wi4041

The Čech-Stone compactification

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2002-2003

Contents

Chapter 1

Topological Spaces

We make an inventory of those properties of metric spaces that depend on the open sets only; its *topological* properties. After this we define topological spaces and look at some examples.

Topological properties

To begin we recall the properties of the family of open sets of a metric space. First we state the definition.

1.1. DEFINITION. A subset U of a metric space X is *open* if for every point p of U there is an $\varepsilon > 0$ such that $B(p, \varepsilon) \subseteq U$.

We denote the family of all open subsets of X, *the topology of* X, by $\mathcal{T}$. The next theorem is from the course *Metric Topology*.

1.2. THEOREM. *The family $\mathcal{T}$ satisfies the following three properties.*

(i) $\varnothing, X \in \mathcal{T}$,

(ii) *if* $U_1, U_2 \in \mathcal{T}$ *then* $U_1 \cap U_2 \in \mathcal{T}$ *and*

(iii) *if* $\{U_i\}_i \subseteq \mathcal{T}$ *then* $\bigcup_i U_i \in \mathcal{T}$.

► **1**. Prove this theorem.

The following is a list of the *topological* properties that occurred in the course *Metric Topology*.

neighbourhood: A set U is a *neighbourhood* of a point p if there is an open set O with $p \in O \subseteq U$.

interior point: A point p is an *interior point* of a set A if A is a neighbourhood of p.

interior: The *interior* of a set is the set of its interior points. Notation $\operatorname{int} A$.

exterior point: A point is an *exterior point* of a set A if it is an interior point of the complement of A.

exterior: The *exterior* of a set is the set of its exterior points. Notation $\operatorname{ext} A$

boundary point: A point is a *boundary point* of a set if it is neither an interior nor an exterior point of the set.

boundary: The *boundary* of a set is the set of its boundary points. Notation ∂A.

closed set: A set is *closed* if it is the complement of an open set.

closure: The *closure* of a set is the union of the set and its boundary. Notation $\operatorname{cl} A$

adherent point: A point is an *adherent point* of a set if every neighbourhood of the point intersects the set.

accumulation point: A point p is an *accumulation point* of a set if every neighbourhood of p contains points of the set different from p..

dense subset: A subset A of a space X is a *dense* subset if $\operatorname{cl} A = X$.

G_δ-set: A set is a *G_δ-set* if it can be written as the intersection of a countable family of open sets.

F_σ-set: A set is a *F_σ-set* if it can be written as the union of a countable family of closed sets.

► **2**. Verify that the family $\mathcal{F}$ of all closed sets has the following properties.
a. $\varnothing, X \in \mathcal{F}$,
b. if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cup F_2 \in \mathcal{F}$ and
c. if $\{F_i\}_i \subseteq \mathcal{F}$ then $\bigcap_i F_i \in \mathcal{F}$.

► **3**. Define for a subset A of a metric space $A^\circ = \bigcup\{U : U$ is open and $U \subseteq A$ and $\bar{A} = \bigcap\{F : F$ is closed and $A \subseteq F$. Prove that $A^\circ = \operatorname{int} A$ and $\bar{A} = \operatorname{cl} A$.

► **4**. In this exercise X is a metric space and $A \subseteq X$. Prove the following formulas/propositions:
a. $\operatorname{cl} A = X \setminus \operatorname{ext} A$,
b. $\operatorname{ext} A = \operatorname{int}(X \setminus A)$,
c. $\partial A = \operatorname{cl} A \setminus \operatorname{int} A$,
d. $x \in \operatorname{cl} A$ if and only if x is an adherent point of A, and
e. A is dense in X if and only if $U \cap A \neq \varnothing$ for every non-empty open subset U of X.

Continuity can also be described in terms of open sets only; the first theorem is from *Metric Topology*.

1.3. THEOREM. *A map $f : X \to Y$ is continuous if and only if for every open subset U of Y the preimage $f^{-1}[U]$ is open in X.*

Continuity in a point can also be formulated in terms of the topology only.

1.4. THEOREM. *Let $f : X \to Y$ be a map between metric spaces. Then: f is continuous in $p \in X$ if and only if for every neighbourhood U of $f(p)$ there is a neighbourhood V of p such that $f[V] \subseteq U$ (or $V \subseteq f^{-1}[U]$).*

► **5**. Prove this theorem.

Homeomorphism is also a topological notion; it is derived from the notion of continuity. A *homeomorphism* between metric spaces is a continuous bijection whose inverse map is also continuous.

The following properties that a metric space may have are also topological.

splittable: A space X is *splittable* if there are two nonempty closed subsets F and G of X such that $F \cap G = \varnothing$ and $X = F \cup G$.

connectedness: A space is *connected* if it is not splittable.

compactness: A space is *compact* if every open cover has a finite subcover.

The Stone-Weierstraß theorem is a topological theorem; in the proof the metric on the space X plays no role, only the compactness of X is needed. Of course the metric on the space $C(X)$ of continuous real-valued functions is important because the theorem states that for every compact space X the corresponding metric space $C(X)$ has a certain property.

We finish with convergence of sequences: a sequence $\langle x_n \rangle_n$ converges to a point x if and only if for every neighbourhood U of x there is an N such that $x_n \in U$ whenever $n \geqslant N$.

The following are truly *metric* properties.

boundedness: A metric space (X, d) is *bounded* if there is a real number M such that $d(x, y) \leqslant M$ for all $x, y \in X$.

total boundedness: A metric space (X, d) is *totally bounded* if for every $\varepsilon > 0$ the open cover $\{B(x, \varepsilon) : x \in X\}$ has a finite subcover.

completeness: A metric space (X, d) is *complete* if every Cauchy-sequence in X converges.

isometry: An *isometry* between two metric spaces is a bijection that preserves distances.

► **6**. Check that these properties are not topological by finding pairs of homeomorphic spaces where one space has the property and the other does not (this will also provide us with homeomorphisms that are not isometries).

Topological Spaces

We will now 'forget' that we made open sets using metrics and we are going to study structures where we only have a family of open sets available. First the definition of a topology.

1.5. Definition. Let X be a set. A *topology* on X is a family $\mathcal{T}$ of subsets of X with the following three properties (quoted directly from Theorem 1.2):

(i) $\varnothing, X \in \mathcal{T}$,

(ii) if $U_1, U_2 \in \mathcal{T}$ then $U_1 \cap U_2 \in \mathcal{T}$, and

(iii) if $\{U_i\}_i \subseteq \mathcal{T}$ then $\bigcup_i U_i \in \mathcal{T}$.

1.6. Definition. A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ a topology on X.

Before we look at a few examples of topological spaces we note that all *topological* notions that were listed above can be used immediately in the context of topological spaces; we know at once when we shall call a map between topological spaces continuous (in a point) or how we define the closure of a subset or when a subset is dense in a topological space.

1.7. Examples.

1. On every set X the family $\mathcal{T}_i = \{\varnothing, X\}$ is a topology; the so-called *indiscrete topology.* This is the minimal topology that we can make on X; check that an indiscrete space is always connected and compact and that every nonempty subset is dense. Every map *to* X is continuous.
2. The other extreme is the *discrete topology*: this is the family $\mathcal{P}(X)$ of *all* subsets of X. This topology can also be defined using the discrete metric. If X has more than one point then its discrete topology not connected; $(X, \mathcal{P}(X))$ is compact if and only if X is finite. Every map *from* X is continuous.

3. Let X be an infinite set. Define
$$\mathcal{T}_{ce} = \{\varnothing\} \cup \{U \subseteq X : X \setminus U \text{ is finite}\}.$$
It is not hard to see that $\mathcal{T}_{ce}$ is a topology; the so-called *cofinite topology.* It follows that a set is closed if and only if it is finite or equal to X. A cofinite space is always connected and compact.
4. The following is a classical example: define a topology $\mathcal{T}_s$ on $\mathbb{R}$ by: $U \in \mathcal{T}_s$ if and only if for every $x \in U$ there is an $\varepsilon > 0$ with $[x, x+\varepsilon) \subseteq U$.[1] Verify that $\mathcal{T}_s$ is indeed a topology. We shall denote the topological space $(\mathbb{R}, \mathcal{T}_s)$ by $\mathbb{S}$. This space is known as the *Sorgenfrey line.*

1.8. *Remark.* A metric space is always assumed to carry its *metric topology* unless we explicitly say otherwise. In particular we always assume that $\mathbb{R}$ is endowed with its natural topology.

► **7**. Let X be an infinite set with its cofinite topology. Prove that every continuous function $f : X \to \mathbb{R}$ is constant.

► **8**. a. Investigate whether the space $\mathbb{S}$ is connected or compact.
b. Show that the 'floor' function defined by $x \mapsto \lfloor x \rfloor$, where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leqslant x\}$, is a continuous function from $\mathbb{S}$ to $\mathbb{R}$.
c. Determine, in $\mathbb{S}$, the interior of $[0, 1]$ and the closure of $(0, 1)$.

Making topological spaces

There are various ways of defining topologies in a set. One way is by specifying a *base.* This notion was already defined in the course *Metric Topology*; we now give its definition in the context of topological spaces.

1.9. DEFINITION. Let $(X, \mathcal{T})$ be a topological space. A *base* for the space (or for the topology) is a subfamily $\mathcal{B}$ of $\mathcal{T}$ with the property that for every $U \in \mathcal{T}$ there is a subfamily $\mathcal{B}'$ of $\mathcal{B}$ such that $U = \bigcup \mathcal{B}'$.

1.10. EXAMPLES.
1. In a metric space the family of all open balls is a base for the topology.
2. In the Sorgenfrey line $\mathbb{S}$ the family of all half-open intervals is a base.

Some families of sets can be bases for topologies and some can't. The following theorem tells us when a family can be a base.

1.11. THEOREM. *Assume $\mathcal{B}$ is a base for a topology $\mathcal{T}$ on the set X. Then $\mathcal{B}$ has the following two properties.*
(B1) *$X = \bigcup \mathcal{B}$; and*
(B2) *If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there is $B \in \mathcal{B}$ with $x \in B \subseteq B_1 \cap B_2$.*
Conversely, if a family $\mathcal{B}$ satisfies these two properties then there is a topology $\mathcal{T}$ for which $\mathcal{B}$ is a base.

[1] In this topology one can approximate numbers only from above; this of trying on shoes: a bit too big is OK, too small is never OK.

Proof. The first property is clear: $X \in \mathcal{T}$.

The second property follows from the fact that the intersection of two open sets is again open: if $B_1, B_2 \in \mathcal{B}$ then there is a subfamily $\mathcal{B}'$ of $\mathcal{B}$ with $B_1 \cap B_2 = \bigcup \mathcal{B}'$. Then choose, given $x \in B_1 \cap B_2$, a $B \in \mathcal{B}'$ with $x \in B$.

Is $\mathcal{B}$ satisfies properties (B1) and (B2) then we let $\mathcal{T}$ be the family of all possible unions of subfamilies of $\mathcal{B}$, so $\mathcal{T} = \{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$. It is not too difficult to verify that $\mathcal{T}$ is a topology on X. To see that $\mathcal{B} \subseteq \mathcal{T}$ observe that $B = \bigcup\{B\}$ for every $B \in \mathcal{B}$. Also, we have defined $\mathcal{T}$ in such a way that $\mathcal{B}$ is automatically a base for $\mathcal{T}$. $\square$

► **9**. Check that the family $\bigl\{[a, b) : a, b \in \mathbb{R} \text{ and } a < b\bigr\}$ satisfies (B1) and (B2).

A second way of creating topologies is via local bases.

1.12. Definition. Let $(X, \mathcal{T})$ be a topological space and $x \in X$. A *local base at x* is a family $\mathcal{B}_x$ of open neighbourhoods of x with the property that for every neighbourhood U of x there is $B \in \mathcal{B}_x$ with $B \subseteq U$.

A local base is also called a *neighbourhood base*.

1.13. Examples.

1. The standard example of a local base is the family of all balls centered at a point in a metric space. If $x \in X$, where (X, d) is a metric space then $\bigl\{B(x, \varepsilon) : \varepsilon > 0\bigr\}$ and $\bigl\{B(x, 2^{-n}) : n \in \mathbb{N}\bigr\}$ local bases at x.
2. If $x \in \mathbb{S}$ then $\bigl\{[x, x + \frac{1}{n}) : n \in \mathbb{N}\bigr\}$ is a neighbourhood base at x.

We can also make topologies by choosing for every point x in a set X a family $\mathcal{B}_x$ and using these as local bases. For this we need to find out what properties such an 'assignment of local bases' must have.

1.14. Theorem. *Assume that in the space $(X, \mathcal{T})$ we have chosen for every $x \in X$ a local base $\mathcal{B}_x$. Then the following properties hold.*

(LB1) *For every x the family $\mathcal{B}_x$ is nonempty and $x \in B$ for every $B \in \mathcal{B}_x$.*

(LB2) *If $B_1, B_2 \in \mathcal{B}_x$ then there is a $B \in \mathcal{B}_x$ such that $B \subseteq B_1 \cap B_2$.*

(LB3) *If $y \in B \in \mathcal{B}_x$ then there is a $D \in \mathcal{B}_y$ such that $D \subseteq B$.*

► **10**. Prove this theorem.

Property (LB3) codes the fact that every element of $\mathcal{B}_x$ is open; it is a neighbourhood of each of its points.

Now assume we have chosen for every point x in a set X a family of subsets such that (LB1), (LB2) and (LB3) of Theorem 1.14 hold. Define $\mathcal{T}$ by: $U \in \mathcal{T}$ if and only if for every $x \in U$ there is a $B \in \mathcal{B}_x$ with $B \subseteq U$.

We verify that $\mathcal{T}$ is indeed a topology and that for every x the family $\mathcal{B}_x$ is a local base (for $\mathcal{T}$) at x.

That $\varnothing \in \mathcal{T}$ is clear (why?) and to see that $X \in \mathcal{T}$ we use (LB1). Property (LB2) helps in establishing that the intersection of two elements of $\mathcal{T}$ belongs two $\mathcal{T}$. That unions of subfamilies of $\mathcal{T}$ again belong to $\mathcal{T}$ is not very hard to check.

Property (LB3) implies that for every x every element of $\mathcal{B}_x$ belongs to $\mathcal{T}$ and it then follows from the definition of $\mathcal{T}$ that $\mathcal{B}_x$ is indeed a local base at x.

1.15. Example. We take $X = \{(x,y) \in \mathbb{R}^2 : y \geqslant 0\}$, the upper half plane. We assign to every point X a local base. For every (x,y) in X we set $\mathcal{B}_{(x,y)} = \{B(x,y,n) : n \in \mathbb{N}\}$, where the sets $B(x,y,n)$ are defined as follows.

For a point (x,y) with $y > 0$ and for $n \in \mathbb{N}$ we define

$$B(x,y,n) = \{(s,t) \in X : \|(s,t) - (x,y)\| < 2^{-n}\},$$

the ordinary open disc around (x,y) with radius 2^{-n}.

For a point of the form $(x,0)$ and for $n \in \mathbb{N}$ we define

$$B(x,0,n) = \{(x,0)\} \cup \{(s,t) \in X : \|(s,t) - (x,2^{-n})\| < 2^{-n}\},$$

the set that consists of the point $(x,0)$ and the open disc with radius 2^{-n} that touches the x-axis at $(x,0)$.

This topological space is known as the *Niemytzki plane*.

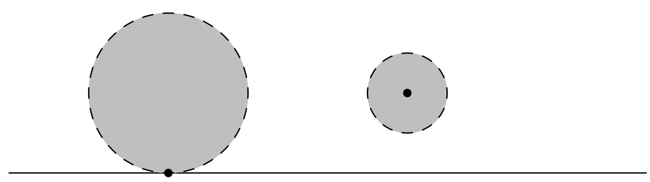


Figure 1. Basic neighbourhoods in the Niemytzki plane

▶ **11**. a. Show that the assignment $(x,y) \mapsto \mathcal{B}_{(x,y)}$ in the Niemytzki plane satisfies the properties from Theorem 1.14.

b. Prove that the Niemytzki plane is connected and that *every* subset of the x-axis is closed.

