2019-11-25

①

The Basic Fraenkel Model

Further properties

- A is Dedekind-finite

  Follows from homework problem 52 c): if f is injective then there is a finite set E such that $f(a) = a$ $(a \in A \setminus E)$

  But then $f[E] \subseteq E$ and so $f[E] = E$.

- $\wp(A)$ is Dedekind-finite

  Homework [Hint: apply problem 52 c)]

The Second Fraenkel Model

! Back to last week: normal ideals. ! ↙ pages ④ and ⑤

Assume A is countably infinite and partition A into pairs: $A = \bigcup_{n \in \omega} P_n$, where $P_n = \{a_n, b_n\}$ and $m \neq n \to P_m \cap P_n = \emptyset$.

- G the subgroup of S_A consisting of those π for which $\pi[P_n] = P_n$ for all n.
- $\mathcal{I} = [A]^{<\aleph_0}$
- So, x is symmetric iff there is a $k \in \omega$ such that IF $\pi(a_i) = a_i$ and $\pi(b_i) = b_i$ for $i < k$ THEN $\pi(x) = x$.
- For all n we have $P_n \in \mathcal{V}$

  because $\mathrm{fix}_G(P_n) = G$, so certainly $\mathrm{sym}_G(P_n) = G$
- $\langle P_n : n \in \omega \rangle$ is in $\mathcal{V}$
  - each pair $\langle n, P_n \rangle$ is in $\mathcal{V}$
  - if $\pi \in G$ then

$$\begin{aligned}\pi(\langle P_n : n \in \omega \rangle) &= \pi(\{\langle n, P_n \rangle : n \in \omega\}) \\ &= \{\langle n, P_n \rangle : n \in \omega\} \\ &= \langle P_n : n \in \omega \rangle\end{aligned}$$

so $\mathrm{sym}_G(\langle P_n : n \in \omega \rangle) = G$.

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CLAIM: THERE IS NO CHOICE FUNCTION FOR $\langle P_n : n \in \omega \rangle$ IN $\mathcal{V}$.

LET $f : \omega \to A$ BE A FUNCTION IN $\mathcal{V}$

AND LET $\{a_0, b_0, \ldots, a_k, b_k\}$ BE ITS SUPPORT

SO IF $\pi(a_i) = a_i$ AND $\pi(b_i) = b_i$ FOR $i \le k$

THEN $\pi(f) = f$

LET $m > k$ AND $\pi = (a_m\ b_m)$

THEN $\pi(f) = f$ AND HENCE $\pi(f(n)) = f(n)$

FOR ALL n. AS $\pi(a_m) \ne a_m$

AND $\pi(b_m) \ne b_m$ WE MUST HAVE

$f(n) \ne a_m$ AND $f(n) \ne b_m$ FOR ALL n.

SO RAN $f \subseteq \bigcup_{i \le k} P_i$

THIS ALSO SHOWS THAT A IS DEDEKIND-FINITE.

NOTE THAT $\mathcal{P}(A)$ IS DEDEKIND-INFINITE [HOMEWORK]

LET $f \in \mathcal{V}$ BE A MAP FROM A TO A.

LET k BE SUCH THAT

IF $\pi(a_i) = a_i$ AND $\pi(b_i) = b_i$ FOR $i \le k$

THEN $\pi(f) = f$.

CLAIM: IF $m > k$ THEN $f[P_m] = P_m$

OR $f[P_m] = \bigcup_{i \le k} P_i$

[HOMEWORK]

SHOW: IF f IS SURJECTIVE THEN f IS INJECTIVE.

2019-11-25

A GENERAL REMARK ABOUT supports

LET A, G AND $\mathcal{F}$, AND HENCE $\mathcal{V}$, BE GIVEN

LET $C \subseteq \mathcal{V}$ BE A CLASS

CALL C SYMMETRIC IF $sym(C) \in \mathcal{F}$

WHERE $sym(C) = \{\pi \in G : \pi''C = C\}$

FORMALLY: C IS GIVEN BY A FORMULA, SO WE ARE TALKING ABOUT A FORMULA φ AND "$x \in C$" MEANS "$\varphi(x)$ HOLDS" AND SO $sym(C)$ IS ACTUALLY $sym(\varphi)$, WHICH IS

$$\{\pi \in G : (\forall x)(\varphi(x) \Leftrightarrow \varphi(\pi(x)))\}.$$

THE FORMULA φ IS SAID TO BE SYMMETRIC IF $sym(\varphi) \in \mathcal{F}$.

