SET THEORY WITH ATOMS.

ENRICH OUR LANGUAGE WITH TWO CONSTANTS:
$\emptyset$ AND A.

THE AXIOMS OF ZFA ARE THOSE OF ZF

EXCEPTIONS:

0: $(\forall x)(x \notin \emptyset)$ $\emptyset$ IS THE EMPTY SET

A: $(\forall z)(z \in A \leftrightarrow (z \neq \emptyset \wedge (\forall x)(x \notin z)))$ A CONSISTS OF ATOMS

EXTENSIONALITY:

$$(\forall x)(\forall y)\big((x \notin A \wedge y \notin A) \rightarrow ((\forall u)(u \in x \leftrightarrow u \in y) \leftrightarrow x = y)\big)$$

[ALL OBJECTS THAT ARE NOT ATOMS
ARE (BEHAVE AS) SETS]

REGULARITY:

$$(\forall x)\big((\exists y)(y \in x) \rightarrow (\exists y)(y \in x \wedge y \cap x = \emptyset)\big)$$

ZFA + "A IS INFINITE" IS CONSISTENT,
PROVIDED ZF IS CONSISTENT.
THIS WILL BE SHOWN IF TIME PERMITS.

FOR EVERY SET S DEFINE

- $\mathcal{P}^0(S) = S$
- $\mathcal{P}^{\alpha+1}(S) = \mathcal{P}^\alpha(S) \cup \mathcal{P}(\mathcal{P}^\alpha(S))$
- $\mathcal{P}^\alpha(S) = \bigcup_{\beta<\alpha} \mathcal{P}^\beta(S)$ (α LIMIT)
- $\mathcal{P}^\infty(S) = \bigcup_{\alpha \in ON} \mathcal{P}^\alpha(S)$

[THIS IS INFORMAL USING THE LANGUAGE OF CLASSES;
IT CAN BE MADE FORMAL USING RECURSION ETC.]

THE UNIVERSE NOW IS $V = \mathcal{P}^\infty(A)$

IT HAS A SUBUNIVERSE $\mathcal{P}^\infty(\emptyset)$ THE KERNEL

THE KERNEL IS A MODEL OF ZF.

$\text{RANK}(x) = \min\{\alpha : x \in \mathcal{P}^\alpha(A)\}$

PERMUTATIONS

IF $\pi : A \to A$ IS A PERMUTATION THEN WE EXTEND ITS ACTION TO ALL OF V BY

- $\pi(\emptyset) = \emptyset$
- $\pi(x) = \{\pi(y) : y \in x\}$ (ALSO DENOTED $\pi''x$)

THIS GIVES US AN $\in$-ISOMORPHISM OF V

- PROVE BY INDUCTION THAT
  - $\pi \circ \sigma$ INDUCES THE COMPOSITION OF THE ACTIONS INDUCED BY σ AND π.
  - THE ACTION INDUCED BY π^{-1} IS THE INVERSE OF THE ACTION INDUCED BY π
- USEFULL FORMULAS / EQUALITIES

  $\pi(\{x, y\}) = \{\pi(x), \pi(y)\}$

  $\pi(\langle x, y\rangle) = \langle \pi(x), \pi(y)\rangle$

  IF f IS A FUNCTION THEN $\pi(f)$ IS A FUNCTION AND

  $\pi(f)(\pi(x)) = \pi(f(x))$

  FOR: $\langle x, y\rangle \in f \iff \langle \pi(x), \pi(y)\rangle \in \pi(f)$.
- IF $x \in P^{\omega}(\emptyset)$ THEN $\pi(x) = x$

  SETS IN THE KERNEL ARE π-INVARIANT FOR EVERY π.
- BY INDUCTION ON COMPLEXITY OF FORMULAS

  $\varphi(x_1, \dots, x_n) \iff \varphi(\pi(x_1), \dots, \pi(x_n))$
- $\mathrm{RANK}(\pi(x)) = \mathrm{RANK}(x)$

IT IS A GOOD EXERCISE TO CHECK/VERIFY SOME/ALL OF THESE STATEMENTS.

