may either "completely fill" one or more finite portions of the interval $b-a$, or there may be no finite

portion of that interval which is "completely filled" by them. A system of points is said to "fill completely" a given interval when, any segment of the interval being taken, however small, one point at least of the system lies on that segment. Thus the *rational* points on any line, *i. e.*, the points of which the abscissæ are rational, completely fill any segment whatever upon the line. We may observe that the assertion, that any given segment of an interval contains at least one point of a given system, is equivalent to the assertion that any given segment contains an infinite number [*i. e.*, a number greater than any that can be assigned] of the points of the system. For we may divide the given segment into as many parts as we please, and each of them must contain at least one point of the system.

10. When the points at which there occur discontinuities surpassing σ completely fill any finite portion of the interval $b-a$, the function $f(x)$ is certainly incapable of integration. For, if l be the total length of the segments which are completely filled, we have evidently $\theta > \sigma l$ for any division of any norm d; *i. e.*, it is impossible that Θ should diminish without limit with d.

But points may exist in an infinite number within a finite interval, without completely filling any portion of that interval. Whenever this happens, it must be possible in any given segment of the interval, however small, to take a finite part such that it shall contain no point of the system; otherwise, the segment in question would be completely filled. We give a few examples of such systems of points, the limits of the interval being in each case 0 and 1. We shall say, for brevity, that points are in close order on any segment when they completely fill it, and in loose order when they do not completely fill it, or any part of it however small.

11. (i.) Let the system of points be defined by the equation $x = \frac{1}{a}$, a being any positive integer. It will be seen, (1) that these points are infinite in number; (2) that they are indefinitely condensed in the vicinity of the origin; (3) that they are in loose order over the whole interval, no segment, even in the immediate vicinity of the origin, being completely filled. For if d be any given quantity, however small, we can always find a finite integral number such that $\frac{1}{m} < d$, and then the finite spaces $\left(\frac{1}{m+1}, \frac{1}{m}\right)$, $\left(\frac{1}{m+2}, \frac{1}{m+1}\right)$, &c. ...... all lie on the segment $(0, d)$, and are all free from points of the system, if we leave their initial and terminal points out of account.

12. (ii.) Let the system of points be defined by the equation $x = \frac{1}{a_1} + \frac{1}{a_2}$, where a_1 and a_2 are any positive integers. Here, it is evi-

dent that the points are indefinitely condensed in the vicinity of each of the points of the system (i). But it can also be shown that they are in loose order over the whole interval from 0 to 1. Let $x = L_1$, $x = L_2$, ($L_1 < L_2$,) be two consecutive points of the system (i); let μ be any positive quantity whatever, and consider the segment $\left(\frac{\mu L_1 + L_2}{\mu+1}, L_2\right)$. If $x = \frac{1}{a_1} + \frac{1}{a_2}$ lies on this segment, we must have $\frac{1}{a_1} \leqq L_1$, $\frac{1}{a_2} \leqq L_1$, because no point of the system (i) lies on the interval (L_1, L_2); and also $\frac{1}{a_1} + \frac{1}{a_2} \geqq \frac{\mu L_1 + L_2}{\mu+1}$; whence $a_1 \leqq \frac{\mu+1}{L_2 - L_1}$, $a_2 \leqq \frac{\mu+1}{L_2 - L_1}$. These inequalities show that, if, from the beginning of any free segment in the system (i), we cut off as small a part as we please (which we may do by taking μ great enough), the remaining portion of that segment will contain only a finite number of points belonging to the system (ii). And this suffices to prove that the points of the system are in loose order; for if d be any segment, however small, situated anywhere in the interval from 0 to 1, we can certainly find on this segment a part free from points of the system (i), and, by what has just been proved, parts of that part will be free from points of the system (ii).

13. (iii.) Let a system of points P_{s+1} be defined by the equation $x = \frac{1}{a_1} + \frac{1}{a_2} + \dots \frac{1}{a_{s+1}}$, where $a_1, a_2, \dots a_{s+1}$ are positive integers. Assuming (what has just been proved for $s=2$) that the system P_s is in loose order over the whole interval from 0 to 1, we shall prove the same thing for the system P_{s+1}. Let $x = L_1$, $x = L_2$ be any two consecutive points of the system P_s; and consider as before the interval $\left(\frac{\mu L_1 + L_2}{\mu+1}, L_2\right)$. If the point P_{s+1}, or $x = \frac{1}{a_1} + \frac{1}{a_2} + \dots \frac{1}{a_{s+1}}$, lies on this interval, we must have, besides the inequality

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_{s+1}} \geqq \frac{\mu L_1 + L_2}{\mu+1},$$

the $s+1$ inequalities included in the formula

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_{s+1}} \leqq L_1 + \frac{1}{a_i},$$

because no point of the system P_s can be between L_1 and L_2. These inequalities give $a_i \leqq \frac{\mu+1}{L_2 - L_1}$, $i = 1, 2, 3 \dots s+1$,

whence we may infer, precisely as in the case in which $s=2$, that the points P_{s+1} are in loose order over the whole of interval from 0 to 1.