1.16. Example. Consider the following subset of $\mathbb{R}^2$:

$$X = \{(0,0)\} \cup \{(2^{-n},0) : n \in \mathbb{N}\} \cup \{(2^{-n},2^{-m}) : n,m \in \mathbb{N}\}.$$

We assign to every point a local base: the points different from $(0,0)$ get their natural neighbourhoods. For the point $(0,0)$ we do something different: for every $n \in \mathbb{N}$ and every function $f : \mathbb{N} \to \mathbb{N}$ we set

$$B(f,n) = \{(0,0)\} \cup \{(2^{-m},0) : m \geqslant n\} \cup \{(2^{-m},2^{-l}) : m \geqslant n, l \geqslant f(m)\}.$$

Then $\mathcal{B}_{(0,0)} = \{B(f,n)\}_{f,n}$.

▶ **12**. a. Show that the assignment Example 1.16 is a valid assignment of local bases.

b. Prove that $(0,0)$ is in the closure of $A = \{(2^{-n},2^{-m}) : n,m \in \mathbb{N}\}$ but that no sequence from A converges to $(0,0)$.

Chapter 2

New Topological Properties

In this chapter we will introduce a number of new topological properties. The first few properties are about the possibility of separating points; these are therefore called *separation properties* or *separation axioms*.

Separation Axioms

When we work with the indiscrete topology then we cannot distinguish points in any way: there is only one nonempty open set and so all points share the same neighbourhoods. We shall formulate a few properties that enable us to separate points better and better.

Separating points from points

The simplest *separation property* is the following:

2.1. DEFINITION. A topological space X is a T_0*-space* if different points have different families of neighbourhoods. In other words: if $x \neq y$ then there is a neighbourhood of x that does not contain y or vice versa.

2.2. EXAMPLES.
1. The simplest T_0-space is $X = \{0, 1\}$ with as its open sets $\varnothing$, $\{0\}$ and X.
2. We get an other example by taking the set $\mathbb{R}$ and the family $\{(a, \infty) : a \in \mathbb{R}\}$ as a base for a topology.

The following theorem implies that in a T_0-space there are at least as many open sets as there are points.

2.3. THEOREM. *A space X is a T_0-space if and only if* $\mathrm{cl}\{x\} \neq \mathrm{cl}\{y\}$ *whenever* $x \neq y$ *in* X.

Proof. If X is a T_0-space and $x \neq y$ then there is, say, a neighbourhood of y that does not contain x; in that case $y \notin \mathrm{cl}\{x\}$.

Conversely, let $x, y \in X$ and assume $x \in \mathrm{cl}\{y\}$. It follows immediately that $\mathrm{cl}\{x\} \subseteq \mathrm{cl}\{y\}$. If also $y \in \mathrm{cl}\{x\}$ then $\mathrm{cl}\{y\} \subseteq \mathrm{cl}\{x\}$ and so $\mathrm{cl}\{x\} = \mathrm{cl}\{y\}$ and hence, by assumption, $x = y$. We find: if $x \neq y$ then $x \notin \mathrm{cl}\{y\}$ or $y \notin \mathrm{cl}\{x\}$; in both cases there is an open set that contains one of the points x and y but not the other. □

► **1**. Find $\mathrm{cl}\{x\}$ for the points in the spaces in examples 2.2.

A better way of separating points is by replacing the word 'or' by 'and' in Definition 2.1.

2.4. DEFINITION. A topological space X is a T_1*-space* if for every two distinct points x and y in X there are neighbourhoods U of x and V of y with $x \notin V$ and $y \notin U$.

A useful characterization of T_1-spaces is the following.

2.5. THEOREM. *A space X is a T_1-space if and only if $\{x\}$ is closed for every $x \in X$.*

Proof. We leave the implication from left to right to the reader.

The other implication follows, given x and y, by taking $U = X \setminus \{y\}$ and $V = X \setminus \{x\}$ respectively. □

The next theorem follows straight from the definitions or from the characterizations.

2.6. THEOREM. *Each T_1-space is a T_0-space.*

2.7. EXAMPLES.

1. Every cofinite topology is T_1; it is in fact the smallest T_1-topology that one can make on a set.
2. A finite T_1-space is discrete (verify).
3. Every metric space is a T_1-space: if $x \neq y$ then let $U = B(x,r)$ and $V = B(y,r)$, where $r = d(x,y)$.

► **2**. Let $(X, \mathcal{T})$ be a topological space. Prove: the space $(X, \mathcal{T})$ is T_1 if and only if $\mathcal{T}_{ce} \subseteq \mathcal{T}$.

► **3**. Show that the spaces in examples 2.2 are not T_1-spaces.

A still better separation of points is obtained as follows.

2.8. DEFINITION. A topological space X is a T_2- or *Hausdorff space* if every two distinct points have disjoint neighbourhoods; so if $x \neq y$ then there are a neighbourhood U of x and a neighbourhood V of y such that $U \cap V = \varnothing$.

It should be clear that every T_2-space is a T_1-space. The following two characterizations make the distinction a bit clearer.

2.9. THEOREM. *Let X be a topological space.*

(i) *X is a T_1-space if and only if for every $x \in X$ we have $\{x\} = \bigcap\{U : U$ is a neighbourhood of $x\}$.*

(ii) *X is a T_2-space if and only if for every $x \in X$ we have $\{x\} = \bigcap\{\operatorname{cl} U : U$ is a neighbourhood of $x\}$.*

2.10. EXAMPLES.

1. Every metric space is a Hausdorff space: if $x \neq y$ then $B(x,r) \cap B(y,r) = \varnothing$ where $r = d(x,y)/2$.
2. The Sorgenfrey line $\mathbb{S}$ is a Hausdorff space: if $x < y$ then $(-\infty, y)$ and $[y, \infty)$ are disjoint neighbourhoods of x and y respectively.
3. Verify that the Niemytzki plane and the space from Example 1.16 are Hausdorff spaces.
4. An infinite set with the cofinite topology is not a Hausdorff space.

The following theorem appears in *Metric Topology* for metric spaces.

2.11. THEOREM. *Let f and g be continuous maps from a topological space X to a Hausdorff space Y. Then the set $\bigl\{x \in X : f(x) = g(x)\bigr\}$ is closed in X.*

► **4**. Prove this theorem. *Hint*: Prove that the complement of the set is open.

► **5**. Make a continuous map f from $\mathbb{R}$ with its natural topology to $(\mathbb{R}, \mathcal{T}_{ce})$ with $\{x : f(x) = x\} = \mathbb{Q}$.

► **6**. a. Prove that in a Hausdorff space every sequence has at most one limit.
b. Show that in $(\mathbb{N}, \mathcal{T}_{ce})$ the sequence $\langle n_n \rangle_n$ converges to every point of $\mathbb{N}$.

Separating points and closed sets

We extend our list of separation properties a bit further. To begin we separate points from closed sets.

2.12. Definition. A space X is a T_3-*space* if for every closed set F in X and every point $x \in X \setminus F$ there are disjoint open sets U and V with $x \in U$ and $F \subseteq V$.

By looking at complements one can characterize the T_3-property in terms of neighbourhoods only.

2.13. Theorem. *A space X is a T_3-space if and only if for every $x \in X$ and every neighbourhood U of x there is a neighbourhood V of x such that $\operatorname{cl} V \subseteq U$.*

► **7**. Prove this theorem.

2.14. Examples.

1. Every metric space has the T_3-property: if U is a neighbourhood of x and $B(x, r) \subseteq U$ then $\operatorname{cl} B(x, r/2) \subseteq U$.
2. The Sorgenfrey line $\mathbb{S}$ is a T_3-space: if U is a neighbourhood of x is and $[x, x+\varepsilon) \subseteq U$ then take $V = [x, x + \varepsilon)$, for $\operatorname{cl} V = V$.

► **8**. a. Show that the Niemytzki plane is a T_3-space. *Hint*: Show that $\operatorname{cl} B(x, y, n+1) \subseteq B(x, y, n)$ for every point (x, y) and every n.
b. Show that the space from Example 1.16 is a T_3-space. *Hint*: Every basic neighbourhood is open-and-closed.

The T_3-property becomes more interesting when we add the T_0-property to it:

2.15. Definition. A topological space X is *regular* if it is both a T_0- and a T_3-space.

In this way we get a strengthening of the Hausdorff property.

2.16. Theorem. *Every regular space is a Hausdorff space.*

Proof. Assume $x \neq y$ in the regular space X, say with $x \notin \operatorname{cl}\{y\}$; now use the T_3-property. □

2.17. Examples.

1. Let $X = [0, 1]$ be the unit interval. Every point $x > 0$ gets its natural neighbourhoods. For the point 0 and each $n \in \mathbb{N}$ we put $B(0, n) = [0, 2^{-n}) \setminus \{2^{-m} : m \in \mathbb{N}\}$. This gives a valid assignment of neighbourhood bases. The resulting space a Hausdorff space. Because $\operatorname{cl} B(0, n) = [0, 2^{-n}]$ for every n the space is not regular (check this).
2. Let $X = \mathbb{R}$ and put

$$\mathcal{T} = \{U \setminus C : U \text{ is open in the natural topology and } C \text{ is countable}\}.$$

Verify that $\mathcal{T}$ is a non-regular Hausdorff topology.

Separating closed sets

The next separation property is the strongest that we shall deal with for the time being. Its definition will not come as a surprise.

2.18. DEFINITION. A space X is a T_4*-space* if for every pair F and G of disjoint closed subsets in X there are disjoint open sets U and V with $F \subseteq U$ and $G \subseteq V$.

2.19. EXAMPLES.

1. Every metric space has the T_4-property: let F and G be disjoint closed subsets in the metric space X. Choose for every $x \in F$ a number $r(x) > 0$ such that $B(x, 3r(x)) \cap G = \varnothing$ and likewise choose $r(x) > 0$ for every $x \in G$. Then put $U = \bigcup\bigl\{B(x, r(x)) : x \in F\bigr\}$ and $V = \bigcup\bigl\{B(x, r(x)) : x \in G\bigr\}$. Check that $U \cap V = \varnothing$ (even $\operatorname{cl} U \cap \operatorname{cl} V = \varnothing$).
2. The spaces from Example 2.2 are T_4-spaces because they have no disjoint closed subsets.
3. The Sorgenfrey line $\mathbb{S}$ is a T_4-space: If F and G are closed and disjoint then choose for $x \in F$ ($x \in G$) an $\varepsilon_x > 0$ such that $[x, x + \varepsilon_x) \cap G = \varnothing$ (or $[x, x + \varepsilon_x) \cap F = \varnothing$). Check that $U = \bigcup\bigl\{[x, x + \varepsilon_x) : x \in F\bigr\}$ and $V = \bigcup\bigl\{[x, x + \varepsilon_x) : x \in G\bigr\}$ are disjoint.

As can be seen from the examples above the combination of T_4 and T_0 is not very interesting. The combination of T_4 and T_1 gives a much richer class of spaces.

2.20. DEFINITION. A topological space X is *normal* if it is both a T_1- and a T_4-space.

Because in a T_1-space points are closed sets the following theorem is clear.

2.21. THEOREM. *Every normal space is regular.*

Not every regular space is normal.

2.22. EXAMPLE. The Niemytzki plane is not normal. To see this we take the following two closed and disjoint sets: $Q = \bigl\{(x, 0) : x \in \mathbb{Q}\bigr\}$ and $P = \bigl\{(x, 0) : x \in \mathbb{P}\bigr\}$ (we use $\mathbb{P}$ to denote the set of irrational numbers). Let $U \supseteq Q$ and $V \supseteq P$ be open subsets; we must show that $U \cap V \neq \varnothing$. This will require some effort.

We begin by writing $\mathbb{P}$ as the union of countably many sets: for each n put

$$P_n = \{x \in \mathbb{P} : B(x, 0, n) \subseteq V\}.$$

We claim: if $x \in \operatorname{cl} P_n$ (with respect to the natural topology of $\mathbb{R}$) then $(x, 0) \in \operatorname{cl} V$ (in the Niemytzki plane).

For let $\langle x_i \rangle_i$ be a sequence in P_n with limit x. We show that $B(x, 0, n) \setminus \bigl\{(x, 0)\bigr\}$ is covered by the family $\bigl\{B(x_i, 0, n) : i \in \mathbb{N}\bigr\}$. Let $(p, q) \in B(x, 0, n) \setminus \bigl\{(x, 0)\bigr\}$ and put $\varepsilon = 2^{-n} - \bigl\|(x, 2^{-n}) - (p, q)\bigr\|$. For every i with $|x_i - x| < \varepsilon$ we have $\bigl\|(x_i, 2^{-n}) - (p, q)\bigr\| < 2^{-n}$ (triangle inequality) and so $(p, q) \in B(x_i, 0, n)$. We see that $B(x, 0, n) \setminus \bigl\{(x, 0)\bigr\} \subseteq V$ and hence that $(x, 0) \in \operatorname{cl} V$.

Now all we have to do is to show that $\operatorname{cl} P_n \cap \mathbb{Q} \neq \varnothing$ for some n: for take q in that intersection and choose $m \geqslant n$ with $B(q, 0, m) \subseteq U$.

Assume $\operatorname{cl} P_n \cap \mathbb{Q} = \varnothing$ for all n. We shall apply Cantor's Nesting theorem to reach a contradiction.

Let $\langle q_n \rangle_n$ be an enumeration of the rational numbers. Choose a closed interval I_1 around q_1 with $I_1 \cap P_1 = \varnothing$ (possible because $q_1 \notin \operatorname{cl} P_1$). Next choose a subinterval J_1

of I_1 with $q_1 \notin J_1$. We continue the recursion: once J_n is found choose a rational number q in the interior of J_n and, because $q \notin \operatorname{cl} P_{n+1}$, a closed interval I_{n+1} around q and disjoint from P_{n+1}, finally then shrink I_{n+1} to an interval J_{n+1} with $q_{n+1} \notin J_{n+1}$.

By Cantor's Nesting theorem there is a point x in $\bigcap_n I_n$. Because we avoided all rational numbers we have $x \notin \mathbb{Q}$; also $\bigcap_n I_n \cap \bigcup_n P_n = \varnothing$ and so $x \notin P$. This is a clear contradiction.

▶ **9**. In Example 2.22 we, implicitly, used the Baire Category Theorem. This theorem states: *if $\langle F_n \rangle_n$ is a sequence of nowhere dense subsets of $\mathbb{R}$ then the complement of $\bigcup_n F_n$ is a dense subset of $\mathbb{R}$.*

A set A is *nowhere dense* if $\operatorname{int} \operatorname{cl} A = \varnothing$.

a. Prove the Baire Category Theorem. *Hint*: Study the proof in Example 2.22 carefully.

b. Prove, using the Baire Category Theorem, that $\mathbb{Q}$ is not a G_δ-set in $\mathbb{R}$.

c. Use the Baire Category Theorem to show that the Niemytzki plane is not normal.

Normal spaces

Normal spaces have many properties that regular spaces do not have; the most important one, for us, is that normal spaces admit many continuous real-valued functions.

Before we formulate the theorem that supplies us with these continuous functions we must reformulate the T_4-property a bit.

First we have a formulation that is analogous to the formulation of the T_3-property in Theorem 2.13.

2.23. Lemma. *A space X is a T_4-space if and only if for every closed set F and every open set U with $F \subseteq U$ there is an open set V such that $F \subseteq V \subseteq \operatorname{cl} V \subseteq U$.*

Proof. Given F and U consider the closed and disjoint sets F and $X \setminus U$. If $O_1 \supseteq F$ and $O_2 \supseteq X \setminus U$ are open and disjoint then $\operatorname{cl} O_1 \subseteq U$.

Conversely, if F and G are disjoint and closed take $U \supseteq F$ with $\operatorname{cl} U \subseteq X \setminus G$ and let $V = X \setminus \operatorname{cl} U$. □

We can strengthen the T_4-property somewhat.

