The Ordered Mostowski Model (The Details)

This is an important week in Set Theory. In 1873 Cantor proved $\mathbb{R}$ is uncountable.

Our assumptions:

- A is a countably infinite set of atoms
- $<$ is a linear order of A in type η, i.e. $(A, <)$ is isomorphic to $\mathbb{Q}$.
- G is the group of order-automorphisms of $(A, <)$, i.e., the order-preserving permutations
- $\mathcal{I}$ is the ideal of finite sets.
- $\mathcal{F}$ the derived normal filter

- $\mathrm{fix}(A) = \{ \mathrm{id}_A \}$ is not in $\mathcal{F}$ so A has no well-order in $\mathcal{V}$
- $\mathrm{sym}(<) = G$ and all its members are (hereditarily) symmetric ($\mathrm{sym}(\langle a, b\rangle) \supseteq \{\pi \in G : \pi(a) = a \wedge \pi(b) = b\}$) so $< \in \mathcal{V}$
- For $E \in \mathcal{I}$ we have $\mathrm{sym}(E) = \mathrm{fix}(E)$ for if $\pi \in \mathrm{sym}(E)$ then $\pi[E] = E$ <u>and</u> π is order-preserving so $\pi(\min E) = \min E, \ldots\ldots$ (this will be useful at some point)

We will see that although AC fails in $\mathcal{V}$ every set in $\mathcal{V}$ has a linear order that is in $\mathcal{V}$.

In fact: If V has a global well-order then $\mathcal{V}$ has a global linear order.

TO PROVE THIS WE NEED THE FOLLOWING FACT ABOUT A, $\mathcal{I}$ AND $\mathcal{V}$.
IF $x \in \mathcal{V}$ THEN THERE IS A MINIMUM SUPPORT FOR x, I.E.,

- $\mathrm{FIX}(E) \leq \mathrm{SYM}(x)$ (E IS A SUPPORT)
- IF F IS A SUPPORT THEN $E \subseteq F$.

THIS WILL FOLLOW FROM
IF E_1 AND E_2 ARE SUPPORTS FOR x THEN $E_1 \cap E_2$ IS A SUPPORT FOR x.

THE PROOF IS JUST ALGEBRA: WE SHOW THAT $\mathrm{FIX}(E_1 \cap E_2)$ IS EQUAL TO THE SUBGROUP $H = \langle\langle \mathrm{FIX}(E_1) \cup \mathrm{FIX}(E_2) \rangle\rangle$ GENERATED BY $\mathrm{FIX}(E_1)$ AND $\mathrm{FIX}(E_2)$.

- FIRST THE CASE WHERE $E_1 \cap E_2 = \emptyset$.

STEP 1 IF $a, b \in A$ THEN THERE IS A $\pi \in H$ SUCH THAT $\pi(a) = b$.
PROOF [NOTES OF LAST WEEK]
INDUCTION ON THE CARDINALITY OF $[a,b] \cap (E_1 \cup E_2)$.

STEP 2 IF $\{a_i : i < n\}$ AND $\{b_i : i < n\}$ ARE SUBSETS OF A SUCH THAT $a_i < a_j$ IFF $b_i < b_j$ THEN THERE IS A $\pi \in H$ SUCH THAT $\pi(a_i) = b_i$ FOR ALL i
PROOF [NOTES OF LAST WEEK]
INDUCTION ON n.

STEP 3 $H = G$
PROOF [NOTES OF LAST WEEK]
GIVEN $\pi \in G$ TAKE $g \in H$ SUCH THAT $g(\pi(a)) = a$ FOR $a \in E_1 \cup E_2$.
THEN $g^{-1}\pi \in H$ AS WELL, AND SO $\pi \in H$.

- THE CASE $E_1 \cap E_2 \neq \emptyset$

$$a_0 \quad a_1 \quad a_2 \quad a_3$$

LET $E = E_1 \cap E_2$ AND WRITE IT
AS $\{a_i : i \leq n\}$ WITH $i < j \to a_i < a_j$
THE INTERVALS $(\leftarrow, a_0), (a_0, a_1) \ldots, (a_n, \to)$
ARE ISOMORPHIC TO $\mathbb{Q}$ AND FOR
EACH INTERVAL I WE HAVE
$$E_1 \cap E_2 \cap I = \emptyset$$
Apply THE FIRST CASE TO EACH INTERVAL.

Now IF E_1 AND E_2 ARE supports FOR x
THEN $\langle\langle \mathrm{FIX}(E_1) \cup \mathrm{FIX}(E_2) \rangle\rangle \subseteq \mathrm{sym}(x)$
AND SO $\mathrm{FIX}(E_1 \cap E_2) \subseteq \mathrm{sym}(x)$.

FIX ONE support E FOR x THEN
$\bigcap \{E \cap F : F$ IS A support FOR $x\}$
IS ALSO A support AND IT IS THE SMALLEST.

THE CLASS
$$\Pi = \{\langle x, E \rangle : x \in \mathcal{V}, E \in \mathcal{I}, E \text{ IS THE SMALLEST support OF } x\}$$
IS SYMMETRIC

ASSUME GLOBAL CHOICE IN V.
FOR $x \in \mathcal{V}$ LET
$$G(x) = \{\pi(x) : \pi \in G\}$$
BE ITS ORBIT.

- $G(x) \in \mathcal{V}$ AND $\mathrm{sym}(G(x)) = G$
- LET $\lhd$ BE A WELLORDER OF THE CLASS
$$\mathcal{O} = \{G(x) : x \in \mathcal{V}\}$$

THEN $\pi(\lhd) = \{\langle \pi(o), \pi(p)\rangle : o, p \in \Theta; o \lhd p\}$
$= \{\langle o, p\rangle : o, p \in \Theta; o \lhd p\}$
$= \lhd$

NOW ASSUME $\langle G(x), M(x)\rangle = \langle G(y), M(y)\rangle$

IF $G(x) = G(y)$ THEN $\pi(x) = y$ FOR SOME π.