14. Let $f(x)$ be a function, which coincides with a given continuous function $\phi(x)$ for all values of x between 0 and 1, except at the points P_{s+1}; and let the difference between $f(x)$ and $\phi(x)$ at those points not

exceed the finite quantity σ. It may be shown that $f(x)$ is integrable between the limits 0 and 1, and that

$$\int_0^1 f(x)\,dx = \int_0^1 \phi(x)\,dx.$$

For, take any small interval from 0 to δ; the points P_1 which lie outside it, between δ to 1, are finite in number and at finite distances from one another. Let there be Δ_1 of them; from each of them measure a space δ_1 to the right; the number of points P_2, lying outside of the measured spaces $\delta+\Delta_1\delta_1$, is necessarily finite; and these points are at finite distances from one another. Let their number be Δ_2, and measure a distance δ_3 to the right of each of them. Proceeding in this way, we shall obtain measured spaces amounting in all to

$$\delta+\Delta_1\delta_1+\Delta_2\delta_2+\ldots\ldots+\Delta_{s+1}\delta_{s+1}=\mathrm{H}.$$

Let ϵ be any given quantity however small; and in the preceding construction let

$$\delta<\frac{\epsilon}{3(s+2)},\ \delta_1<\frac{\epsilon}{3(s+2)\Delta_1},\ \delta_2<\frac{\epsilon}{3(s+2)\Delta_2},\ \ldots\ldots\ \delta_{s+1}<\frac{\epsilon}{3(s+2)\Delta_{s+1}};$$

we shall thus have $\mathrm{H}<\frac{1}{3}\epsilon$. Let d be the least of the spaces $\delta, \delta_1, \delta_2 \ldots \delta_{s+1}$; it may be shown that, in any division of norm d, the sum of the segments containing points P_{s+1} cannot exceed 3H. For all the points P_{s+1} lie on the measured spaces; and supposing (which is the most unfavourable case) that one of those spaces begins and ends with a point P_{s+1}, we can at most triple it, by imagining a segment equal to d placed on each side of it. Thus, in every division of norm d, the sum of the segments containing the points of discontinuity is less than ϵ; whence we infer, by Riemann's theorem, that $\int_0^1 f(x)\,dx$ has the same value as $\int_0^1 \phi(x)\,dx$.

15. (iv.) Let m be any given integral number greater than 2. Divide the interval from 0 to 1 into m equal parts; and exempt the last segment from any subsequent division. Divide each of the remaining $m-1$ segments into m equal parts; and exempt the last segment of each from any subsequent division. If this operation be continued *ad infinitum*, we shall obtain an infinite number of points of division P upon the line from 0 to 1. These points are in loose order: for if d be any segment however small, situated anywhere in the interval from 0 to 1, we may take an index k which satisfies the inequality $\frac{1}{m}<\frac{1}{2}d$; and then determine a segment of the type $\left(\frac{a}{m^k}, \frac{a+1}{m^k}\right)$ lying entirely on the segment d. But this segment is either itself an exempted segment or its m^{th} part is so. It will be seen that, after k operations, the

sum of the exempted segments amounts to $1-\left(1-\frac{1}{m}\right)^k$; so that, as k increases without limit, the points of division P occur upon segments which occupy only an infinitesimal portion of the interval from 0 to 1. And it may be inferred that a function, having any finite discontinuities at the points P, would be integrable. For, if d be any given small quantity, let the index k be determined by the inequalities $\frac{1}{m^k} > d > \frac{1}{m^{k+1}}$; the number N of excepted segments which surpass $\frac{1}{m^k}$ is

$$1+(m-1)+(m-1)^2+\ldots\ldots+(m-1)^{k-2};$$

and the sum of the remaining segments is

$$\left(1-\frac{1}{m}\right)^{k-1}.$$

It is evident that in any division of norm d, the sum of the segments containing points P cannot exceed

$$\left(1-\frac{1}{m}\right)^{k-1}+2\mathrm{N}d.$$

But, as d decreases, and k increases, without limit, $\left(1-\frac{1}{m}\right)^{k-1}$ and $2\mathrm{N}d$, which is less than $\frac{2\mathrm{N}}{m^k}$, both decrease without limit; *i.e.*, in any division of norm d, the sum of the segments containing points of discontinuity diminishes without limit with d; and the function is integrable.