2.24. Lemma. *If X is a T_4-space and F and G are closed and disjoint in X then there are open sets $U \supseteq F$ and $V \supseteq G$ such that $\operatorname{cl} U \cap \operatorname{cl} V = \varnothing$.*

Proof. First choose disjoint open sets $O_1 \supseteq F$ and $O_2 \supseteq G$. Then choose $U \supseteq F$ with $\operatorname{cl} U \subseteq O_1$ and let $V = O_2$. □

The next theorem is known as Urysohn's Lemma.

2.25. Theorem. *A space X is a T_4-space if and only if for each pair of closed and disjoint sets F and G there is a continuous function $f : X \to [0,1]$ with $f \restriction F \equiv 0$ and $f \restriction G \equiv 1$.*

Proof. From right to left is clear: given the continuous function f we take $U = f^{-1}\big[[0, \frac{1}{2})\big]$ and $V = f^{-1}\big[(\frac{1}{2}, 1]\big]$.

The other implication will take more effort. First we see what we will actually need. If $f : X \to [0,1]$ is a function as required then for every $r \in (0,1)$ we have an open set:

$U_r = f^{-1}\bigl[[0,r)\bigr]$. This family of open sets completely determines the function f because for every x and every r we have

$$f(x) < r \text{ if and only if } x \in U_r,$$

and so

$$f(x) = \begin{cases} \inf\{r : x \in U_r\}, & \text{if } x \in \bigcup_r U_r \text{ and} \\ 1, & \text{if } x \notin \bigcup_r U_r. \end{cases} \tag{$*$}$$

The family $\{U_r : 0 < r < 1\}$ has another property, namely:

$$\text{if } s < r \text{ then } \operatorname{cl} U_s \subseteq U_r. \tag{$**$}$$

Thus we see: a continuous function from X to $[0,1]$ determines a family of open sets with property $(**)$ and this family determines the function via formula $(*)$.

Inspired by this we try and make a family $\{U_r : 0 < r < 1\}$ of open sets that satisfies $(**)$, then define a function via formula $(*)$ and then *show that this function is continuous.*

We begin by making U_r for each $r \in \mathbb{Q} \cap (0,1)$. For this we take an enumeration $\langle q_n \rangle_n$ of $\mathbb{Q} \cap [0,1]$ with $q_0 = 0$ and $q_1 = 1$. To start the construction we pick two open sets U_0 and U_1 such that

$$F \subseteq U_0 \subseteq \operatorname{cl} U_0 \subseteq U_1 \subseteq X \setminus G.$$

Next we choose an open set U_{q_2} such that

$$\operatorname{cl} U_0 \subseteq U_{q_2} \subseteq \operatorname{cl} U_{q_2} \subseteq U_1.$$

Now assume $n > 2$ and that U_{q_i} is found for all $i < n$ such that

$$\text{if } i, j < n \text{ and } q_i < q_j \text{ then } \operatorname{cl} U_{q_i} \subseteq U_{q_j}. \tag{$\dagger$}$$

We determine U_{q_n} in such a way that $(\dagger)$ also holds for $i, j \leqslant n$. To this end we look where q_n lies with respect to the q_i with $i < n$; take $i_0, i_1 < n$ with $q_{i_0} < q_n < q_{i_1}$ and such that no q_i met $i < n$ lies in the interval (q_{i_0}, q_{i_1}). Now we can choose an open set U_{q_n} with

$$\operatorname{cl} U_{q_{i_0}} \subseteq U_{q_n} \subseteq \operatorname{cl} U_{q_n} \subseteq U_{q_{i_1}}.$$

In the end we have a family $\bigl\{U_q : q \in \mathbb{Q} \cap [0,1]\bigr\}$ of open sets with property $(**)$. Define $U_r = \bigcup_{q \leqslant r} U_q$ all other $r \in [0,1]$. This family also satisfies $(**)$: if $r < s$ then pick p and q in $\mathbb{Q}$ with $r < p < q < s$, then $\operatorname{cl} U_r \subseteq \operatorname{cl} U_p \subseteq U_q \subseteq U_s$.

Now define $f : X \to [0,1]$ via formula $(*)$. We claim that f is continuous. Let $x \in X$ and let (r,s) be an interval around $f(x)$. Choose p and q with $r < p < f(x) < q < s$ and set $U = U_q \setminus \operatorname{cl} U_p$; U is open and because $p < f(x) < q$ we have $x \in U$. Also $f[U] \subseteq (r,s)$: if $y \in U$ then $p \leqslant f(y) \leqslant q$.

Finally: if $x \in F$ then $x \in U_r$ for all r, so $f(x) = 0$ and if $x \in G$ then $x \notin \bigcup_r U_r$ hence $f(x) = 1$. □

▶ **10**. For metric spaces it is easy to find a function as in Theorem 2.25; verify that

$$f(x) = \frac{d(x,F)}{d(x,F) + d(x,G)}$$

is as required.

Using Urysohn's Lemma we can give a nice description of closed G_δ-sets (and hence also of open F_σ-sets).

► **11**. Let X be a T_4-space. A closed set F in X is a G_δ-set if and only if there is a continuous function $f : X \to \mathbb{R}$ such that $F = \{x : f(x) = 0\}$. *Hint*: From right to left is easy. From left to right: assume $F = \bigcap_n O_n$ and choose for each n a continuous $f_n : X \to [0,1]$ with $f_n \restriction F \equiv 0$ and $f_n \restriction (X \setminus O_n) \equiv 1$. Consider $f = \sum_n 2^{-n} f_n$.

Chapter 3

Compactness and Products

In this chapter we discuss two things: the class of compact Hausdorff spaces and products of topological spaces. The definition of compactness is just as in the course *Metric Topology*; however, on its own compactness is not that interesting, combined with Hausdorffness it becomes a very powerful topological property.

Compactness

First we repeat the definition of compactness.

3.1. Definition. A topological space is *compact* if every open cover of that space has a finite subcover.

3.2. Examples.

1. Every finite space is compact.
2. Every space with the cofinite topology is compact.
3. The Sorgenfrey line, the Niemytzki plane and the space from Example 1.16 are not compact.
4. Every closed and bounded interval in $\mathbb{R}$ is compact, with respect to the natural topology.

Simple properties

Some of the following properties of compact spaces are already known from *Metric Topology*.

3.3. Theorem. *Let $f : X \to Y$ be a continuous and surjective map, where X is a compact space, then Y is compact too.*

3.4. Theorem. *Every closed subspace of a compact space is compact.*

The theorem that compact subspaces of metric spaces are closed does not hold for general topological spaces.

3.5. Example. Consider $\mathbb{N}$ with the cofinite topology and take the subspace $2\mathbb{N}$. The $2\mathbb{N}$ is compact (because it also has the cofinite topology) but is not closed.

We do have the following theorem.

3.6. Theorem. *Let X be a Hausdorff space and Y a compact subspace of X, then Y is closed in X.*

Proof. The proof is instructive enough to go through completely.

Let $x \in X \setminus Y$; we seek a neighbourhood U of x that is disjoint from Y. Choose for every $y \in Y$ a neighbourhood U_y of x and a neighbourhood V_y of y such that $U_y \cap V_y = \varnothing$.

This gives us an open cover of Y: the family $\{V_y : y \in Y\}$. Take a finite subcover, say $\{V_{y_1}, \ldots, V_{y_k}\}$.

Now let $U = \bigcap_{i=1}^k U_{y_i}$ and $V = \bigcup_{i=1}^k V_{y_i}$; then U and V are disjoint, $x \in U$ and $Y \subseteq V$.

Not only did we find a neighbourhood of x that is disjoint from Y, but we even found disjoint neighbourhoods of x and Y. $\square$

The proof of this theorem gives us the following result almost for free.

3.7. THEOREM. *Every compact Hausdorff space is regular.*

If we study the proof even better then we will also be able to prove the following.

3.8. THEOREM. *Every compact Hausdorff space is normal.*

Another theorem from *Metric Topology* says that a continuous bijection from a compact metric space to any other space is automatically a homeomorphism; this results holds, with the same proof, also for compact Hausdorff spaces.

3.9. THEOREM. *Let $f : X \to Y$ be a continuous bijection, where X is compact and Y a Hausdorff space. Then f is a homeomorphism.*

Proof. To prove continuity of f^{-1} we must show that $f[A]$ is closed in Y whenever A is closed in X.

Well, A is compact hence $f[A]$ is compact and so $f[A]$ is closed. $\square$

This theorem tells us something about the location of compact Hausdorff topologies among other topologies. Indeed, let $\mathcal{T}$ and $\mathcal{S}$ be topologies on the same set X. Assume $\mathcal{T}$ is compact, that $\mathcal{S}$ is Hausdorff and that $\mathcal{S} \subseteq \mathcal{T}$. The the theorem implies $\mathcal{T} = \mathcal{S}$ because the identity map is a continuous bijection from $(X, \mathcal{T})$ to $(X, \mathcal{S})$.

► **1**. Derive from the remark above that a compact Hausdorff topology is *minimal Hausdorff* and *maximal compact*: every topology that is a proper subfamily is no longer Hausdorff and every topology of which it is a proper subfamily is no longer compact.

Subspaces

The class of compact Hausdorff spaces has many more pleasant properties. To be able to formulate these we must introduce a few more notions.

First a topological property that is characteristic of all subspaces of compact Hausdorff spaces: every subspace of every compact Hausdorff space has this property and every space with this property is in fact a subspace of a (suitable) compact Hausdorff space.

The property is not obvious, until it is pointed out to you. Take a subspace A of a compact Hausdorff space X. Take in A a closed subset F (closed in A) and a point x in $A \setminus F$. There is a closed subset F^+ of X with $F = A \cap F^+$. Next, because $X$is normal, we can find a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f \restriction F^+ \equiv 1$. The restriction of f to A does the same for x and F.

It follows that every subspace of every compact Hausdorff space has the property from the following definition.

3.10. DEFINITION. A topological space X is a $T_{3\frac{1}{2}}$-*space* if for every closed set F in X and every point x in $X \setminus F$ there is a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f \restriction F \equiv 1$.

A space that is both a T_0- and a $T_{3\frac{1}{2}}$-space is called *completely regular* or a *Tychonoff space*.

► **2**. Prove that every normal space is completely regular and that every completely space is regular.

3.11. EXAMPLES.

1. Every normal space is completely regular (why?), hence every metric space is completely regular; prove this directly.
2. The Sorgenfrey line is therefore also completely regular; this can be seen directly: if F is closed and $x \notin F$ then take $y > x$ with $[x,y) \cap F = \varnothing$. The function f defined by
$$f(p) = \begin{cases} 0 & \text{if } x \leqslant p < y \text{ and} \\ 1 & \text{otherwise} \end{cases}$$
clearly is as required.

3.12. EXAMPLE. The Niemytzki plane is completely regular; because the space is not normal we must show this directly. For points above the x-axis we can use the natural metric on $\mathbb{R}^2$ to define the needed continuous functions.

Take a point $(x,0)$ on the x-axis and define for every $r \in (0,1]$
$$U_r = \bigl\{(x,0)\bigr\} \cup \bigl\{(p,q) : \|(p,q) - (x,r)\| < r\bigr\}.$$
verify that each U_r is open and that $\operatorname{cl} U_r \subseteq U_s$ whenever $r < s$. As in the proof of Urysohn's Lemma we define
$$f(u) = \begin{cases} \inf\{r : u \in U_r\}, & \text{if } u \in \bigcup_r U_r \text{ and} \\ 1, & \text{if } u \notin \bigcup_r U_r. \end{cases}$$
This gives is a continuous function f with $f(x,0) = 0$ and $f(p,q) = 1$ for $(p,q) \notin U_1$. By rescaling this function we can find for every neighbourhood U of $(x,0)$ a function g with $g(x,0) = 0$ and $g(p,q) = 1$ for $(p,q) \notin U$.

3.13. EXAMPLE. We complete the picture by describing a space that is regular but not completely regular. For this we define a topology on the upper half plane.

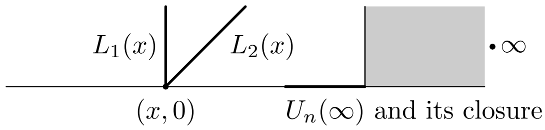
The points above the x-axis will be isolated, i.e., if z is not on the x-axis then $\bigl\{\{z\}\bigr\}$ is a local base at z.

For a point $(x,0)$ on the x-axis we make basic neighbourhoods as follows: first put $L_1(x) = \bigl\{(x,y) : 0 \leqslant y \leqslant 1\bigr\}$ (the vertical line segment of length 1 based at $(x,0)$) and $L_2(x) = \bigl\{(x+y,y) : 0 \leqslant y \leqslant 1\bigr\}$ (the line segment from $(x,0)$ to $(x+1,1)$). Next we set $L_x = L_1(x) \cup L_2(x)$. As a local base at $(x,0)$ we take $\mathcal{B}_x = \{L_x \setminus F : F$ is finite and $x \notin F\}$. The space we obtain in this way is regular, even completely regular, because every basic neighbourhood is open-and-closed.

We add one more point ∞ to our space, with basic neighbourhoods

$$U_n(\infty) = \{\infty\} \cup \bigl\{(x,y) : x \geqslant n\bigr\} \qquad n \in \mathbb{N}.$$

Check that $\operatorname{cl} U_{n+1} = U_{n+1} \cup \bigl\{(x,0) : n < x \leqslant n+1\bigr\}$; the space M that we get is still regular: $\operatorname{cl} U_{n+1} \subseteq U_n$.



The space is not completely regular. For let $f : M \to [0,1]$ be continuous with $f(x,0) = 0$ $x \leqslant 0$. We prove that $f(\infty) = 0$.

To this end we first observe the following: for every $x \in \mathbb{R}$ there is a countable subset of A_x of L_x such that if $p \in L_x \setminus A_x$ then $f(p) = f(x,0)$; indeed, for every n there is a finite set F_n such that $|f(p) - f(x,0)| < 2^{-n}$ for $p \in L_x \setminus F_n$, let $A_x = \bigcup_n F_n$.

Using this we prove: for every n there are at most countably many $x \in (-\infty, n]$ with $f(x,0) \neq 0$. This is true, by assumption, for $n = 0$. Assume the statement is true for some $n \geqslant 0$ and, using this, choose a sequence $\langle x_i \rangle_i$ in $(n-1, n)$ that converges to n and satisfies $f(x_i, 0) = 0$ for all i.

The set $\bigcup_i \bigl(L_2(x_i) \cap A_{x_i}\bigr)$ is countable and hence so is its projection A onto the x-axis. Take $x \in [n, n+1) \setminus A$; the line $L_1(x)$ intersects $L_2(x_i) \setminus A_{x_i}$ for all but finitely many i. Therefore every neighbourhood of $(x,0)$ intersects all but finitely many $L_2(x_i) \setminus A_{x_i}$, that is, every neighbourhood of $(x,0)$ has points p with $f(p) = 0$; this implies $f(x,0) = 0$.

▶ **3**. Add an extra point $-\infty$ to the space M from Example 3.13, with basic neighbourhoods $U_n(-\infty) = \{-\infty\} \cup \bigl\{(x,y) : x \leqslant -n\bigr\}$. This new space we call M^+.

Prove that $f(\infty) = f(-\infty)$ for every continuous function $f : M^+ \to \mathbb{R}$.

▶ **4**. Now study the article [1972] by van Douwen or do Problem 2.7.17 in Engelking's book [1989].

Products

We repeat: every subspace of every compact Hausdorff space is completely regular and for every completely regular space there is a compact Hausdorff space that contains the given space as a subspace.

For the proof of the second part of the last sentence will require some new notions, one of which is the product of a family of topological spaces. Finite products are easiest, they are defined much like $\mathbb{R}^n$.

Finite products

First we define the product of finitely many sets, then we bring the topologies into play.

3.14. Definition. Let X_1, X_2,…, X_n be a finite number of sets. The *product* of these sets is the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$ with $x_i \in X_i$ $(1 \leqslant i \leqslant n)$.

We write this product as $X_1 \times X_2 \times \cdots \times X_n$ or $\prod_{i=1}^n X_i$.