IF φ OR C IS SYMMETRIC THEN SO IS

$$C \cap \mathcal{P}^\alpha(A) = \{x \in \mathcal{P}^\alpha(A) : \varphi(x)\}$$

FOR EVERY α AND SO

$$C_\alpha = \{x \in \mathcal{P}^\alpha(A) : \varphi(x)\}$$

IS IN $\mathcal{V}$

LIKEWISE IF $x \in \mathcal{V}$ THEN ALSO $C \cap x \in \mathcal{V}$

LEMMA:

LET G BE A SUBGROUP OF S_A AND LET $\mathcal{I}$ BE A NORMAL IDEAL ON A. LET $\mathcal{V}$ BE THE CORRESPONDING PERMUTATION MODEL

THE CLASS

$$Supp = \{\langle E, x\rangle : E \in \mathcal{I},\ x \in \mathcal{V},\ fix(E) \subseteq sym(x)\}$$

IS SYMMETRIC

PROOF: IF $\pi \in G$ THEN

$fix(\pi(E)) = \pi \cdot fix(E) \cdot \pi^{-1}$ AND

$sym(\pi(x)) = \pi \cdot sym(x) \cdot \pi^{-1}$

So $\langle E, x\rangle \in Supp$ IFF $\langle \pi(E), \pi(x)\rangle \in Supp$

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④

THE ORDERED MOSTOWSKI MODEL

LET A BE COUNTABLY INFINITE AND LET $<$ (FROM V) BE A LINEAR ORDER OF A ISOMORPHIC TO THE RATIONALS.

- G IS THE GROUP OF ALL ORDER-PRESERVING PERMUTATIONS OF A
- $\mathcal{I}$ IS THE FAMILY OF FINITE SUBSETS OF A

- $\mathrm{FIX}(A) = \{ \mathrm{ID}_A \}$ IS NOT IN THE FILTER DERIVED FROM $\mathcal{I}$;
- $\mathrm{sym}(<) = G$ SO $<$ IS IN $\mathcal{V}$
- WE SEE A HAS A LINEAR ORDER BUT NO WELL-ORDER.
- IN FACT THERE IS A LINEAR ORDER OF THE WHOLE CLASS $\mathcal{V}$ [ASSUMING V HAS A WELL-ORDER]

  IN PARTICULAR WE GET:

  "EVERY SET HAS A LINEAR ORDER + ¬AC"

  IS CONSISTENT WITH ZFA.

WE NEED TO ANALYZE THE ACTION OF THE GROUP G.

LEMMA 1 LET E AND F BE FINITE AND DISJOINT SUBSETS OF A AND LET $a, b \in A$.

THERE IS A $\pi \in H = \langle\langle \mathrm{FIX}_G(E) \cup \mathrm{FIX}_G(F) \rangle\rangle$ SUCH THAT $\pi(a) = b$.

PROOF WLOG $a \neq b$ AND, SAY, $a < b$.

TAKE c AND d SUCH THAT $c < a < b < d$

$$[c,d] \cap (E \cup F) = [a,b] \cap (E \cup F)$$

LET $n = |[a,b] \cap (E \cup F)|$

WE USE INDUCTION ON n.

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$n = 0$

TAKE AN ISOMORPHISM $\sigma : [c,d] \to [c,d]$

SUCH THAT $\sigma(a) = b$. [USE $\mathbb{Q}$]

DEFINE $\varphi = \sigma \cup \{ \langle x,x \rangle : x \leq c \vee x \geq d \}$

THEN $\varphi \in G$, $\varphi(a) = b$ AND

$\varphi \in \mathrm{FIX}(E \cup F) \subseteq H$.

$n \to n+1$ CASE 1a $a \in E$

TAKE $e > a$ SUCH THAT

$[c, e] \cap (E \cup F) = \{a\}$

TAKE AN ELEMENT φ φ OF G SUCH

THAT $\varphi(x) = x$ $x \leq c \vee x \geq e$

$\varphi(a) > a$

NOTE: $\varphi \in \mathrm{FIX}(F) \subseteq H$.

$|[\varphi(a), b] \cap (E \cup F)| = n$.

WE GET $\psi \in H$

SUCH THAT $\psi(\varphi(a)) = b$.

AND $\psi\varphi \in H$

CASE 1b $b \in E$ SIMILAR:

$\varphi(b) < b$ AND
$\psi(a) = \varphi(b)$ SO
$\varphi^{-1}(\psi(a)) = b$

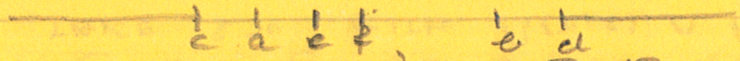
FIRST $\varphi \in \mathrm{FIX}(F)$ THEN $\psi \in H$.