16. (v.) Let us now, as in the last example, divide the interval from 0 to 1 into m equal parts, exempting the last segment from any further division; let us divide each of the remaining $m-1$ segments by m^2, exempting the last segment of each segment; let us again divide each of the remaining $(m-1)(m^2-1)$ segments by m^3, exempting the last segment of each segment; and so on continually. After $k-1$ operations we shall have

$$\mathrm{N}=1+(m-1)+(m-1)(m^2-1)+\ \ldots\ +(m-1)(m^2-1)\ldots(m^{k-2}-1)$$

exempted segments, of which the sum will be

$$1-\left(1-\frac{1}{m}\right)\left(1-\frac{1}{m^2}\right)\ldots\left(1-\frac{1}{m^{k-1}}\right).$$

This sum, when k is increased without limit, approximates to the finite limit $1-\mathrm{E}\left(\frac{1}{m}\right)$; where $\mathrm{E}\left(\frac{1}{m}\right)$ is the Eulerian product $\prod_1^\infty\left(1-\frac{1}{m^k}\right)$, and is certainly different from zero. The points of division Q exist in loose order over the whole interval. For, if d be any small segment of that interval, and if $\frac{1}{m^{\frac{1}{2}k(k-1)}} < \frac{1}{2}d$, a segment of the type $\left(\frac{a}{m^{\frac{1}{2}k(k-1)}}, \frac{a+1}{m^{\frac{1}{2}k(k-1)}}\right)$ can be found lying entirely on the segment d, and this segment is either itself exempted, or its $\left(\frac{1}{m^k}\right)^{th}$ part is exempted. But a function having finite discontinuities at the points Q would be incapable of integration. For, if d be any norm, and $\delta < \frac{1}{m^{\frac{1}{2}k(k-1)}} < d$, in the division

$$\delta + \frac{i}{m^{\frac{1}{2}k(k-1)}}, \quad i = 0, 1, 2, 3, \ldots\ldots$$

(which is a division of norm d), the sum of the segments containing points of discontinuity is

$$\left(1-\frac{1}{m}\right)\left(1-\frac{1}{m^2}\right)\ldots\ldots\left(1-\frac{1}{m^{k-1}}\right)+\frac{N}{m^{\frac{1}{2}k(k-1)}},$$

which approximates to the finite limit $E\left(\frac{1}{m}\right)$ when d is diminished, and k is increased without limit.

17. The result obtained in the last example deserves attention, because it is opposed to a theory of discontinuous functions, which has received the sanction of an eminent geometer, Dr. Hermann Hankel, whose recent death at an early age is a great loss to mathematical science. In an interesting memoir ("Untersuchungen ueber die unendlich oft oscillirenden und unstetigen Functionen," Tübingen, 1870), Dr. Hankel has laid down the distinction, here adopted from him, between a system of points which completely fill a segment, and a system of points which do not completely fill any segment, but lie in loose order. [The term employed by Dr. Hankel is "zerstreut"; the use of the equivalent English words "dispersed" or "scattered" has been avoided in the present note, because they might seem to exclude the sort of condensation in the vicinity of a finite or infinite number of points, which, as we have seen in the examples (i.), (ii.), (iii.), may present itself in the case of systems of points in loose order.] Dr. Hankel then asserts (see p. 26) that, when a system of points is in loose order on a line, the line may be so divided as to make the sum of the segments containing the points less than any assignable line. The proof of this assertion is, in effect, as follows:—Divide the line into segments, of which each contains a point of the system, and imagine each segment to be diminished to its n^{th} part, yet so as still to have upon it the point of the system which it contained before. The sum of the segments can thus be made less than the n^{th} part of the whole line; *i.e.*, less than any line that can be assigned, because we may suppose n as great as we please. It must be conceded that this demonstration is rigorous, if the number of points in the system is finite; but the construction indicated ceases to convey any clear image to the mind, as soon as the number of points becomes infinite. If we are allowed to divide the line from 0 to 1, in example (iii.), in such a manner as to include every point $(P_{k,1})$ in a seg-