3.15. EXAMPLES.

1. According to this definition $\mathbb{R}^n$ is indeed the product of n copies of $\mathbb{R}$.
2. $\mathbb{R} \times \mathbb{Q}$ is the set of points in the plane whose second coordinate is rational.

The natural topology of $\mathbb{R}^n$ has the family of all open blocks as a base. We use this idea to define a topology on other products.

3.16. DEFINITION. Let $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$, …, $(X_n, \mathcal{T}_n)$ be a finite family of topological spaces.

An *open block* in $\prod_{i=1}^n X_i$ is a set of the form $\prod_{i=1}^n U_i$ where U_i is open in X_i.

3.17. LEMMA. *The family of all open blocks is a base for a topology on* $\prod_{i=1}^n X_i$.

Proof. Verify that the intersection of two open blocks is again an open block and that the open blocks cover the whole product. Then apply Theorem 1.11. □

This topology is the product topology.

3.18. DEFINITION. Let $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$, …, $(X_n, \mathcal{T}_n)$ be a finite family of topological spaces.

The *product topology* on $\prod_{i=1}^n X_i$ is the topology that has the family of all open blocks as a base.

The set $\prod_{i=1}^n X_i$ with the product topology is called the *product* of the spaces $(X_1, \mathcal{T}_1)$, …, $(X_n, \mathcal{T}_n)$.

We used the situation in $\mathbb{R}^n$ as inspiration for the definition of the product topology. There is an other reason to define the product topology the way we did: the projections from the product to the factors are continuous and the product topology is the smallest topology that does this.

3.19. DEFINITION. Let $X = \prod_{i=1}^n X_i$ be a product of n many sets and $i \leqslant n$. The map $\pi_i : X \to X_i$ defined by $\pi_i(x_1, \ldots, x_n) = x_i$ is the *projection on the i-th coordinate or factor*.

3.20. THEOREM. *Let $X = \prod_{i=1}^n X_i$ be a product of n many topological spaces. Then every projection $\pi_i : X \to X_i$ is continuous. Every other topology that makes the projections continuous is larger than the product topology.*

Proof. It is easy to see that each π_i is continuous: if $U \subseteq X_i$ is open then $\pi_i^{-1}[U]$ is an open block (what are the factors?).

Conversely, assume $\mathcal{T}$ is a topology that makes the projections continuous. It follows at once that for every i and every open subset U of X_i the open block $\pi_i^{-1}[U]$ belongs to $\mathcal{T}$. Every intersection of finitely many such open blocks also belongs to $\mathcal{T}$ (because $\mathcal{T}$ is a topology); but in this way we see that all open blocks belong to $\mathcal{T}$. But then arbitrary unions of open blocks belong to $\mathcal{T}$ as well, which is what we were trying to prove.. □

Using this theorem we can show that continuity of a map *to* a product is the same as coordinate-wise continuity.

3.21. THEOREM. *Let $X = \prod_{i=1}^n X_n$ be a product of n many topological spaces. Then: a map $f : Y \to X$ is continuous if and only if all compositions $\pi_i \circ f$ are continuous.*

Proof. If f is continuous, then certainly every composition $\pi_i \circ f$ is continuous.

Conversely, assume each composition $\pi_i \circ f$ is continuous. Let $y \in Y$, and let $U = \prod_i U_i$ be a basic neighbourhood of $f(y)$. Now for $z \in Y$ we have: $f(z) \in U$ if and only if for each i the i-th coordinate of $f(z)$ belongs to U_i. That i-th coordinate is $\pi_i\bigl(f(z)\bigr)$.

Thus we find

$$f^{-1}[U] = \bigcap_{i=1}^{n} (\pi_i \circ f)^{-1}[U_i],$$

which shows that $f^{-1}[U]$ is a neighbourhood of y. □

In general we shall denote the composition $\pi_i \circ f$ by f_i.

Conversely we can make, given maps $f_i : Y \to X_i$, a map f from Y to X: define $f(y) = \bigl(f_1(y), f_2(y), \ldots, f_n(y)\bigr)$. The map f is called the *diagonal* of the maps f_1, f_2, …, f_n. We denote the diagonal as $f = \triangle_{i=1}^{n} f_i$ or $f = f_1 \triangle f_2 \triangle \cdots \triangle f_n$.

From *Metric Topology* we know that every closed and bounded block in $\mathbb{R}^n$ is compact. A careful analysis of the proof of that result yields the following theorem.

3.22. Theorem. *The product of a finite number of compact spaces is compact.*

One of the proofs that $\mathbb{R}^2$ is connected can be generalised.

3.23. Theorem. *The product of finitely many connected spaces is connected.*

► **5**. Prove for each of the following properties that the product has it if each of the factors has it.

T_0, T_1, T_2, regular, completely regular and 'having a countable base'.

► **6**. The product of the Sorgenfrey line with itself is not normal. *Hint*: Consider the subsets $Q = \bigl\{(q, -q) : q \in \mathbb{Q}\bigr\}$ and $P = \bigl\{(p, -p) : p \in \mathbb{P}\bigr\}$ of the anti-diagonal. Show that Q and P are closed. Modify the argument for the Niemytzki plane from Example 2.22.

► **7**. The product of finitely many metrizable topological spaces is metrizable. *Hint*: Define, by analogy with $\mathbb{R}^n$, a metric on $\prod_{i=1}^{n} X_i$ by

$$d\bigl((x_1, \ldots, x_n), (y_1, \ldots, y_n)\bigr) = \sum_{i=1}^{n} d_i(x_i, y_i),$$

where d_i is a metric on X_i that induces the topology.

Infinite products

Our first task is to define what a product of an arbitrary family of spaces should be. The answer is, after some thought, fairly obvious; if we have finitely many sets, n say, then their product consists of ordered n-tuples of points where the i-th coordinate comes from the i-th set.

Such an n-tuple is in fact a function with domain $\{1, 2, \ldots, n\}$ that picks for every i a point in X_i — a choice function.

3.24. Definition. Let $\{X_t\}_{t\in T}$ be a family of sets. The *product* of these sets is defined as the set of all choice functions for that family; we denote the product as $\prod_{t\in T} X_t$.

Thus, $x \in \prod_{t\in T} X_t$ if and only if x is a function with domain T and such that $x(t) \in X_t$ for all t. To reinforce the idea of coordinates we write x_t in stead of $x(t)$ and $x = (x_t)_{t\in T}$.

Next we assume that each X_t is a topological space, with topology $\mathcal{T}_t$. The question now is how we shall topologize $\prod_{t\in T} X_t$. We let a natural demand guide us, namely that the projections should be continuous and that we should use a few open sets as possible; compare Theorem 3.20.

3.25. DEFINITION. Let $X = \prod_{t\in T} X_t$ be a product of a family of sets and let $s \in T$. The map $\pi_s : X \to X_t$ defined by $\pi_s\bigl((x_t)_{t\in T}\bigr) = x_s$ is called the *projection onto the s-th coordinate or factor*.

If we want every projection to be continuous then the preimage $\pi_t^{-1}[U]$ should be open for every t and every open subset U of X_t we shall call a set of this form an *open strip*. Furthermore finite intersection of open strips should be open as well. We are lead to consider a special kind of open blocks that we shall call *finite open blocks*.

3.26. DEFINITION. Let $X = \prod_{t\in T} X_t$ be the product of a family of topological spaces. A *finite open block* in X is a set of the form $\prod_{t\in T} U_t$, where U_t is an open subset of X_t for every t *and* where for at most finitely many t we have $U_t \neq X_t$.

► **8**. Verify that a set is an open block if and only if it is the intersection of finitely many open strips.

Observe that the full product is a finite open block and that the intersection of two finite open blocks is again a finite open block. The family of all finite open blocks is therefore a base for a topology — this will be the product topology.

3.27. DEFINITION. Let $X = \prod_{t\in T} X_t$ be the product of a family of topological spaces. The *product topology* on X is the topology that has the family of finite open blocks as a base.

The set X with the product topology is called the *product* of the family of the spaces $\bigl\{(X_t, \mathcal{T}_t) : t \in T\bigr\}$.

The validity of the next theorem was simply forced by the definition.

3.28. THEOREM. *Let $X = \prod_{t\in T} X_t$ be the product of a family of topological spaces. Then every projection $\pi_t : X \to X_t$ is continuous. Every other topology that makes all projections continuous is larger than the product topology.*

The proof can be found above and is completely analogous to that of Theorem 3.20. Theorem 3.21, in appropriate form, is also valid.

3.29. THEOREM. *Let $X = \prod_{t\in T} X_t$ be the product of a family of topological spaces. Then: a map $f : Y \to X$ is continuous if and only if all compositions $\pi_t \circ f$ are continuous.*

The proof offers no new problems because we used *finite* open blocks to define the topology.

Finally we can turn a family of maps $f_t : Y \to X_t$ into a map f from Y to X via $f(y) = \bigl(f_t(y)\bigr)_{t\in T}$. We call f the *diagonal* of the maps $\{f_t\}_{t\in T}$. The notation is the same: $f = \triangle_{t\in T} f_t$.

► **9**. Prove, for each of the following properties, that a product has it if every factor has it. T_0, T_1, T_2, regular and completely regular.

► **10**. Let $X = \prod_{n\in\mathbb{N}} X_n$ be the product of a countable family of topological spaces. Prove:

a. If every X_n has a countable base then so does X.

b. If every X_n is metrizable then so is X.

► **11**. Prove that every product connected spaces is connected.

► **12**. There is another, seemingly obvious, way of topologizing a product. Here we use *all* open blocks, i.e., unrestricted products of the form $\prod_{t\in T} U_t$ where each U_t is open X_t. This topology is called the *box topology*.

a. Verify that the family of all open blocks does indeed serve as a base for a topology.

This topology does not have as many nice properties as the product topology. Consider the product $X = \prod_{n\in\mathbb{N}} X_n$, where $X_n = [0,1]$ for each n.

b. The diagonal $\triangle_{n\in\mathbb{N}} \mathrm{Id}_n$ is not continuous; $\mathrm{Id}_n : [0,1] \to X_n$ is the identity map. *Hint*: The range carries, as a subspace of X, the discrete topology.

c. The box topology on X is not connected and not metrizable.

Chapter 4

Tychonoff's Theorem

Tychonoff's Theorem, which states that the product of compact spaces is again compact, is one of the most important theorem from topology. It appears in many places in mathematics; we will use it in the construction of the Čech-Stone compactification.

Before we can prove Tychonoff's theorem we must know a bit more about compact spaces and we also need to introduce a few extra notions.

To see what we need we go through the proof for the case of finitely many spaces.

Let X and Y be compact spaces and let $\{U_i \times V_i : i \in I\}$ be a cover of $X \times Y$ by open blocks. The crucial step is to find, for each $x \in X$, a neighbourhood U_x of x such that the strip $U_x \times Y$ can be covered by finitely many of the blocks $U_i \times V_i$.

This uses compactness of Y: there are finitely many i, say i_1, i_2, …, i_n, such that $\{x\} \times Y \subseteq \bigcup_{k=1}^n U_{i_k} \times V_{i_k}$. Then take $U_x = \bigcap_k U_{i_k}$; the strip is also covered by these blocks.

The open cover $\{U_x : x \in X\}$ of X then has a finite subcover; using this we can find a finite subcover of $\{U_i \times V_i : i \in I\}$.

What happened is that we turned an arbitrary open cover, via blocks, into a cover by strips and a cover by strips is in fact a cover of one of the factors.

This works for more than two factors as well. A cover of $X \times Y \times Z$ will first be reduced to a cover by strips of the form $U \times Z$, with U open in $X \times Y$; then the resulting cover of $X \times Y$ is reduced to a cover by strips of the form $V \times Y$ and we are done.

We would like to do the same for arbitrary products, but this will not be as easy as in the case of finitely many factors: there is usually no 'last' factor that we can use as a starting point.

There is an important tool that will enable us to reduce the case for arbitrary products to the case of finitely many factors: this tool is called filter.

Filters and ultrafilters

We shall reach Tychonoff's Theorem in a roundabout way. First we show how to do convergence in arbitrary topological spaces, then we prove the analogue of the theorem that a metric space is compact if and only if every sequence has a converging subsequence, and we finish by showing that this convergence property is preserved under products.

The proper generalisation of sequences is given by filters — see Definition 4.7 and Example 4.8.1 for the connection.

Filters

4.1. Definition. Let X be a set. A family $\mathcal{F}$ of subsets of X is a *filter* on X if

(i) $\varnothing \notin \mathcal{F}$,

(ii) if $F_1, F_2 \in \mathcal{F}$ then there is $F_3 \in \mathcal{F}$ with $F_3 \subseteq F_1 \cap F_2$, and
(iii) if $F \in \mathcal{F}$ and $F \subseteq G$ then $G \in \mathcal{F}$.

4.2. Examples.

1. If $\langle x_n \rangle_n$ is a sequence X then the family $\mathcal{F}$ defined by, $F \in \mathcal{F}$ if and only if there is an $N \in \mathbb{N}$ with $\{x_n : n \geqslant N\} \subseteq F$, a filter on X.
2. If X is an infinite set then $\mathcal{F} = \{F : X \setminus F \text{ is finite}\}$ is a filter on X, the cofinite or Fréchet filter.
3. If $x \in X$ then $\mathcal{F}_x = \{F : x \in F\}$ is a filter on X.
4. If X is a topological space and $x \in X$ then the family $\mathcal{U}_x$ of all neighbourhoods of x is a filter, the *neighbourhood filter* of x.

We can also describe a filter by specifying a base for it.

4.3. Definition. Let X be a set. A family $\mathcal{B}$ of subsets of X is a *filter base* on X if the family $\mathcal{F} = \{F : \text{there is a } B \in \mathcal{B} \text{ with } B \subseteq F\}$ is a filter. We call $\mathcal{B}$ a *base* for the filter $\mathcal{F}$.

We indicate a base for each filter from 4.2.

4.4. Examples.

1. The family of all 'tails' of the sequence $\langle x_n \rangle_n$ is a base for the corresponding filter; a *tail* is a set of the form $\{x_n : n \geqslant N\}$.
2. The cofinite filter has no obvious base, except when $X = \mathbb{N}$: then we can take the tails of $\mathbb{N}$.
3. The family $\bigl\{\{x\}\bigr\}$ is a base for $\mathcal{F}_x$.
4. Every local base at x is a base for $\mathcal{U}_x$.

► **1**. Show that a family $\mathcal{B}$ is a filter base if and only if it satisfies (i) and (ii) of Definition 4.1.

An example that is important in connection with compactness is the following.

4.5. Example. Let $\{A_i : i \in I\}$ be a cover of a set X without finite subcover; then the family of all sets of the form $X \setminus \bigcup_{i \in F} A_i$, with F finite, a filter base on X. The corresponding filter $\mathcal{F}$ has an empty intersection because $\bigcap \mathcal{F} = X \setminus \bigcup_{i \in I} A_i = \varnothing$. A filter whose intersection is empty is called a *free filter*.

Using this example it is not hard to see that the following theorem must be true.

4.6. Theorem. *A space X is compact if and only if for every filter $\mathcal{F}$ on X we have $\bigcap\{\operatorname{cl} F : F \in \mathcal{F}\} \neq \varnothing$.*

Proof. If $\mathcal{F}$ is a filter then consider $\mathcal{U} = \{X \setminus \operatorname{cl} F : F \in \mathcal{F}\}$; no finite subfamily of $\mathcal{U}$ covers X (why?). because X is compact the family $\mathcal{U}$ cannot be an open cover. But $X \setminus \bigcup \mathcal{U} = \bigcap \mathcal{F}$.

Conversely, if X has an open cover $\mathcal{U}$ without a finite subcover then we make a filter $\mathcal{F}$ as in Example 4.5. But then we'd have $\bigcap\{\operatorname{cl} F : F \in \mathcal{F}\} = \varnothing$. □

► **2**. Let $\langle x_n \rangle_n$ be a sequence in a space X and $\mathcal{F}$ the associated filter. Prove that $\langle x_n \rangle_n$ converges to x if and only $\mathcal{U}_x \subseteq \mathcal{F}$.

This inspires the following definition.