CASE 2a $a \in F$ SYMMETRY:

NOW $\varphi \in \mathrm{FIX}(E)$ AND $\psi \in H$

CASE 2a $b \in F$ SIMILAR

CASE 3 : $a, b \notin E \cup F$.



LET $e = \min\{ x \in E \cup F : a < x \}$

THEN $a < e < b$ BECAUSE $n+1 > 0$

TAKE $g > f > e$ SUCH THAT $g \notin E \cup F$

AND $(a, g) \cap (E \cup F) = \{e\}$

TAKE $\varphi \in G$ SUCH THAT $\varphi(a) = f$

$\varphi(x) = x$ $(x \leq c \vee x \geq g)$

THEN $\varphi \in \mathrm{FIX}(F)$ IF $e \in E$

AND $\varphi \in \mathrm{FIX}(E)$ IF $e \in F$

APPLY THE CASE n TO $\varphi(a)$ AND b.

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LEMMA 2 LET E AND F BE FINITE AND DISJOINT. LET $\{a_i : i<n\}$ AND $\{b_i : i<n\}$ BE SUBSETS OF A SUCH THAT

$$a_i < a_j \quad \text{AND} \quad b_i < b_j \quad \text{WHENEVER } i<j.$$

THEN THERE IS A $\varphi \in H = \langle\langle FIX(E) \cup FIX(F) \rangle\rangle$ SUCH THAT $\varphi(a_i) = b_i$ FOR ALL $i<n$

PROOF LEMMA 1 GIVES US THE CASE $n=1$.

TO GO FROM n TO $n+1$ TAKE $\varphi \in H$ SUCH THAT $\varphi(a_i) = b_i$ FOR $i<n$.

- NOW WE MUST MOVE $\varphi(a_n)$ TO b_n.
  CLEARLY $b_{n-1} < \varphi(a_n)$
  LOOK AT THE INTERVAL $I = (b_{n-1}, \rightarrow)$
- APPLY LEMMA 1 TO
  - THE SET I
  - THE SETS $E \cap I$ AND $F \cap I$
  - THE POINTS $\varphi(a_n)$ AND b_n

  TO GET $\psi \in G$ SUCH THAT
  - $\psi(x) = x \quad x \leq b_{n-1}$
  - $\psi \restriction I \in \langle\langle FIX(E \cap I) \cup FIX(F \cap I) \rangle\rangle$
  - $\psi(\varphi(a_n)) = b_n$

  THEN $\psi \in H$ AND $\psi\varphi \in H$ AND $\psi\varphi(a_i) = b_i$ FOR $i \leq n$.

LEMMA 3 LET E AND F BE FINITE AND DISJOINT THEN $G = \langle\langle FIX(E) \cup FIX(F) \rangle\rangle$.

PROOF (IF $E = \emptyset$ OR $F = \emptyset$ THEN WE'RE DONE.)

LET $\pi \in G$ AND CONSIDER $\pi[E \cup F]$

WE CAN ENUMERATE $E \cup F$ AS $\{b_i : i<n\}$ AND $\pi[E \cup F]$ AS $\{a_i : i<n\}$ SUCH THAT $a_i < a_j$ AND $b_i < b_j$ WHENEVER $i<j$

LEMMA 2 GIVES US $\varphi \in \langle\langle FIX(E) \cup FIX(F) \rangle\rangle$ SUCH THAT $\varphi(a_i) = b_i$ FOR ALL i,

NOW NOTE THAT $(\varphi \circ \pi)(b_i) = b_i$ FOR ALL i
SO THAT $\varphi \circ \pi \in FIX(E \cup F)$.
WE FIND THAT $\pi = \varphi^{-1} \circ (\varphi \circ \pi)$ IS IN ⟨
IN $\langle\langle FIX(E) \cup FIX(F) \rangle\rangle$.