SYMMETRY.

CONSIDER A SUBGROUP, G, OF THE PERMUTATION GROUP OF A.

A FAMILY $\mathcal{F}$ OF SUBGROUPS OF G IS A NORMAL FILTER ON G IF

- $G \in \mathcal{F}$
- IF $H \in \mathcal{F}$ AND $H \leq K$ THEN $K \in \mathcal{F}$
- IF $H, K \in \mathcal{F}$ THEN $H \cap K \in \mathcal{F}$
- IF $\pi \in G$ AND $H \in \mathcal{F}$ THEN $\pi H \pi^{-1} \in \mathcal{F}$.
- FOR EVERY $a \in A$ THE GROUP $\{\pi : \pi(a) = a\}$ IS IN $\mathcal{F}$.

FOR $x \in V$ LET

$$\mathrm{SYM}_G(x) = \{\pi \in G : \pi(x) = x\}.$$

NOTE, THIS MEANS THAT $\pi \in \mathrm{SYM}_G(x)$ IFF

$$\{\pi(y) : y \in x\} = x$$

EXERCISE

LET f BE A FUNCTION AND LET $\pi \in \mathrm{SYM}_G(f)$.
SHOW: IF $x \in \mathrm{DOM}\, f$ AND $\pi(x) = x$
THEN $\pi(f(x)) = f(x)$.

GIVEN G AND $\mathcal{F}$ SAY THAT x IS SYMMETRIC (WITH RESPECT TO G AND $\mathcal{F}$) IF $\mathrm{SYM}_G(x) \in \mathcal{F}$.

WE LET $\mathcal{V}$ (OR BETTER: $\mathcal{V}_{G,\mathcal{F}}$) BE THE CLASS OF HEREDITARILY SYMMETRIC OBJECTS, SO

$$x \in \mathcal{V} \iff \mathrm{SYM}_G(x) \in \mathcal{F} \wedge x \subseteq \mathcal{V}.$$

[HOMEWORK: THIS IS WELL-DEFINED].

TERMINOLOGY: $\mathcal{V}$ IS A PERMUTATION MODEL.

THEOREM: $\mathcal{V}$ IS A TRANSITIVE MODEL OF ZFA; IN ADDITION $P^\infty(\emptyset) \subseteq \mathcal{V}$ AND $A \in \mathcal{V}$.

PROOF

- TRANSITIVE, BY DEFINITION
- IF $x, y \in \mathcal{V}$ THEN $\{x, y\} \in \mathcal{V}$
  FOR - $\mathrm{sym}_G(\{x,y\}) \supseteq \mathrm{sym}_G(x) \cap \mathrm{sym}_G(y)$
  - $\{x, y\} \subseteq \mathcal{V}$ IS CLEAR
- IF $x \in \mathcal{V}$ THEN $\bigcup x \in \mathcal{V}$
  IF $\pi(x) = x$ THEN
  $$\pi(\bigcup x) = \bigcup\{\pi(y) : y \in x\} = \bigcup x$$
  AND SO $\mathrm{sym}_G(x) \subseteq \mathrm{sym}_G(\bigcup x)$.
  AND $y \subseteq \mathcal{V}$ IF $y \in x$ SO $\bigcup x \subseteq \mathcal{V}$.
- VERIFY THE OTHER AXIOMS YOURSELF.
- WE HAVE $\mathrm{sym}_G(A) = G$
- WE HAVE $\mathrm{sym}_G(x) = G$ FOR $x \in P^\infty(\emptyset)$.