4.7. DEFINITION. Let X be a topological space, $\mathcal{F}$ a filter on X and $x \in X$. We say the the filter $\mathcal{F}$ *converges to the point* x if $\mathcal{U}_x \subseteq \mathcal{F}$.

4.8. EXAMPLES.

1. Exercise 2 now says: a sequence $\langle x_n \rangle_n$ converges to a point x if and only if the corresponding filter converges to x.
2. Certainly the filter $\mathcal{U}_x$ converges to x; it is the smallest filter to do so.
3. The filter $\mathcal{F}_x$ converges to x too.

► **3**. Let $\mathcal{F}$ be a filter on a space X and let $x \in X$. Prove
  a. If $x \in \bigcap\{\operatorname{cl} F : F \in \mathcal{F}\}$ then there is a filter $\mathcal{G}$ with $\mathcal{G} \supseteq \mathcal{F}$ that converges to x.
  Hint: $\mathcal{G} = \{U \cap F : U \in \mathcal{U}_x, F \in \mathcal{F}\}$
  b. If there is a filter $\mathcal{G}$ with $\mathcal{G} \supseteq \mathcal{F}$ that converges to x then $x \in \bigcap\{\operatorname{cl} F : F \in \mathcal{F}\}$.

We call a filter $\mathcal{G}$ *finer* than the filter $\mathcal{F}$ if $\mathcal{F} \subseteq \mathcal{G}$; we also call $\mathcal{F}$ *coarser* than $\mathcal{G}$. Exercise 3 proves following theorem.

4.9. THEOREM. *Let $\mathcal{F}$ be a filter on a topological space X and $x \in X$. Then $x \in \bigcap\{\operatorname{cl} F : F \in \mathcal{F}\}$ if and only if there is a filter $\mathcal{G}$ that is finer than $\mathcal{F}$ and that converges to x.*

Now we can formulate and prove the promised analogue of the theorem that a metric space is compact if and only if every sequence in that space has a converging subsequence.

4.10. THEOREM. *A topological space is compact if and only if for every filter on the space there is a finer filter that converges.*

Proof. Combine Theorems 4.6 and 4.9. □

Using convergence of filters we can describe other topological notions as well.

► **4**. Let A be a subset of a topological space X and $x \in X$. Prove: $x \in \operatorname{cl} A$ if and only if there is a filter $\mathcal{F}$ with $A \in \mathcal{F}$ that converges to x.

Next we look at continuity. For this we define what the image of a filter under a map is. This is quite straightforward: if $f : X \to Y$ is a map and $\mathcal{F}$ a filter on X then $\{f[F] : F \in \mathcal{F}\}$ a filter base on Y (check this); we denote the filter that it generates as $f(\mathcal{F})$ and we call it the image of $\mathcal{F}$ under f.

► **5**. Let $f : X \to Y$ be a map between topological spaces. Then:
  a. If $x \in X$ then f is continuous at x if and only if for every filter on X that converges to x its image under f converges to $f(x)$.
  b. The map f is continuous if and only if for every convergent filter on X its image under f converges too (to the correct limit).

We also need to know that convergence in a product is the same as coordinatewise convergence.

4.11. THEOREM. *Let $\mathcal{F}$ be a filter on a product $X = \prod_{t \in T} X_t$ of topological spaces. Then: $\mathcal{F}$ converges to $x = (x_t)_{t \in T}$ if and only if for every t the filter $\pi_t(\mathcal{F})$ converges to x_t.*

Proof. Because projections are continuous we clearly have 'only if'.

Conversely, assume $\pi_t(\mathcal{F})$ converges to x_t for all t. Let U be a basic neighbourhood of x, determined by $U_{t_1}, U_{t_2}, \ldots, U_{t_n}$.

For each i we have $U_{t_i} \in \pi_{t_i}(\mathcal{F})$, so there is an $F_i \in \mathcal{F}$ with $\pi_{t_i}(F_i) \subseteq U_{t_i}$. But then $F_i \subseteq \pi_{t_i}^{-1}[U_{t_i}]$ and so $\pi_{t_i}^{-1}[U_{t_i}] \in \mathcal{F}$ for each i.

It follows that $U \in \mathcal{F}$, because $U = \bigcap_{i=1}^{n} \pi_{t_i}^{-1}[U_{t_i}]$. □

► **6**. Prove that a topological space is a Hausdorff space if and only if every filter converges to at most one point.

Ultrafilters

We need a special kind of filters: ultrafilters.

4.12. Definition. An *ultrafilter* is a filter for which there is no finer filter.

In other words: if for every filter $\mathcal{G}$ with $\mathcal{F} \subseteq \mathcal{G}$ we have $\mathcal{F} = \mathcal{G}$ ($\mathcal{F}$ is a *maximal* filter) then we call $\mathcal{F}$ an ultrafilter.

We need some characterizations of ultrafilters.

4.13. Theorem. *Let $\mathcal{F}$ be a filter on a set X; then the following are equivalent.*

(i) *$\mathcal{F}$ is an ultrafilter;*

(ii) *for every subset A of X we have: if $A \cap F \neq \varnothing$ for all $F \in \mathcal{F}$ then $A \in \mathcal{F}$;*

(iii) *for any two subsets A and B of X we have: if $A \cup B \in \mathcal{F}$ then $A \in \mathcal{F}$ or $B \in \mathcal{F}$; and*

(iv) *for every subset A van X either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.*

Proof. (i) ⇒ (ii): assume $A \cap F \neq \varnothing$ for all $F \in \mathcal{F}$. Then $\{A \cap F : F \in \mathcal{F}\}$ is a filter base (check) and the filter $\mathcal{G}$ that it generates is finer than $\mathcal{F}$ and $A \in \mathcal{G}$; but then $\mathcal{G} = \mathcal{F}$ and so $A \in \mathcal{F}$.

(ii) ⇒ (iii): assume $A \notin \mathcal{F}$. Then there is no $F \in \mathcal{F}$ with $F \subseteq A$ and so $F \cap X \setminus A \neq \varnothing$ for all $F \in \mathcal{F}$. But then $X \setminus A \in \mathcal{F}$ and so $(X \setminus A) \cap A \cup B \in \mathcal{F}$. Now observe $(X \setminus A) \cap A \cup B = B \setminus A \subseteq B$.

(iii) ⇒ (iv): apply (iii) to A and $X \setminus A$.

(iv) ⇒ (i): let $\mathcal{G}$ be a filter that is finer than $\mathcal{F}$ and let $A \in \mathcal{G}$ be arbitrary. Then $X \setminus A \notin \mathcal{F}$ because A intersects every element of $\mathcal{F}$ and it does not intersect $X \setminus A$. It follows that $A \in \mathcal{F}$. We see that $\mathcal{G} \subseteq \mathcal{F}$. □

► **7**. Prove: if $\mathcal{F}$ is an ultrafilter on X and $f : X \to Y$ a map then the image $f(\mathcal{F})$ is also an ultrafilter.

The following theorem now follows easily from Theorem 4.10.

4.14. Theorem. *In a compact space every ultrafilter converges.*

The converse is also true: if in a topological space every ultrafilter converges then this space is compact. The proof of this is not as easy as that of Theorem 4.10; making an ultrafilter is much harder than making just a finer filter. There is one kind of ultrafilter with a simple description.

4.15. Example. If $x \in X$ then $\mathcal{F}_x = \{A \subseteq X : x \in A\}$ is an ultrafilter.

This is the only type of ultrafilters that have an 'easy' description. The other ultrafilters — the free ultrafilters — are in some sense almost indescribable. To illustrate this we mention the following theorem, where we use the map $\phi : \mathcal{P}(\mathbb{N}) \to [0,1]$, defined $\phi(A) = \sum_{n \in A} 2^{-n}$, to transform an ultrafilter into a subset of $[0,1]$.

4.16. THEOREM. *Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Then the subset $U = \{\phi(A) : A \in \mathcal{U}\}$ of $[0,1]$ is not Lebesgue-measurable.*

The Axiom of Choice

In order to create ultrafilters we need something that enables us to get from 'finite' to 'infinite' in one step. That 'something' is the *Axiom of Choice*.

AXIOM OF CHOICE. If $\{X_t : t \in T\}$ is a nonempty family of nonempty sets then the product $\prod_{t \in T} X_t$ is nonempty as well.

This looks like it should be true (obviously), but that is not really the case. The point is that the axiom does not tell is how to *make* even *one single point* of $\prod_{t \in T} X_t$.

► **8**. Let $\mathcal{A}$ be the family of all nonempty subsets of $\mathbb{R}$. Exhibit one point of $\prod\{A : A \in \mathcal{A}\}$.

4.17. *Remark.* The Axiom of Choice is not a theorem. We shall not make any attempt at proving it and since the 1960's we know that that cannot be done, nor is it possible to prove that it is false. The following is an attempt at elucidating how this can be.

The original definition of a set was, in the words of Georg Cantor: Unter einer "Menge" verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unsrer Anschauung oder unseres Denkens (welche die "Elemente" von M genannt werden) zu einem Ganzen. In spite of this essentially empty definition a lot of useful work was done with sets but certain contradictions started to creep into the theory of sets: the entity $A = \{x : x \notin x\}$ is a set according to Cantor's definition but it gives rise to a contradiction: neither $A \in A$ nor $A \notin A$ can be true.

In the beginning of the twentieth century mathematicians started to think about what one can and what one cannot do with sets. The result was a list of rules (axioms) that one should abide by when dealing with sets.

For example, given two sets x and y one can form a new set that has precisely x and y as its elements: $\{x, y\}$. Another rule says that for every set x there is a set z such that $z = \bigcup x$. These two together allow us to form $x \cup y$ for any two sets: $x \cup y = \bigcup\{x, y\}$.

When you work within these (fairly natural) rules then you will not encounter weird entities like A above or 'the set of all sets' anymore; these simply cannot occur anymore.

The Axiom of Choice takes a special place among these axioms because it is, unlike the other ones, clearly not constructive. As in the case of Euclid's Fifth Postulate a lot of work went into attempts to derive the Axiom of Choice from the others as well into attempts at disproving it. As mentioned above these attempts were doomed to fail: adding the Axiom of Choice or its negation to the other axioms will not lead to contradictions.

The majority of mathematicians uses the Axiom of Choice without any reservations and we join them. If you want to know more: get a book on *Set Theory* from the Library and start reading.

We give two other statements that are equivalent to Axiom of Choice and that have proved very useful in mathematics. We need a few more definitions.

4.18. DEFINITION. Let X be a set. A *partial order* on X is a relation on X, written suggestively as $\preccurlyeq$, with the following three properties.

(i) $x \preccurlyeq x$ for every $x \in X$;

(ii) if $x \preccurlyeq y$ and $y \preccurlyeq x$ then $x = y$ for all x and y; and

(iii) if $x \preccurlyeq y$ and $y \preccurlyeq z$ then $x \preccurlyeq z$ for all x, y and z.

These properties are common to the relations $\leqslant$ on $\mathbb{R}$ and $\subseteq$ on families of sets.
The order $\leqslant$ on $\mathbb{R}$ has an extra property that $\subseteq$ misses:

4.19. DEFINITION. A *linear order* is a partial order with the following extra property: if $x, y \in X$ then $x \preccurlyeq y$ or $y \preccurlyeq x$.

The best we can have is a linear order as on $\mathbb{N}$.

4.20. DEFINITION. A *well-order* is a partial order with the property that every nonempty subset has a smallest element. (A well-ordered set is automatically a linearly ordered set.)

4.21. EXAMPLES.

1. Every family of sets is partially ordered by $\subseteq$.
2. The set $\mathbb{R}$ is linearly ordered by $\leqslant$.
3. The set $\mathbb{N}$ well-ordered by $\leqslant$.
4. Define $\preccurlyeq$ on $\mathbb{R}^2$ by $(x, y) \preccurlyeq (u, v)$ if and only if $x \leqslant u$ and $y \geqslant v$; then $\preccurlyeq$ is a partial order that is not linear.
5. Define $\preccurlyeq$ on $\mathbb{R}^2$ by $(x, y) \preccurlyeq (u, v)$ if and only if $x < u$ of $x = u$ and $y \leqslant v$; then $\preccurlyeq$ is a linear order on $\mathbb{R}^2$. It is called the *lexicographic order*.
6. The set $\mathbb{N}^2$ is well-ordered by the lexicographic order.

We get the following two statements.

WELLORDERING THEOREM. Every set can be well-ordered.

ZORN'S LEMMA. If X is a partially ordered set in which every linearly ordered subset has an upper bound then X has a *maximal element*, which is an element x such that there is no $y \neq x$ with $x \preccurlyeq y$.

Just like the Axiom of Choice the Well-ordering Theorem and Zorn's Lemma are not constructive; they do not say where the well-order or the maximal element come from. They just say that they are there.

The words 'theorem' and 'lemma' are used because the statements were derived from the Axiom of Choice. One can prove (using only the other axioms of set theory) that the Axiom of Choice, the Well-ordering Theorem and Zorn's lemma are equivalent.

Some of the consequences of the Axiom of Choice are:

I. Every vector space has a base.

II. In a ring with 1 every ideal is contained in a maximal ideal.

III. The Hahn-Banach Theorem: if V is a vector space, C a convex subset of V and W a subspace of V with $W \cap C = \varnothing$ then there is a subspace U of V of codimension 1 such that $W \subseteq U$ and $U \cap C = \varnothing$ (codimension 1 means that there is a vector x in $V \setminus U$ such that $U \cup \{x\}$ spans V).

We need the following theorem:

4.22. THEOREM (Ultrafilter Theorem). *Let X be a set and $\mathcal{F}$ a filter on X. Then there is an ultrafilter $\mathcal{U}$ on X that is finer than $\mathcal{F}$.*

Proof. We apply Zorn's Lemma. For this we consider the family

$$\mathfrak{F} = \{\mathcal{G} : \mathcal{G} \text{ is a filter that is finer than } \mathcal{F}\}.$$

The family $\mathfrak{F}$ is partially ordered by $\subseteq$. Let $\mathfrak{F}' \subseteq \mathfrak{F}$ be nonempty and linearly ordered by $\subseteq$ and put $\mathcal{G} = \bigcup \mathfrak{F}'$.

Clearly $\mathcal{H} \subseteq \mathcal{G}$ for every $\mathcal{H} \in \mathfrak{F}'$; we claim that $\mathcal{G}$ is a filter.

(1) $\varnothing \notin \mathcal{G}$ because $\varnothing \notin \mathcal{H}$ for every $\mathcal{H} \in \mathfrak{F}'$.

(2) If $G_1, G_2 \in \mathcal{G}$ then choose $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathfrak{F}'$ with $G_1 \in \mathcal{H}_1$ and $G_2 \in \mathcal{H}_2$. Now either $\mathcal{H}_1 \subseteq \mathcal{H}_2$ or $\mathcal{H}_2 \subseteq \mathcal{H}_1$ holds, for example the former. Then $G_1, G_2 \in \mathcal{H}_2$ and so $G_1 \cap G_2 \in \mathcal{H}_2$. But then $G_1 \cap G_2 \in \mathcal{G}$.

(3) If $G \in \mathcal{G}$ and $G \subseteq H$ then pick $\mathcal{H} \in \mathfrak{F}'$ with $G \in \mathcal{H}$; then $H \in \mathcal{H}$ and so $H \in \mathcal{G}$.

We see that $\mathcal{G}$ is a filter and hence an upper bound for $\mathfrak{F}'$.

The partially ordered set $\mathfrak{F}$ satisfies the assumptions of Zorn's Lemma, hence there is a maximal element $\mathcal{G}$. Then $\mathcal{G}$ is a filter, $\mathcal{G}$ is finer than $\mathcal{F}$ and every other filter that is finer than $\mathcal{G}$ belongs to $\mathfrak{F}$ and must be equal to $\mathcal{G}$.

Conclusion: $\mathcal{G}$ is an ultrafilter that is finer than $\mathcal{F}$. □

Now we can prove the converse of Theorem 4.14.

4.23. THEOREM. *If in a topological space every ultrafilter converges then the space is compact.*

Proof. All difficulties are behind us. By the Ultrafilter Theorem for every there is a finer ultrafilter that, by assumption, converges. Now apply Theorem 4.10. □

Tychonoff's Theorem

Now we can prove Tychonoff's Theorem.