ment of its own, these segments, in the vicinity of the points $P_{\prime\prime}$, will have to be less than any line that can be assigned; and, if such a mode of division is admissible, it is difficult to see why it should not also be considered admissible so to divide the line as to include every rational point in a segment of its own: in which case Dr. Hankel's proposition would extend to systems of points in close order, as well as to systems in loose order. But whether we do or do not admit the truth of Dr. Hankel's proposition, the use which he makes of it (p. 31) to establish the applicability of Riemann's criterion to a certain class of functions would seem to be erroneous. To prove that Riemann's condition of integrability is satisfied for a given discontinuous function, we have to show that, given any finite quantity d, however small, the sum of the segments, which, *in any division whatever of norm* d, contain the points of discontinuity, is evanescent with d. And it is evident that this cannot be shown, if we confine ourselves to considering modes of division in which some of the segments are from the very beginning assumed to be less than any quantity that can be assigned.

While, therefore, we may safely admit the theorem that no function can be integrable which has discontinuities, surpassing a given quantity σ, at an infinite number of points forming a system in close order; the converse assertion that, when the system of points of discontinuity is in loose order, the function is integrable, would seem to be established by no satisfactory demonstration, and to be negatived by the result obtained in example (v.)

18. Another proposition, contained in the same memoir (p. 28), appears open to a similar objection. It may be admitted that a function $f(x)$ having discontinuities, which surpass a given quantity σ however small, only at points which form a system in loose order, is necessarily continuous over finite portions of any interval however small. But it would seem to be untrue that such a function is necessarily continuous in the vicinity of any one of its points of discontinuity. If, for example, $f\left(\frac{1}{a_1}+\frac{1}{a_2}\right)=1$, and $f(x)=0$, for every other value of x, it is evident that, however small the given quantity ϵ may be, the difference $f\left(\frac{1}{a_1}+\epsilon\right)-f\left(\frac{1}{a_1}+\delta\right)$ oscillates an infinite number of times between the values 0 and 1, as δ decreases from ϵ to 0; *i.e.*, the function $f(x)$ is discontinuous in the vicinity of the point $\frac{1}{a_1}$ to the right.

19. We add a few remarks which may serve still further to illustrate the meaning and use of Riemann's theorem.

(i.) The problem, "Given a system of points upon an interval (a, b), to find, among all divisions of norm d, that in which the segments containing the points have the maximum sum," is perfectly determinate. We may say that a point of the system is *isolated*, when it is separated from the next preceding and next following point by a distance $> 2d$. Similarly a group of points may be said to be isolated, when the distance between any two consecutive points of the group is less than $2d$, but the distance between the extreme points of the group, and those which immediately precede and follow it, is greater than $2d$. It is evident that, for any given value of d, the given system of points resolves itself into a finite number of isolated groups. The first and last point of each group determine a segment; on either side of each of these segments, and on either side of each isolated point, we may place a segment equal to d. The sum of the segments thus obtained is the maximum sum required.

It will be observed that in this solution each point of the system is regarded as double; *i.e.*, as capable of affecting two segments at once, one on each side of it. If the discontinuity of a function at any point can be removed by changing the value of the function at that point only, for example, if $f(x-0) = a$, $f(x) = a+\sigma$, $f(x+0) = a$, the point must be regarded as single (its contribution to the difference Θ of Art. 2 would be only $\sigma \times d$). But if the values of the function preceding and following the point of discontinuity are different (*i.e.*, if $f(x-0) = a$, $f(x+0) = a+\sigma$), the point of discontinuity produces a double effect, its contribution to the difference Θ being $2\sigma \times d$. Similarly, in the case of functions which, like $\cos\left(\frac{\pi}{x}\right)$ in the vicinity of the origin, admit of an infinite number of maxima and minima within a finite interval, the contribution to Θ of each point at which there is a maximum, or minimum, is two-fold. For the practical application of Riemann's criterion, the distinction between points producing a one-fold effect and those producing a two-fold effect is immaterial.

20. (ii.) When a function, which is discontinuous but never infinite, does not admit of integration between the limits a and b, the symbol $\int_a^b f(x)\,dx$ becomes indeterminate. But the maximum and minimum values attributable to that symbol are perfectly determinate; and if it should become advisable to attribute a definite value to the symbol, we might select for that purpose the arithmetical mean between these two extreme values. If, for continually decreasing values of d, we calculate the corresponding maximum values of the sum S of Art. 2, these values will, as shall now be shown, converge to a determinate limit A. And similarly the successive minimum values of S will converge to a determinate limit B, different from A in the case under consideration.