SOURCES OF FILTERS:

OUR EXAMPLES COME FROM IDEALS

$\mathcal{I} \subseteq P(A)$ IS A <u>NORMAL</u> IDEAL ON A IF

- $\emptyset \in \mathcal{I}$
- IF $E \in \mathcal{I}$ AND $F \subseteq E$ THEN $F \in \mathcal{I}$
- IF $E, F \in \mathcal{I}$ THEN $E \cup F \in \mathcal{I}$
- IF $\pi \in G$ AND $E \in \mathcal{I}$ THEN $\pi[E] \in \mathcal{I}$
- FOR ALL $a \in A$ WE HAVE $\{a\} \in \mathcal{I}$, SO $[A]^{<\omega} \subseteq \mathcal{I}$.

FOR ANY x LET $\mathrm{fix}_G(x) = \{\pi : (\forall y \in x)(\pi(y) = y)\}$
(A SUBGROUP OF G.

GIVEN $\mathcal{I}$ DEFINE

$$\mathcal{F} = \{ H \leq G : (\exists E \in \mathcal{I})(\mathrm{FIX}_G(E) \leq H) \}.$$

(THE FILTER GENERATED BY $\{\mathrm{FIX}_G(E) : E \in \mathcal{I}\}$)

WE'LL START WITH A, G AND $\mathcal{I}$, DEFINE $\mathcal{F}$ AND THEN TAKE $\mathcal{V} = \mathcal{V}_{G,\mathcal{F}}$.

IN THIS CASE

x IS SYMMETRIC IFF

THERE IS $E \in \mathcal{I}$ SUCH THAT $\mathrm{FIX}_G(E) \leq \mathrm{SYM}_G(x)$.

COMMON TERMINOLOGY: E IS A <u>SUPPORT</u> OF x

<u>WE</u> ASSUME ZFA + AXIOM OF CHOICE. FROM NOW ON.

IN PARTICULAR $\mathcal{P}^\infty(\emptyset)$ SATISFIES ZFC, SO EVERY MEMBER OF $\mathcal{P}^\infty(\emptyset)$ HAS A WELL-ORDER IN $\mathcal{P}^\infty(\emptyset)$.

CONCLUSION

$x \in \mathcal{V}$ HAS A WELL-ORDER IN $\mathcal{V}$

IFF

THERE IS AN $f \in \mathcal{V}$ THAT IS AN INJECTIVE MAP FROM x TO $\mathcal{P}^\infty(\emptyset)$

HOMEWORK:

a) IF $f \in \mathcal{V}$ AND $f : x \to \mathcal{P}^\infty(\emptyset)$ IS INJECTIVE THEN $\mathrm{SYM}_G(f) \leq \mathrm{FIX}_G(x)$, AND SO $\mathrm{FIX}_G(x) \in \mathcal{F}$.

b) IF $x \in \mathcal{V}$ AND $\mathrm{FIX}_G(x) \in \mathcal{F}$. THEN x HAS A WELLORDER IN $\mathcal{V}$.

OUR FIRST MODEL: THE BASIC FRAENKEL MODEL

LET A BE COUNTABLY INFINITE,
LET G BE THE GROUP OF ALL PERMUTATIONS OF A,
LET $\mathcal{I}$ BE THE FAMILY OF FINITE SUBSETS OF A,
AND $\mathcal{F}$ IS DEFINED FROM $\mathcal{I}$ AS ABOVE.
THUS $x \in \mathcal{V}$ IF AND ONLY IF
THERE IS A FINITE SUBSET E OF A
SUCH THAT $\pi(x) = x$ WHENEVER $\pi(a) = a$ $(a \in E)$.

- THERE IS NO FINITE SET $E \subseteq A$ SUCH THAT $\mathrm{FIX}_G(E) \subseteq \mathrm{FIX}_G(A) = \{\mathrm{ID}_A\}$.
  CONCLUSION: THERE IS NO WELL-ORDER OF A THAT IS IN $\mathcal{V}$.
- PROVE: IF $n \in \omega$ AND IF $f : n \to A$ IS FROM V THEN $f \in \mathcal{V}$.
  [SO A IS INFINITE IN $\mathcal{V}$]