4.24. THEOREM (Theorem). *A product of topological spaces is compact if and only if every factor is compact.*

Proof. The continuity of the projections ensures that every factor is compact if the whole product is.

Now assume every factor X_t of the product $X = \prod_{t \in T} X_t$ is compact. Let $\mathcal{F}$ be an ultrafilter on X. For every t the image filter $\pi_t(\mathcal{F})$ is an ultrafilter (Exercise 7) and hence convergent, say with limit x_t.

By Theorem 4.11 $\mathcal{F}$ converges to the point $(x_t)_{t \in T}$. □

We shall give another proof of Tychonoff's Theorem; this proof will be more involved but it will teach us a few new things. In the proof we reduce the case of a general open cover of the product to that of a cover by open strips.

Second proof of Tychonoff's Theorem. Let $\mathcal{U}$ be a family of finite open blocks in $X = \prod_{t\in T} X_t$ such that no finite subfamily of $\mathcal{U}$ covers X. Make a filter $\mathcal{F}$ as in Example 4.5, so a base for $\mathcal{F}$ is the family $\{X \setminus \bigcup \mathcal{U}' : \mathcal{U}' \text{ is a finite subfamily of } \mathcal{U}\}$. Let $\mathcal{G}$ be an ultrafilter that is finer than $\mathcal{F}$.

Let $U \in \mathcal{U}$; U is the intersection of finitely many open strips: $U = \bigcap_{i=1}^n \pi_{t_i}^{-1}[U_{t_i}]$. As $\mathcal{G}$ is an ultrafilter and $X \setminus U \in \mathcal{G}$ there must be a t_i with $X \setminus \pi_{t_i}^{-1}[U_{t_i}] \in \mathcal{G}$. It follows that for every $U \in \mathcal{U}$ we can find a strip U^+ such that $U \subseteq U^+$ and $X \setminus U^+ \in \mathcal{G}$.

Make such a simultaneous choice $U \mapsto U^+$ of strips (Axiom of Choice) and consider the family $\mathcal{U}^+ = \{U^+ : U \in \mathcal{U}\}$.

Because $U \subseteq U^+$ for all U we get $\bigcup \mathcal{U} \subseteq \bigcup \mathcal{U}^+$.

For every finite subfamily $\mathcal{V}$ of $\mathcal{U}^+$ we have $X \setminus \bigcup \mathcal{V} \in \mathcal{G}$, so no finite subfamily of $\mathcal{U}^+$ is a cover.

For $t \in T$ let $\mathcal{U}_t$ be the subfamily of $\mathcal{U}^+$ that consists of strips of the form $\pi_t^{-1}[O]$ with O open in X_t and let $\mathcal{V}_t = \{O : \pi_t^{-1}[O] \in \mathcal{U}_t\}$. Because no finite subfamily of $\mathcal{U}_t$ covers the product X no finite subfamily of $\mathcal{V}_t$ will cover the space X_t. Because X_t is compact $\mathcal{V}_t$ does not cover X_t.

Choose (simultaneously) $x_t \in X_t \setminus \bigcup \mathcal{V}_t$ for every t. The point $x = (x_t)_{t\in T}$ is not covered by $\mathcal{U}^+$ and hence certainly not by $\mathcal{U}$. □

This proof actually proves another theorem; to formulate that result we need another definition.

4.25. Definition. A *subbase* for a topology $\mathcal{T}$ is a subfamily $\mathcal{S}$ of $\mathcal{T}$ such that the family of intersections of finite subfamilies of $\mathcal{S}$ is a base for $\mathcal{T}$.

Note that $\varnothing$ is finite and a subfamily of every family; if $\mathcal{S}$ is a subbase then $\bigcap \varnothing$ will also belong to the corresponding base. Now

$$\bigcap \varnothing = \bigl\{x \in X : \forall S \in \varnothing\, [x \in S]\bigr\} = X,$$

so: whether $\mathcal{S}$ covers the space or not, the family of finite intersections is always a cover.

4.26. Examples.

1. The family of all open strips is a subbase for the product topology.
2. The family of all intervals of the form $(-\infty, b)$ and (a, ∞) is a subbase for the natural topology of $\mathbb{R}$.

A subbase is usually employed to create a topology out of a family of sets that one really wants to be open. For this one first takes all finite intersections of elements of the family and uses the family thus obtained as a base for the desired topology. This is in fact the way we constructed the product topology: the open strips needed to be open for the projections to be continuous.

The theorem alluded to above is Alexander's Subbase Lemma.

4.27. Theorem (Alexander's Subbase Lemma). *Let X be a topological space and $\mathcal{S}$ a subbase for the topology. Then: X is compact if and only if every open cover of X by elements of $\mathcal{S}$ has a finite subcover.*

Proof. The proof parallels that of our second proof of Tychonoff's Theorem: let $\mathcal{B}$ denote the family of all intersections of finite subfamilies of $\mathcal{S}$. Then $\mathcal{B}$ is a base. Let $\mathcal{U} \subseteq \mathcal{B}$ be a cover without finite subcover and make a filter $\mathcal{F}$ as before. Take an ultrafilter $\mathcal{G}$ that is finer than $\mathcal{F}$ and choose, using $\mathcal{G}$, for every $U \in \mathcal{U}$ a $S_U \in \mathcal{S}$ such that $U \subseteq S_U$ and $X \setminus S_U \in \mathcal{G}$.

Then $\{S_U : U \in \mathcal{U}\}$ is a cover of X by elements of $\mathcal{S}$ without a finite subcover. $\square$

This theorem enables us to give a very short proof that every closed and bounded interval in $\mathbb{R}$ is compact.

► **9**. Let $[a, b]$ be a closed and bounded interval in $\mathbb{R}$ and let

$$\mathcal{U} = \big\{[a, x_\lambda) : \lambda \in \Lambda\big\} \cup \big\{(y_\mu, b] : \mu \in M\big\}.$$

be a subbasic open cover of $[a, b]$.

a. Verify that Λ and M are not empty.

Let $x = \sup_\lambda x_\lambda$.

b. There must be a μ with $y_\mu < x$.

c. There must be a λ with $x_\lambda > y_\mu$.

d. $\big\{(x_\lambda, b], [a, y_\mu)\big\}$ is a finite subcover of $\mathcal{U}$.

Chapter 5

The Čech-Stone compactification

We now have all the ingredients that we need to make the Čech-Stone compactification. We first show how to make for every completely regular space a compact space of which it is a subspace. Next we will see how to recognize the Čech-Stone compactification among all other compactifications.

The construction

We already know that every subspace of every compact Hausdorff space is completely regular. We now prove the converse.

For a completely regular space X, we let $\mathcal{C}_X$ denote the set of all continuous functions from X to $[0,1]$.

5.1. Theorem. *Let X be a completely regular space. There are a compact Hausdorff space Y and an embedding $F : X \to Y$.*

Proof. We abbreviate $\mathcal{C}_X$ by $\mathcal{C}$. We take, for every $f \in \mathcal{C}$ a copy $[0,1]_f$ of $[0,1]$.

This gives us a family of maps $f : X \to [0,1]_f$, with its diagonal map

$$F = \mathop{\triangle}_{f\in\mathcal{C}} f : X \to [0,1]^{\mathcal{C}}.$$

Because F is a diagonal map of continuous maps, it is continuous. The space $[0,1]^{\mathcal{C}}$ is compact by Tychonoff's Theorem and Hausdorff by Exercise 9. It remains to prove that $F : X \to F[X]$ is a homeomorphism.

First: F is injective. For if $x \neq y$ in X then there is a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f(y) = 1$. Then the f-th coordinate of $F(x)$ is 0 and the f-th coordinate of $F(y)$ is 1, so $F(x) \neq F(y)$.

Second: U be open in X, to show $F[U]$ is open in $F[X]$ we take $x \in U$ and seek an open set O in $[0,1]^{\mathcal{C}}$ such that $F(x) \in O \cap F[X] \subseteq F[U]$. Take a continuous $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in X \setminus U$ and let $O = \pi_f^{-1}\bigl[[0,1)\bigr]$.

The f-th coordinate of $F(x)$, which is $f(x)$, is 0, so $F(x) \in O$. If y is such that $F(y) \in O$ then we must have $f(y) < 1$, hence $y \in U$; we find $O \cap F[X] \subseteq F[U]$. □

We have seen that a space is completely regular if and only if it can be embedded into a compact Hausdorff space. If $i : X \to Y$ is such an embedding then the points of Y that are not in the closure of $i[X]$ are not very important: such a point has a neighbourhood that is disjoint from $i[X]$.

This brings us to the following definition.

5.2. Definition. A *compactification* of a completely regular space X consists of compact Hausdorff space Y and an embedding $i : X \to Y$ such that $i[X]$ is a dense subset of Y.

In the situation of Definition 5.2 the spaces X and $i[X]$ are homeomorphic, so we never make a formal distinction between these two spaces and simply assume that X is a subspace of Y.

In Theorem 5.1 we implicitly constructed one compactification:

5.3. DEFINITION. Let X be a completely regular space and $\beta = \triangle_{f\in\mathcal{C}} f$ the embedding of X into $[0,1]^{\mathcal{C}_X}$.

The closure of $\beta[X]$ in $[0,1]^{\mathcal{C}_X}$ is a compactification of X; we call it the *Čech-Stone compactification* of X and we denote it as βX.

Properties of βX

We shall derive some properties of βX, such as its place among the other compactifications and some of its characterizations.

βX is the largest compactification

The word 'largest' in this sentence requires some explanation.

5.4. DEFINITION. Let Y and Z be two compactifications of the completely regular space X. We call Y a *larger* compactification of X than Z if there is a continuous map f from Y to Z such that $f(x) = x$ for all $x \in X$. We write $Y \trianglerighteq Z$ or $Z \trianglelefteq Y$.

► **1**. Prove: if $Y \trianglerighteq Z$ and $f : Y \to Z$ is the continuous map with $f(x) = x$ for $x \in X$ then $f[Y] = Z$.

The relation $\trianglerighteq$ is almost a partial order but not quite.

5.5. THEOREM. *Assume Y and Z are compactifications of X with $Y \trianglerighteq Z$ and $Z \trianglerighteq Y$. Then Y and Z are homeomorphic and the homeomorphism h can be chosen in such a way that $h(x) = x$ for all $x \in X$.*

Proof. There are continuous maps $f : Y \to Z$ and $g : Z \to Y$ with the properties that $f(x) = x$ and $g(x) = x$ for all $x \in X$.

Consider $f \circ g$, this is a map from Z to Z such that $(f \circ g)(x) = x$ for all x in X. But $f \circ g$ is continuous and X is dense in Z, so $(f \circ g)(z) = z$ for all $z \in Z$; see Theorem 2.11.

Likewise $(g \circ f)(y) = y$ for all $y \in Y$. The map f is the desired homeomorphism. □

We call two compactifications *equivalent* if the situation of the theorem above occurs. We never distinguish between equivalent compactifications.

Now we prove that βX is the largest compactification.

5.6. THEOREM. *Let Y be a compactification of the completely regular space X. Then $\beta X \trianglerighteq Y$.*

Proof. The point of the proof is that every continuous function from Y to $[0,1]$ determines a continuous function from X to $[0,1]$: its restriction to X.

For every $f \in \mathcal{C}_Y$ the restriction $f \restriction X$ belongs to $\mathcal{C}_X$. We can use this to make a projection from $[0,1]^{\mathcal{C}_X}$ onto $[0,1]^{\mathcal{C}_Y}$:

$$\pi_{X,Y} : (x_f)_{f\in\mathcal{C}_X} \mapsto (x_{f\restriction X})_{f\in\mathcal{C}_Y}.$$

(This looks pretty impressive but it is quite straightforward.)

Consider also the diagonal map $i = \triangle_{f\in\mathcal{C}_Y} : Y \to [0,1]^{\mathcal{C}_Y}$ and carefully check that $i(x) = \pi_{X,Y}\bigl(\beta(x)\bigr)$ for all $x \in X$.

Let F be the restriction of $\pi_{X,Y}$ to βX. We already know that $F\bigl(\beta(x)\bigr) = i(x)$ for all $x \in X$ (upon identification of X, $\beta[X]$ and $i[X]$ this means $F(x) = x$ for all x).

It remains to prove that $F[\beta X] \subseteq i[Y]$ (Exercise 1 will then imply $F[\beta X] = i[Y]$.

So let $x \in \beta X$ and consider the filter $\mathcal{G} = \{U \cap X : U \in \mathcal{U}_x\}$ on X. In βX this generates a filter $\mathcal{G}^\beta$ that converges to x. The image filter $F(\mathcal{G}^\beta)$ converges, in $[0,1]^{\mathcal{C}_Y}$, to $F(x)$; but $X \in \mathcal{G}$, so $F(x) \in \operatorname{cl} F[X] = \operatorname{cl} i[X]$. Thus $F[\beta X] \subseteq \operatorname{cl} i[X]$ To finish we note that $i[Y]$ is compact, hence closed in $[0,1]^{\mathcal{C}_Y}$. We find that $\operatorname{cl} i[X] \subseteq i[Y]$. □

This gives us the first characterization of βX: if Y is a compactification of X such that $Y \trianglerighteq Z$ for every other compactification of X then Y is (equivalent with) βX.

Extending functions

The next theorem gives us one of the most important characterizations of βX. To avoid misunderstanding: unless explicitly mentioned otherwise, X is always assumed to be a completely regular space.

5.7. Theorem. *If f is a continuous function from X to $[0,1]$ then there exists a continuous function $\beta f : \beta X \to [0,1]$ such that $\beta f \restriction X = f$.*

Conversely: every compactification with this extension property is equivalent to βX.

Proof. The first part is easy: restrict the projection onto the f-th coordinate to βX.

The second part is also not very difficult. Let Y be a compactification with the extension property. Denote, for every $f \in \mathcal{C}_X$ the (unique!) extension of f to Y by $\hat{f}$.

Verify that the diagonal map $\triangle_{f\in\mathcal{C}_X} \hat{f} : Y \to [0,1]^{\mathcal{C}_X}$ maps Y onto βX. This shows that $Y \trianglerighteq \beta X$ and hence that Y and βX are equivalent. □

The next theorem is proved by rescaling closed intervals in $\mathbb{R}$ down to $[0,1]$.

5.8. Theorem. *If $f : X \to \mathbb{R}$ is a bounded continuous function then f has a continuous extension $\beta f : \beta X \to \mathbb{R}$.*

By checking the proof of Theorem 5.6 carefully we get the following theorem.

5.9. Theorem. *Let $f : X \to K$ be a continuous map, where K is a compact Hausdorff space. Then f has a continuous extension $\beta f : \beta X \to K$.*

We can summarize the results above in one theorem.

5.10. Theorem. *Let X be a completely regular space and Y a compactification of X. Then the following are equivalent.*

(i) $Y = \beta X$.
(ii) *For every continuous function $f : X \to [0,1]$ there is a continuous extension $\hat{f} : Y \to [0,1]$.*
(iii) *For every bounded continuous function $f : X \to \mathbb{R}$ there is a continuous extension $\hat{f} : Y \to \mathbb{R}$.*
(iv) *For every continuous map $f : X \to K$, where K is compact Hausdorff, there is a continuous extension $\hat{f} : Y \to K$.*

We can characterize normal spaces by the way they are embedded in βX.

5.11. THEOREM. *A completely regular space X is normal if and only if for each pair of disjoint closed sets F and G in X their closures in βX are disjoint.*

Proof. Sufficiency is clear.

For the proof of necessity we use Urysohn's Lemma: choose a continuous function $f : X \to [0,1]$ with $f \restriction F \equiv 0$ and $f \restriction G \equiv 1$ and consider the values of βf on the closures of F and G. □

For normal spaces we can characterize βX in terms of closed sets. The following exercise may come in useful in the following proof.

► **2**. Let D be a dense subset of a space X. Then $\operatorname{cl} O = \operatorname{cl}(U \cap D)$ for every open subset O of X.