The difference A—B is, of course, the limit of the successive differences Θ.

From the two sets of inequalities

$$A(d_1) > A(d_2) > A(d_3) > \ldots\ldots,$$
$$B(d_1) < B(d_2) < B(d_3) < \ldots\ldots,$$

combined with the inequality $A(d_n) > B(d_n)$, which holds for any value of n however great, we infer that each of the two series,

$$A(d_1)-A(d_2),\quad A(d_2)-A(d_3),\quad A(d_3)-A(d_4),\ \ldots\ldots,$$
$$B(d_2)-B(d_1),\quad B(d_3)-B(d_2),\quad B(d_4)-B(d_3),\ \ldots\ldots,$$

consists of positive terms, and that, however many terms of either series we add together, we can never surpass $A(d_1)-A(d_\infty)$ in the first, and $B(d_\infty)-B(d_1)$ in the second; *i.e.*, in neither of them can we ever surpass $A(d_1)-B(d_1)$. But if a series of positive terms be such that the sum of any number of its terms, however great, can never surpass a given finite quantity, the sum of the first n terms of the series converges to a finite and determinate limit, when n is increased without limit (see Riemann, Vorlesungen, pp. 39, 40). The sums $A(d_1)-A(d_n)$, $B(d_n)-B(d_1)$, therefore converge to finite and determinate limits; or, which is the same thing, the two series of terms

$$A(d_1),\quad A(d_2),\quad A(d_3),\ \ldots\ldots,$$
$$B(d_1),\quad B(d_2),\quad B(d_3),\ \ldots\ldots,$$

converge to finite and determinate limits.

If, for example, the function $f(x)$ have the value σ_1 at every point of the system considered in Art. 16, example (v.), and the value $\sigma_2 < \sigma_1$ at every other point; we shall find

$$B = \sigma_2,\quad A = \sigma_2 + (\sigma_1 - \sigma_2) \times E\left(\frac{1}{m}\right).$$

21. (iii.) Riemann's criterion of integrability is applicable to the case of any multiple integral extended over a finite space. For example, in the case of a triple integral, we must imagine the whole space of the integration divided into small spaces such that any one of them could be comprehended within a sphere of a diameter d; and any such division into spaces is a division of norm d. The criterion of integrability, then, is that, in any division whatever of norm d, the sum of the spaces in which the ordinate-differences surpass a given quantity σ, must diminish without limit with d. The ordinate-difference of any space is, of course, the difference between the greatest and least values of the function within the space.

Considering, for simplicity, the case of two dimensions only, we observe that the space of integration may not only contain points of discontinuity finite or infinite in number, but may be intersected by curves of discontinuity. The function may have values differing by a finite quantity on either side of such a curve; or its values at points along the curve may be discontinuous, or both of these kinds of discontinuity may be combined at the same curve. If L (σ), the total length of the curves at which the discontinuities surpass σ, be finite, the function can be integrated over the given space; since, if we draw curves parallel to the curves of discontinuity and at a distance d from them on either side, the area of the channel-like spaces thus obtained will be $2d$L (σ), and will surpass the greatest sum of spaces, including the curves in any division of norm d. But the function may be integrable even if the total length of the curves of discontinuity is infinite; because an infinite number of contiguous curves may be enclosed in one and the same channel. And, provided that the curves can all be included in channels of which the length is L, and of which the breadth δ is comminuent with d, the condition that L $\times$ δ should be comminuent with d, will suffice to ensure the integrability of the function.*

On the Higher Singularities of Plane Curves.

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THE ordinary singularities of a plane curve are its double points and double tangents, its stationary points and stationary tangents; or, as they have been also called, its nodes and links, its cusps and inflexions. The fundamental theorem, that any of the so-called higher singularities of a plane curve may be regarded as equivalent to a certain number of ordinary singularities of each of these four kinds, has been enunciated by Professor Cayley, who has also given a method for determining in every case the four indices δ, τ, κ, ι, proper to any given singularity.

Several enquiries, which appear to possess some interest, are suggested by this theorem. Among them we may mention the two following—

(1). It is important to prove that the indices of singularity, as defined by Professor Cayley, satisfy the equations of Plücker; and that the "genus" or "deficiency" of the plane curve is correctly given by these indices.

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* This Paper, though it was not read, was offered to the Society and accepted in the usual manner.