THEOREM LET E AND F BE FINITE.
THEN $FIX(E \cap F) = \langle\langle FIX(E) \cup FIX(F) \rangle\rangle$.
PROOF THE SET $E \cap F$ SPLITS A INTO INTERVALS
$(\leftarrow, a_0), (a_0, a_1), \ldots, (a_n, \rightarrow)$
FOR EACH SUCH INTERVAL I WE HAVE
$(E \cap I) \cap (F \cap I) = \emptyset$.
LET $\pi \in FIX(E \cap F)$; FOR EACH I WE HAVE
$\pi \restriction I \in \langle\langle FIX(E \cap I) \cup FIX(F \cap I) \rangle\rangle$
WRITE $\pi \restriction I = \varphi^I_{m_I} \circ \varphi^I_{m-1} \circ \cdots \circ \varphi^I_1 \circ \varphi^I_0$
WHERE $\varphi^I_i \in FIX(E \cap I)$ IFF i IS ODD
$\varphi^I_i \in FIX(F \cap I)$ IFF i IS EVEN
WLOG $m_I = m_J$ FOR ANY TWO INTERVALS
(JUST ADD IDENTITY FUNCTIONS)
LET $\varphi_i = \bigcup_I \varphi^I_i \cup \{\langle x, x \rangle : x \in E \cap F\}$ $i \leq m$
THEN $\pi = \varphi_m \circ \varphi_{m-1} \circ \cdots \circ \varphi_1 \circ \varphi_0$
AND $\varphi_i \in FIX(E)$ IF i IS ODD,
$\varphi_i \in FIX(F)$ IF i IS EVEN.

COROLLARY: IF x IS SYMMETRIC AND
$FIX(E_1) \subseteq sym(x)$ AND $FIX(E_2) \subseteq sym(x)$
THEN $FIX(E_1 \cap E_2) \subseteq SYM(x)$.
CONSEQUENTLY x HAS A *SMALLEST* SUPPORT.

THE CLASS $\{\langle x, E \rangle : E$ IS *THE* *SMALLEST* SUPPORT OF $x\}$
IS SYMMETRIC.

2019-11-25

IF E IS A SUPPORT OF x AND $\pi \in SYM(E)$
THEN $\pi(x) = x$
FOR π IS ORDER-PRESERVING SO $\pi[E] = E$
IMPLIES $\pi \in FIX(E)$ AND SO $\pi(x) = x$.

THERE IS A SYMMETRIC CLASS FUNCTION F
THAT IS INJECTIVE FROM $\mathcal{V}$ TO $On \times]$.

- WE ASSUME GLOBAL CHOICE; THIS HOLDS IN L.
- FOR $x \in \mathcal{V}$ CONSIDER $G(x) = \{\pi(x) : \pi \in G\}$
- $sym(G(x)) = G$ FOR ALL x.
- FOR ALL x AND y WE HAVE
  $G(x) = G(y)$ OR $G(x) \cap G(y) = \emptyset$
- FROM GLOBAL CHOICE WE GET
  $F_1 : \mathcal{V} \longrightarrow On$
  SUCH THAT - $G(x) \cap G(y) = \emptyset \rightarrow F_1(x) \neq F_1(y)$
  - $G(x) = G(y) \rightarrow F_1(x) = F_1(y)$
- DEFINE $F_2 : \mathcal{V} \longrightarrow]$ BY
  $F_2(x)$ IS THE SMALLEST SUPPORT OF x.
- NOW $F = \{\langle x, F_1(x), F_2(x)\rangle : x \in \mathcal{V}\}$
  IS SYMMETRIC: IF $\langle x, F_1(x), F_2(x)\rangle \in F$
  THEN $\langle \pi(x), \pi(F_1(x)), \pi(F_2(x)\rangle \in F$
  FOR - $\pi(F_1(x)) = F_1(x) = F_1(\pi(x))$
  - IF E IS THE SMALLEST SUPPORT OF x
    THEN $\pi(E)$ IS THE SMALLEST SUPPORT
    OF $\pi(x)$.
- IF $\langle F_1(x), F_2(x)\rangle = \langle F_1(y), F_2(y)\rangle$
  THEN $G(x) = G(y)$, SO $y = \pi(x)$ FOR SOME π.
  BUT THEN $F_2(x) = F_2(y) = \pi(F_2(x))$
  BUT THEN $x = \pi(x) = y$.

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FINALLY: $\mathcal{J}$ HAS A LINEAR ORDER:

$E \triangleleft F$ IFF $\min(F \triangle E) \in F$

(LEXICOGRAPHIC ORDER OF CHARACTERISTIC FUNCTIONS).

SO $On \times \mathcal{J}$ HAS A LINEAR ORDER:

$\langle \alpha, E \rangle \alpha \langle \beta, F \rangle \leftrightarrow \alpha \in \beta$ OR

$\alpha = \beta$ AND $E \triangleleft F$

IN THIS MODEL WE HAVE THE AXIOM OF CHOICE FOR FAMILIES OF WELL-ORDERABLE SETS.