5.12. THEOREM. *Let X be a normal space and Y a compactification of X. Then the following are equivalent.*

(i) $Y = \beta X$.

(ii) *For any two closed subsets F and G of X we have* $\operatorname{cl}_Y F \cap \operatorname{cl}_{\beta X} G = \operatorname{cl}_Y(F \cap G)$.

(iii) *For any two disjoint closed subsets F and G of X we have* $\operatorname{cl}_Y F \cap \operatorname{cl}_Y G = \varnothing$.

Proof. We already know that (i) ⇒ (iii).

We prove that (iii) ⇒ (ii). Assume $y \notin \operatorname{cl}_Y(F \cap G)$.

Choose a neighbourhood U of y and a neighbourhood V of $\operatorname{cl}_Y(F \cap G)$ such that $U \cap V = \varnothing$. The sets $F \setminus V$ and $G \setminus V$ are closed in X and disjoint, hence their closures in Y are disjoint. So, for example, $y \notin \operatorname{cl}_Y(F \setminus V)$. But then also $y \notin \operatorname{cl}_Y F$.

Finally we show that (ii) ⇒ (i) by showing that $Y \trianglerighteq \beta X$. We define $f : Y \to \beta X$ as follows: let $y \in Y$ and consider the filter $\mathcal{F}_y = \{U \cap X : U \in \mathcal{U}_y\}$ on X.

We claim that there is exactly one point in $\bigcap\{\operatorname{cl}_{\beta X} F : F \in \mathcal{F}_y\}$. Assume x and z are distinct points in the intersection and choose neighbourhoods U of x and V of z with disjoint closures. In X we have: $F \cap U \neq \varnothing$ and $F \cap V \neq \varnothing$ for all $F \in \mathcal{F}_y$. This holds in Y as well and so $y \in \operatorname{cl}_Y U \cap \operatorname{cl}_Y V$. But $\operatorname{cl} U \cap \operatorname{cl} V = \varnothing$ in X, a contradiction.

We define $f(y)$ to be the unique point in $\bigcap\{\operatorname{cl}_{\beta X} F : F \in \mathcal{F}_y\}$. Observe that $x \in F$ for all $F \in \mathcal{F}_x$ and hence $f(x) = x$ for all $x \in X$.

It remains to prove that f is continuous. Let $y \in Y$ and let $O \ni f(y)$ be open. Choose an open neighbourhood U of $f(y)$ with $\operatorname{cl}_{\beta X} U \subseteq O$ and then an open neighbourhood V of y such that $\operatorname{cl}_{\beta X}(V \cap X) \subseteq U$ (why is this possible?). Now show $f[V] \subseteq \operatorname{cl}_{\beta X} U$. □

For an arbitrary completely regular spaces we can do something similar where we replace closed sets by zero-sets.

5.13. DEFINITION. A subset A of a topological X is called a *zero-set* if there is a continuous function $f : X \to [0,1]$ with $A = \{x : f(x) = 0\}$. The complement of a zero-set is called a *cozero-set*[1]

We get the following theorem.

5.14. THEOREM. *Let X be a completely regular space and Y a compactification of X. The following are equivalent:*

[1]Engelking [1989] calls these sets *functionally closed* and *functionally open* respectively.

(i) $Y = \beta X$.

(ii) *For any two zero-sets F and G of X we have* $\mathrm{cl}_Y(F \cap G) = \mathrm{cl}_Y F \cap \mathrm{cl}_Y G$.

(iii) *For any two disjoint zero-sets F and G of X we have* $\mathrm{cl}_Y F \cap \mathrm{cl}_Y G = \varnothing$.

► **3**. a. Prove: if x is a point in a completely regular space X and U is a neighbourhood of x then there is a zero-set F such that $x \in \operatorname{int} F$ and $F \subseteq U$.

b. Prove: if F and G are disjoint zero-sets then there is a continuous function $f : X \to [0,1]$ with $F = \{x : f(x) = 0\}$ and $G = \{x : f(x) = 1\}$.

c. Prove Theorem 5.14.

Chapter 6

Applications in combinatorics

In this chapter we show how ultrafilters and $\beta\mathbb{N}$ can be used to prove combinatorial results. We shall give proofs of three well-known theorems. Each of these says, in its own way: *if you put infinitely many things in finitely many pots then (at least) one of these will be very full.*

Ramsey's Theorem

For Ramsey's Theorem we need some notation and a definition. For a set X and $r \in \mathbb{N}$ we let $[X]^r$ denote the family of all r element subsets of X.

6.1. Definition. The expression

$$\omega \to (\omega)^r_m$$

means: If we divide $[\mathbb{N}]^r$ into m pieces $I_1, \ldots, I_m$ then then are an infinite subset X of $\mathbb{N}$ and an $i \leqslant m$ such that $[X]^r \subseteq I_i$.

Here ω is the cardinal number of $\mathbb{N}$ (the number of elements, countably infinite). We could have put natural numbers, M and k say, in place of the two ω's. The meaning remains the same, with $\mathbb{N}$ replaced by $\{0, \ldots, M-1\}$ and 'infinite' by 'k element'.

Intuitively the formula $M \to (k)^r_m$ says this: if we colour each choice of r elements from $M = \{0, 1, \ldots, M-1\}$ with one of m colours then there is a subset A of M of size k such that all choices from A have the same colour. A set like A is called a *homogeneous set* for the colouring.

▶ **1**. Prove that $6 \to (3)^2_2$.

Ramsey's Theorem says that the formula from Definition 6.1 always holds.

6.2. Theorem (Ramsey, infinite version). *For every r and m in $\mathbb{N}$ the formula*

$$\omega \to (\omega)^r_m.$$

holds.

Proof. We prove this by induction on r. For $r = 1$ there is nothing to prove: if $\mathbb{N} = I_1 \cup \cdots \cup I_m$ then at least one of the sets I_i is infinite.

The step from r to $r+1$ will be a bit more complicated. Assume $[\mathbb{N}]^{r+1} = I_1 \cup \cdots \cup I_m$. We shall divide $[\mathbb{N}]^r$ into m pieces and apply the induction hypothesis to this. To do this we choose a free ultrafilter u on $\mathbb{N}$.

For $F = \{x_1, \ldots x_r\} \in [\mathbb{N}]^r$ (with $x_1 < \cdots < x_r$) and define $A_{F,i}$ (for $1 \leqslant i \leqslant m$) as follows:

$$A_{F,i} = \bigl\{n > x_r : \{x_1, \ldots, x_r, n\} \in I_i\bigr\}.$$

Then $\bigcup_i A_{F,i} = \{n : n > x_r\}$, so there is an i such that $A_{F,i} \in u$. Let i_F be the smallest index with this property.

We make a strictly increasing sequence $\langle h_n \rangle_n$ in $\mathbb{N}$, as follows: for $n \leqslant r$ put $h_n = n$. Once $h_1, h_2, \ldots, h_n$ are found consider the intersection of all sets A_{F,i_F} with $F \in \bigl[\{h_1, \ldots, h_n\}\bigr]^r$; this set belongs to u and hence it is infinite. Choose h_{n+1} in this intersection. In the end put $H = \{h_n : n \in \mathbb{N}\}$.

Let $F \in [H]^r$. Then all $r+1$-element subsets from H that have F as their initial segment belong to the same set: namely I_{i_F}. We use this to divide $[H]^r$ in m pieces: F goes into J_{i_F}.

The induction hypothesis gives us an infinite subset K of H and an $i \leqslant m$ such that $i_F = i$ for all $F \in [K]^r$; but that means that every $r+1$-element subset from K belongs to I_i. □

There is also a finite form of Ramsey's Theorem.

6.3. Theorem (Ramsey, finite version). *For every k, r and m there is a natural number M such that*

$$M \to (k)^r_m.$$

Proof. Assume there are r, k and m for which no such M exists. So, for every M there is a colouring $[M]^r = I_1^M \cup \cdots \cup I_m^M$ without a homogeneous set with k elements.

For each such colouring we make a colouring of $[\mathbb{N}]^r$: if $\max F < M$ then it keeps its colour, otherwise F gets colour 0.

This defines a function $x_M : [\mathbb{N}]^r \to \{0, 1, \ldots, m\}$, which is also a point in the product of $[\mathbb{N}]^r$ many factors $\{0, 1, \ldots, m\}$. This product is a compact and metrizable space. The sequence $\langle x_M \rangle_M$ has a converging subsequence $\langle x_{M_p} \rangle_p$, say with limit x. This point determines a colouring of $[\mathbb{N}]^r$ with only the colours $1, \ldots, m$.

This colouring has no k-element homogeneous set, leave alone an infinite one. This contradicts Theorem 6.2. □

► **2**. Prove: for every $k \in \mathbb{N}$ we have $\binom{2k-2}{k-1} \to (k)^2_2$.

Hindman's Theorem

Hindman's Theorem reads as follows:

6.4. Theorem (Hindman). *If $\mathbb{N} = A_1 \cup \cdots \cup A_n$ then there are an i and an infinite subset B of A_i such that for every finite number of different elements $b_1, \ldots, b_p$ of B the sum $b_1 + \cdots + b_p$ belongs to A_i.*

The proof of this theorem requires a fair amount of machinery. It will give us the opportunity to get better acquainted with $\beta\mathbb{N}$.

$\beta\mathbb{N}$ as a space of ultrafilters

To begin we show that the points of $\beta\mathbb{N}$ are, in fact, nothing but the ultrafilters on $\mathbb{N}$. To see this we must realize that $\mathbb{N}$ is normal and that every subset of $\mathbb{N}$ is closed (with respect to the natural topology). From this point on $\operatorname{cl} A$ always denotes the closure of A in $\beta\mathbb{N}$.

6.5. LEMMA. *Let $x \in \beta\mathbb{N}$. Then $u_x = \{A \subseteq \mathbb{N} : x \in \operatorname{cl} A\}$ is an ultrafilter on $\mathbb{N}$.*

Proof. Because $\operatorname{cl}(A \cap B) = \operatorname{cl} A \cap \operatorname{cl} B$ for any two subsets of $\mathbb{N}$, it follows at once that $A \cap B \in u_x$ if $A, B \in u_x$. Certainly if $A \in u_x$ and $A \subseteq B$ then $B \in u_x$. Also $\operatorname{cl} \varnothing = \varnothing$, so $\varnothing \notin u_x$. We see that u_x is a filter.

Because $A \cap (\mathbb{N} \setminus A) = \varnothing$ and $\operatorname{cl} A \cup \operatorname{cl}(\mathbb{N} \setminus A) = \beta\mathbb{N}$ we find that either $x \in \operatorname{cl} A$ or $x \in \operatorname{cl}(\mathbb{N} \setminus A)$. It follows that u_x is an ultrafilter. □

The converse is also true.

6.6. LEMMA. *If u is an ultrafilter on $\mathbb{N}$ then $\bigcap_{A \in u} \operatorname{cl} A$ consists of exactly one point x_u.*

Proof. If $x \neq y$ in $\beta\mathbb{N}$ then choose neighbourhoods $U \ni x$ and $V \ni y$ with $U \cap V = \varnothing$. Then, for example, $U \cap \mathbb{N} \notin u$ and so $\mathbb{N} \setminus U \in u$. But then $x \notin \operatorname{cl}(\mathbb{N} \setminus U)$ and so $x \notin \bigcap_{A \in u} \operatorname{cl} A$. □

6.7. THEOREM. *The operations from the previous two lemma's are each other's inverses: $x = x_{u_x}$ and $u = u_{x_u}$ for every $x \in \beta\mathbb{N}$ and every ultrafilter u on $\mathbb{N}$.*

Proof. Left to the reader. □

There is also a canonical base for $\beta\mathbb{N}$.

6.8. THEOREM. *The family $\{\operatorname{cl} A : A \subseteq \mathbb{N}\}$ is a base for the topology of $\beta\mathbb{N}$.*

Proof. To begin: because $\operatorname{cl} A \cap \operatorname{cl}(\mathbb{N} \setminus A) = \varnothing$ and $\operatorname{cl} A \cup \operatorname{cl}(\mathbb{N} \setminus A) = \beta\mathbb{N}$ the set $\operatorname{cl} A$ is open, being the complement of a closed set.

Now let $x \in \beta\mathbb{N}$ and let O be a neighbourhood of x. Take an open neighbourhood V of x with $\operatorname{cl} V \subseteq U$ and let $A = \mathbb{N} \cap V$. Then $\operatorname{cl} A = \operatorname{cl} V$, so $x \in \operatorname{cl} A \subseteq U$. □

The considerations above lead to an alternative description of $\beta\mathbb{N}$: the underlying set is the set of all ultrafilters on $\mathbb{N}$. For $A \subseteq \mathbb{N}$ we define $\bar{A} = \{u : A \in u\}$. The family $\{\bar{A} : A \subseteq \mathbb{N}\}$ serves as a base for the topology of $\beta\mathbb{N}$.

► **3**. Verify that the operations from Lemmas 6.5 and 6.6 transform $\operatorname{cl} A$ into $\bar{A}$ and conversely.

► **4**. What ultrafilters correspond to the points of $\mathbb{N}$?

Adding ultrafilters

For the proof of Hindman's Theorem we need to know how to add the elements of $\beta\mathbb{N}$.

This in fact quite straightforward: let $n \in \mathbb{N}$ and consider the map ρ_n from $\mathbb{N}$ to $\mathbb{N}$ defined by $\rho_n(m) = m + n$. We can extend this to a map $\beta\rho_n : \beta\mathbb{N} \to \beta\mathbb{N}$. We simply write $u + n$ instead of $\beta\rho_n(u)$ for $u \in \beta\mathbb{N}$.

Next we take, for a fixed $u \in \beta\mathbb{N}$, the map $\lambda_u : \mathbb{N} \to \beta\mathbb{N}$ defined by $\lambda_u(n) = u + n$; this map we also extend to a map $\beta\lambda_u$ and we simply write $u + v$ for $\beta\lambda_u(v)$.

We also write ρ_n and λ_u instead of $\beta\rho_n$ and $\beta\lambda_u$ respectively.

This addition is associative.

6.9. THEOREM. *For every three elements u, v and w of $\beta\mathbb{N}$ we have $u + (v + w) = (u + v) + w$.*

Proof. We use the continuity of the maps λ. First observe that

$$u + (v + w) = \lambda_u(v + w) = \lambda_u\bigl(\lambda_v(w)\bigr) = (\lambda_u \circ \lambda_v)(w)$$

and

$$(u + v) + w = \lambda_{u+v}(w).$$

So by continuity of the λs it suffices to prove that $u + (v + n) = (u + v) + n$ for all $n \in \mathbb{N}$.

Fix n. Now we have

$$u + (v + n) = \lambda_u\bigl(\rho_n(v)\bigr) = (\lambda_u \circ \rho_n)(v)$$

and

$$(u + v) + n = \rho_n\bigl(\lambda_u(v)\bigr) = (\rho_n \circ \lambda_u)(v).$$

Because ρ_n is also continuous, it suffices to show that $(\lambda_u \circ \rho_n)(m) = (\rho_n \circ \lambda_u)(m)$ for every m. But $(\lambda_u \circ \rho_n)(m) = u + (m + n)$ and $(\rho_n \circ \lambda_u)(m) = (u + m) + n$. We also fix m and recalculate $u + (m + n)$ and $(u + m) + n$:

$$u + (m + n) = \rho_{m+n}(u)$$

and

$$(u + m) + n = \rho_n\bigl(\rho_m(u)\bigr) = (\rho_n \circ \rho_m)(u).$$

We see that we must check that $\rho_{m+n}(l) = (\rho_n \circ \rho_m)(l)$ for all l. But this is easy because $\rho_{m+n}(l) = l + (m + n) = (l + m) + n = (\rho_n \circ \rho_m)(l)$. □

► **5**. Prove that $u + n = n + u$ for every $n \in \mathbb{N}$ and every $u \in \beta\mathbb{N}$.

Unfortunately $+$ is not commutative on $\beta\mathbb{N}$.