- $M(x)$ IS THE SMALLEST SUPPORT OF x
- SO $\pi(M(x))$ IS THE SMALLEST SUPPORT OF y
- HENCE $\pi(M(x)) = M(y) = M(x)$
- BUT $\mathrm{Sym}(M(x)) = \mathrm{Fix}(M(x)) \leq \mathrm{sym}(x)$
  HENCE $y = \pi(x) = x$.

SO $x \longmapsto \langle G(x), M(x)\rangle$ IS INJECTIVE

ORDER $\mathcal{V}$ LEXICOGRAPHICALLY

$x \prec y$ IF $G(x) \lhd G(y)$ OR
$G(x) = G(y)$ AND
$M(x) < M(y)$
(LEXICOGRAPHICALLY: $\min(M(y) \Delta M(x)) \in M(y)$).

[IF YOU DON'T LIKE GLOBAL CHOICE DO THIS FOR EACH INDIVIDUAL SET IN $\mathcal{V}$].

HOW TO BUILD MODELS OF ZFA?

WE CAN BUILD THESE MODELS, "SIMPLY" BY DESIGNATING SOME SETS TO BE ATOMS.

ASSUME ZF(C) WE BUILD A MODEL FOR ZFA + A IS INFINITE (+ AC).

LET $C = \{c_n : n \in \omega\}$ BE AN INFINITE FAMILY OF INFINITE SUBSETS OF ω (E.G. $c_n = \{o \cdot 2^n : o \text{ IS ODD}\}$).

RECURSIVELY LET

$$P_0 = C$$
$$P_{\alpha+1} = P_\alpha \cup (\mathcal{P}(P_\alpha) \setminus \{\emptyset\})$$
$$P_\alpha = \bigcup_{\beta<\alpha} P_\beta \qquad (\alpha \text{ LIMIT})$$

AND $P = \bigcup_{\alpha \in ON} P_\alpha$

- INTERPRET $\emptyset$ BY c_0
- INTERPRET A BY $\{c_n : 0 < n < \omega\}$

- $\neg(\exists x)(x \in \emptyset)$ HOLDS IN P AS THERE IS NO $x \in P$ SUCH THAT $x \in c_0$
- $(\forall z)(z \in A \leftrightarrow (z \neq \emptyset \wedge \neg(\exists x)(x \in z)))$ HOLDS IN P (SAME REASON)
- IF X AND Y ARE NOT IN A THEN
  $(\forall u)(u \in X \leftrightarrow u \in Y) \leftrightarrow X = Y$
  $\leftarrow$ CLEAR
  $\rightarrow$ INDUCTION ON RANK.
- IF $(\exists x)(x \in S)$ THEN $(\exists x \in S)(x \cap S = \emptyset)$
  TAKE x OF MINIMAL RANK
  (x MIGHT BE AN ATOM).
- VERIFY PAIRING, UNION, POWER SET, YOURSELF
- INFINITY: $I = \bigcup_{\alpha \in \omega} P_\alpha$ WORKS:
  $c_0 \in P_0$, $c_0 \cup \{c_0\} \in P_1$ ...
  IF $x \in P_n$ THEN $x \cup \{x\} \in P_{n+1}$
  NOTE THAT THE P_α ARE TRANSITIVE IN THAT IF $y \in x \in P_\alpha$ <u>AND</u> $y, x \in P$ THEN $y \in P_\alpha$

- FOR SEPARATION AND REPLACEMENT USE THE CLASS P AS A PARAMETER OR RATHER ADD ITS DEFINING FORMULA TO EACH FORMULA UNDER CONSIDERATION. BECAUSE WE RELATIVIZE TO $\underline{P}$

APPROACH WITHOUT ATOMS, BUT ALSO WITHOUT FOUNDATION / REGULARITY

WE ASSUME V SATISFIES $ZF(C)$.

TAKE A (CLASS) BIJECTION $F: V \to V$

THE STRUCTURE $\langle V, E \rangle$ WHERE

$x E y$ MEANS $x \in F(y)$

IS A DIRECTED-GRAPH TYPE MODEL FOR $ZF(C)_0$, I.E. FOR SET THEORY WITHOUT FOUNDATION [HOMEWORK]

[OF COURSE IF $F = ID_V$ WE HAVE $E = \in$ SO THEN WE DO HAVE FOUNDATION]

IF $F(0) = 1$, $F(1) = 0$ AND $F(x) = x$ FOR ALL OTHER x THEN

$$\langle V, E \rangle \models (\exists x)(x = \{x\})$$

[NOTE $\langle V, E \rangle \models$ "1 IS THE EMPTY SET"]

TAKING $F(n) = \{n\}$, $F(\{n\}) = n$ FOR $n \in \omega$ AND $F(x) = x$ OTHERWISE YIELDS A MODEL OF $ZF(C)_0$ PLUS "THERE IS AN INFINITE SET A SUCH THAT $x = \{x\}$ FOR ALL $x \in A$"

THUS WE GET THE CONSISTENCY OF ZFC_0 + "THERE IS AN INFINITE SET A SUCH THAT $x = \{x\}$ FOR ALL $x \in A$"

BUILD U BY $U_0 = A$, $U_{\alpha+1} = P(U_\alpha)$

$U_\alpha = \bigcup_{\beta < \alpha} U_\beta$ (α LIMIT)

- U CONTAINS THE WELL-FOUNDED SETS
- A AND ITS PERMUTATIONS CAN BE USED TO CREATE PERMUTATION MODELS TOO.