Let $T = \{2^n : n \in \mathbb{N}\}$ and choose two distinct free ultrafilters u and v with $T \in u$ and $T \in v$. Also choose disjoint sets $A, B \subseteq T$ with $A \in u$ and $B \in v$.

► **6**. Prove $u + v \neq v + u$.

a. $(T + n) \cap (T + m)$ is finite when $n \neq m$.

Define $A_n = (A + n) \setminus \bigcup_{i<n}(T + i)$ and similarly $B_n = (B + n) \setminus \bigcup_{i\leqslant n}(T + i)$.

b. $A_n \cap A_m = \varnothing$, $A_n \cap B_m = \varnothing$ and $B_n \cap B_m = \varnothing$ for all n and m.

c. $u + n \in \operatorname{cl} A_n$ and $v + n \in \operatorname{cl} B_n$ for all n.

Let $A^+ = \bigcup_n A_n$ and $B^+ = \bigcup_n B_n$.

d. $u + v \in \operatorname{cl}\{u + n : n \in \mathbb{N}\} \subseteq \operatorname{cl} A^+$.

e. $v + u \in \operatorname{cl}\{v + n : n \in \mathbb{N}\} \subseteq \operatorname{cl} B^+$.

f. $u + v \neq v + u$.

► **7**. a. Prove: if ρ_u is continuous then $u + v = v + u$ for all v.

b. Deduce that, with T as in the previous exercise, the map ρ_u is not continuous for any $u \in \operatorname{cl} T \setminus \mathbb{N}$.

Idempotents $\beta\mathbb{N} \setminus \mathbb{N}$

We concentrate on the points in $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$. We will look for a point $u \in \mathbb{N}^*$ with $u = u + u$; such a point we call an *idempotent ultrafilter*.

We can find such a point in very general situations.

6.10. DEFINITION. A *semigroup* is a set with an associative binary operation.

6.11. EXAMPLES.

1. $\mathbb{N}$ with $+$ is a semigroup.
2. $\{n \in \mathbb{N} : n \geqslant 25$ and n is even$\}$ is with the operation $+$ also a semigroup.
3. $\mathbb{N}^*$ with the operation $+$ is a semigroup.

6.12. DEFINITION. Let X be a compact Hausdorff space with an associative binary operation $*$. We call X a *semi-topological semigroup* if for every $x \in X$ the map $\lambda_x : X \to X$ defined by $\lambda_x(y) = x * y$ is continuous.

The semigroup $\mathbb{N}^*$ is semi-topological. The next theorem says that we can expect to find many idempotent elements

6.13. THEOREM. *Every semi-topological semigroup X has an idempotent element.*

Proof. We shall apply Zorn's Lemma to the collection

$$\mathcal{A} = \{A \subseteq X : A \text{ is closed and nonempty and } A * A \subseteq A\}.$$

The family $\mathcal{A}$ is nonempty: $X \in \mathcal{A}$. If $\mathcal{A}'$ is a chain in $\mathcal{A}$ then $A = \bigcap \mathcal{A}'$ is nonempty (by compactness) and closed. Take $x, y \in A$; for every $B \in \mathcal{A}'$ we have $x, y \in B$ and hence also $x * y \in B$, we find that $x * y \in A$.

By Zorn's Lemma (applied upside-down) there is a *minimal* element A in $\mathcal{A}$. We show that A consists of one point only.

Let $x \in A$ and consider $x * A = \lambda_x[A]$. This set is closed (λ_x is continuous and X is compact Hausdorff) and a subset of A. We show $x * A \in \mathcal{A}$; take $y, z \in A$, then $(x * y) * (x * z) = x * (y * x * z) \in x * A$. We get $x * A = A$ by minimality of A. In particular there is a $y \in A$ with $x * y = x$.

Consider $C = \{y \in A : x * y = x\}$. We have just seen that C is nonempty, C is closed because λ_x is continuous. If $y, z \in C$ then also $x * (y * z) = (x * y) * z = x * z = x$, so $y * z \in C$. We find that $C \in \mathcal{A}$ and by minimality of A we get $C = A$. But then, in particular, $x * x = x$ and, by minimality again $A = \{x\}$. □

6.14. COROLLARY. *There is a $u \in \mathbb{N}^*$ with $u + u = u$.*

The proof of the theorem

Now we can prove Hindman's Theorem. First a bit of notation. For $A \subseteq \mathbb{N}$ and $k \in \mathbb{N}$ we write

$$A - k = \{n : n + k \in A\}.$$

► **8**. If $u \in \beta\mathbb{N}$ and $k \in \mathbb{N}$ then $u + k = \{A : A - k \in u\}$.

The next proposition finishes the proof of Hindman's Theorem.

6.15. PROPOSITION. *Let $u \in \mathbb{N}^*$ be an idempotent ultrafilter and $A \in u$. Then A has an infinite subset B, such that every finite sum of elements of B belongs to A.*

Proof. For each element C of u we write $C^{\#} = \{n : C - n \in u\}$. Verify that $C^{\#} \in u$ for every $C \in u$ (use that $u + u = u$).

We make a sequence of elements of u and a sequence of natural numbers as follows. Put $A_0 = A$ and choose $b_0 \in A_0 \cap A_0^{\#}$.

When A_n and $b_n \in A_n \cap A_n^{\#}$ are found we put $A_{n+1} = A_n \cap (A_n - b_n)$ and we choose $b_{n+1} \in A_{n+1} \cap A_{n+1}^{\#}$ larger than b_n. Observe that $A_{n+1} \in u$ because $b_n \in A_n \cap A_n^{\#}$.

In the end put $B = \{b_n : n \in \mathbb{N}\}$. We prove: if $n_0 < n_1 < \cdots < n_k$ then $b_{n_0} + b_{n_1} + \cdots + b_{n_k} \in A_{n_0}$. We use induction on k. If $k = 0$ then it says $b_{n_0} \in A_{n_0}$, which holds by construction.

If $k > 0$ then, by the inductive assumption, $b = b_{n_1} + \cdots + b_{n_k} \in A_{n_1}$. Further $A_{n_1} \subseteq A_{n_0+1} \subseteq A_{n_0} - b_{n_0}$, so $b \in A_{n_0} - b_{n_0}$, in other words $b_{n_0} + b \in A_{n_0}$. □

Use the ideas in the proof of Theorem 6.3 to prove the following theorem.

6.16. Theorem. *For every two natural numbers k and l there is a natural M such that whenever $M = A_1 \cup \cdots \cup A_k$ then there are an $i \leqslant n$ and a subset B of A_i with l elements with the property that the property that every sum of different elements of B belongs to A_i.*

Van der Waerden's Theorem

Van der Waerden's Theorem reads as follows.

6.17. Theorem. *If $\mathbb{N} = A_1 \cup \cdots \cup A_n$ then there is an $i \leqslant n$ such that A_i contains arbitrarily long arithmetic progressions.*

An *arithmetic progression* is a sequence of the form $\{a + bi : i \leqslant n\}$.

Again we shall use idempotent elements in semigroups to get the result. We shall need a special kind of idempotents.

Minimal idempotent elements in semigroups

Let X be a semi-topological semigroup (X is compact Hausdorff). We define a partial order on the idempotent elements.

6.18. Definition. Let x and y be idempotent elements of X and define

$$x \preccurlyeq y \text{ is and only if } x = x * y = y * x.$$

6.19. Lemma. *The relation $\preccurlyeq$ is a partial order on the set of idempotent elements of X.*

Proof. Clearly $x \preccurlyeq x$. If $x \preccurlyeq y$ and $y \preccurlyeq x$ then $x = x * y = y * x = y$. If $x \preccurlyeq y$ and $y \preccurlyeq z$ then $x = x * y = x * (y * z) = (x * y) * z = x * z$ and $x = y * x = (y * z) * x = (z * y) * x = z * (y * x) = z * x$ and so $x \preccurlyeq z$. □

We shall prove that for every idempotent x there is a minimal idempotent y (with respect to $\preccurlyeq$) with $y \preccurlyeq x$. It follows that semigroups as $\mathbb{N}^*$ have minimal idempotent elements. We shall also show that every element of every minimal idempotent ultrafilter contains arbitrarily long arithmetic progressions. This then will complete the proof of Van der Waerden's Theorem.

We need to investigate a special kind of subsets of semigroups.

6.20. DEFINITION. A subset I of a semigroup X is called a *left ideal* (*right ideal*) if for every $x \in X$ we have $x * I \subseteq I$ ($I * x \subseteq I$). An ideal that is both a left and a right ideal is called a *two-sided ideal*.

► **9**. For every $x \in X$ the set $x * X = \{x * y : y \in X\}$ is always a right ideal and $X * x$ is always a left ideal.

► **10**. $\mathbb{N}^*$ is a two-sided ideal in $\beta\mathbb{N}$.

We need minimal right ideals. These exist in abundance. We assume tacitly that every ideal under consideration is nonempty.

6.21. LEMMA. *Every right ideal contains a minimal right ideal.*

Proof. Let R be a right ideal. For any $x \in R$ the set $x * X = \lambda_x[X]$ is a closed right ideal and $x * X \subseteq R$, so every right ideal contains a *closed* right ideal.

As in the proof of Theorem 6.13 we take a minimal element A in the family $\mathcal{R}$ of all closed right ideals that are contained in R.

Then A is also a minimal right ideal: any smaller ideal would contain another closed right ideal. □

The proof of this lemma gives us some more information.

6.22. PROPOSITION. *Every minimal right ideal is closed and $x * X = x * R = R$ for all $x \in R$.*

Proof. That R is closed is clear. The second part of the proposition follows from the facts that $x * R \subseteq x * X \subseteq R$ and that $x * R$ is a right ideal: if $y \in R$ and $z \in X$ then $(x * y) * z = x * (y * z) \in x * R$. □

We can recognize minimal idempotents by their position in the semigroup.

6.23. THEOREM. *The following are equivalent for an idempotent element x.*

(i) *x is minimal.*

(ii) *x belongs to a minimal right ideal.*

(iii) *$x * X$ is a minimal right ideal.*

Proof. (i) ⇒ (iii): We know that $x * X$ is a right ideal. Assume $R \subseteq x * X$ is a minimal right ideal. Then R is a semi-topological semigroup and so it contains an idempotent element y. Put $z = y * x$. Then $z \in R$ because R is a right ideal. We show that z is an idempotent element and $z \preccurlyeq x$. Choose $r \in X$ with $x * r = y$ (because $R \subseteq x * X$). Then $z = y * x = x * r * x$ and so

$$z * z = (y * x) * (x * r * x) = y * x * r * x = y * z = y * y * x = y * x = z.$$

Next

$$z * x = y * x * x = y * x = z$$

and

$$x * z = x * x * r * x = x * r * x = z,$$

hence $z \preccurlyeq x$. But this implies $z = x$ and so $x * X \subseteq R$ (because $x = z \in R$).

(iii) ⇒ (ii): This follows because $x * x = x$ and so $x \in x * X$.

(ii) $\Rightarrow$ (i): Let R be a minimal right ideal with $x \in R$ and assume $y \preccurlyeq x$. Because $y = x * y$ it follows that $y \in R$ and also $y * X = R$. Choose $r \in X$ with $x = y * r$, then $x = y * r = y * y * r = y * x = y$. □

Now we can prove that there are enough minimal idempotent elements:

6.24. Theorem. *For every idempotent element x there is a minimal idempotent element y with $y \preccurlyeq x$.*

Proof. Take a minimal right ideal $R \subseteq x * X$ and an idempotent y in R. Take $z = y * x$ and follow the proof of (i) $\Rightarrow$ (iii) in Theorem 6.23; z is idempotent, $z \in R$ so z is minimal and $z \preccurlyeq x$. □

The final ingredient that we need is the following theorem.

6.25. Theorem. *If R is a minimal right ideal and I some two-sided ideal then $R \subseteq I$.*

Proof. First take $x \in R$ and $y \in I$, then $x * y \in R \cap I$ because R is a right ideal and I is a two-sided (hence left) ideal. So $R \cap I$ is a nonempty right ideal; but R is minimal so $R \subseteq R \cap I$. □

In particular every two-sided ideal contains every minimal idempotent elements. This is exactly the fact that we need for the proof of Van der Waerden's Theorem.

The proof of the theorem

We have seen that it suffices to prove the following proposition.

6.26. Proposition. *If $u \in \mathbb{N}^*$ is a minimal idempotent element then every element of u contains arbitrarily long arithmetic progressions.*

Let $k \in \mathbb{N}$. We prove in a sequence of lemmas that every element of u contains an arithmetic progression of length k.

We work in the semigroup $S = \beta\mathbb{N}^k$ and consider the following subsets:

(i) $E^- = \bigl\{\bigl(a, a+d, \ldots, a+(k-1)d\bigr) : a, d \in \mathbb{N}\bigr\}$,
(ii) $I^- = \bigl\{\bigl(a, a+d, \ldots, a+(k-1)d\bigr) : a, d \in \mathbb{N}, d > 0\bigr\}$,
(iii) $E = (\mathbb{N}^*)^k \cap \operatorname{cl} E^-$ and
(iv) $I = (\mathbb{N}^*)^k \cap \operatorname{cl} I^-$.

(We take closures in $\beta\mathbb{N}^k$.) Observe that I^- contains all 'true' arithmetic progressions and E^- also the constant ones. Also observe that we only have to show that the point $\vec{u} = (u, \ldots, u)$ belongs to I. For, in that case whenever $A \in u$ the set A^k is a neighbourhood of $\vec{u}$, this intersects I^- and every point in the intersection gives us an arithmetic progression.

This we do as follows:

We prove that E is a semi-topological semigroup, that I is a two-sided ideal of E and that $\vec{u}$ is a minimal idempotent element of E. Application of Theorem 6.25 gives the desired result

6.27. Lemma. *E is a semi-topological semigroup and I is a two-sided ideal of E.*

Proof. Because E is closed S it is compact.

Let $\vec{p}, \vec{q} \in E$. We prove that $\vec{p}+\vec{q} \in E$ and show simultaneously that $\vec{p}+\vec{q} \in I$ when $\vec{p} \in I$ or $\vec{q} \in I$.

Let U be a neighbourhood of $\vec{p}+\vec{q}$ and choose, because $\lambda_{\vec{p}}$ is continuous, a neighbourhood V of $\vec{q}$ such that $\lambda_{\vec{p}}[V] \subseteq U$. Choose a and d in $\mathbb{N}$ such that $\vec{x} = \bigl(a, \ldots, a + (k-1)d\bigr) \in V$ (if $\vec{q} \in I$ we choose $d > 0$). This shows $\vec{p} + \vec{x} \in U$. Continuity of $\rho_{\vec{x}}$ gives us a neighbourhood O of $\vec{p}$ such that $O + \vec{x} \subseteq U$. Next choose b and e in $\mathbb{N}$ such that $\vec{y} = \bigl(b, \ldots, b + (k-1)e\bigr) \in O$ (if $\vec{p} \in I$ take $e > 0$). Then $\vec{y} + \vec{x} = \bigl(b + a, \ldots, (b+a) + (k-1)(e+d)\bigr)$ is an element of $U \cap E^-$. If $\vec{p} \in I$ or $\vec{q} \in I$ then $e + d > 0$ and so $\vec{y} + \vec{x}$ even belongs to $U \cap I^-$.

Because U was arbitrary we find $\vec{p}+\vec{q} \in E$ (and $\vec{p}+\vec{q} \in I$ if $\vec{p} \in I$ or $\vec{q} \in I$). $\square$

6.28. LEMMA. *Every point of the form $(x, \ldots, x)$, with $x \in \mathbb{N}^*$, is in E.*

Proof. This is clear because every point of the form $(n, \ldots, n)$, with $n \in \mathbb{N}$, belongs to E^-. $\square$

6.29. LEMMA. *The point $\vec{u}$ is a minimal idempotent element of E.*

Proof. This also clear: if $(r_1, \ldots, r_k)$ is an idempotent then every r_i is an idempotent and if $(r_1, \ldots, r_k) \preccurlyeq \vec{u}$ then $r_i \preccurlyeq u$ for every$i$ and so $r_i = u$. $\square$

► **11**. Formulate and prove the finite version of Van der Waerden's Theorem.

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