THE JECH-SOCHOR EMBEDDING THEOREM

LET $\mathcal{U}$ BE A MODEL OF ZFA + AC, LET A BE ITS SET OF ATOMS, M ITS KERNEL, AND LET α BE AN ORDINAL IN $\mathcal{U}$.
FOR EVERY PERMUTATION MODEL $\mathcal{V} \subseteq \mathcal{U}$ THERE IS A SYMMETRIC EXTENSION N OF M AND A SET $\tilde{A} \in N$ SUCH THAT

$$\mathcal{P}^{\alpha}(A)^{\mathcal{U}} \text{ IS } \in\text{-ISOMORPHIC TO } \mathcal{P}^{\alpha}(\tilde{A})^{N}$$

SYMMETRIC EXTENSION:
ONE CAN EXTEND MODELS OF SET THEORY BY EMPLOYING THE METHOD OF FORCING.
VERY ROUGHLY:
ONE TAKES A PARTIAL ORDER $\mathbb{P} \in M$ AND A SPECIAL <u>GENERIC</u> SUBSET G OF $\mathbb{P}$.
THE GENERIC EXTENSION IS OBTAINED AS $\{ R[G] : R \text{ A RELATION}; R \in M \}$

NOTE $R[G] = \{ x : (\exists p \in G)(\langle p, x\rangle \in R) \}$
(SO VERY OFTEN $R[G] = \emptyset$ ....)

THAT EXTENSION IS DENOTED $M[G]$.
AN AUTOMORPHISM π OF $\mathbb{P}$ CAN BE MADE TO ACT ON THE CLASS OF RELATIONS, BY $\in$-INDUCTION:

$$\pi(R) = \{ \langle \pi(p), \pi(y) \rangle : \langle p, y \rangle \in R, p \in \mathbb{P} \}$$

GIVEN A GROUP $\mathcal{G}$ OF AUTOMORPHISMS AND A NORMAL FILTER OF SUBGROUPS ONE DEFINES SYMMETRIC AND HEREDITARILY SYMMETRIC RELATIONS
THE SYMMETRIC EXTENSION

$$N = \{ R[G] : R \text{ IS HER. SYMM.} \}$$

COHEN USED $\mathbb{P} = \{p : p$ IS A FINITE FUNCTION; $\operatorname{dom} p \subseteq \omega\times\omega$; $\operatorname{ran} p \subseteq 2\}$

ORDERING $p \le q$ MEANS $p \supseteq q$

G GENERIC : - IF $p, q \in G$ THEN THERE IS $r \in G$ SUCH THAT $r \le p, q$

- $\forall \langle n,m\rangle\ \exists p \in G\ \langle n,m\rangle \in \operatorname{dom} p$
- AND MORE

$\bigcup G$ IS A FUNCTION THAT DETERMINES <u>NEW</u> SUBSETS OF ω: $x_n = \{m : \bigcup G(n,m) = 1\}$

LOOK AT PERMUTATIONS OF ω: IF $\pi \in S_\omega$ THEN IT ACTS ON $\mathbb{P}$:

$\operatorname{dom} \pi(p) = \{\langle \pi(n), m\rangle : \langle n,m\rangle \in \operatorname{dom} p\}$

$\pi(p)(\pi(n), m) = p(n,m)$

$\mathcal{F}$ IS THE FILTER GENERATED BY $\{\operatorname{fix}(e) : e \in [\omega]^{<\omega}\}$

THE SET $A = \{x_n : n \in \omega\}$ IS IN N BUT $\langle x_n : n \in \omega\rangle$ IS NOT

IN FACT

$N \models$ "A HAS NO WELL-ORDER"

EVEN $N \models$ "THERE IS NO INJECTIVE $f : \omega \to A$"

NOTE THAT A IS A SUBSET OF $\{0,1\}^\omega$ AND HENCE OF $[0,1]$

THE CHARACTERISTIC FUNCTION OF A IS SEQUENTIALLY CONTINUOUS BUT NOT CONTINUOUS.

[IT TAKES A BIT OF WORK, BUT STILL ...]

HOW DOES THE PROOF WORK?

TAKE κ REGULAR SUCH THAT $\kappa > |\mathcal{P}^{\alpha}(A)|$
MUCH LIKE COHEN DID
BUILD A MATRIX $U = \{x_{a,\xi} : \xi \in \kappa, a \in A\}$
OF SUBSETS OF κ [BY FORCING]
AND IDENTIFY $a \in A$ WITH
$\tilde{a} = \{x_{a,\xi} : \xi \in \kappa\}$
AND SO $\tilde{A} = \{\tilde{a} : a \in A\}$
IS THE SET OF COLUMNS IN THE MATRIX.

WE DO THIS FOR THE MOSTOWSKI-MODEL
THE PROPERTIES OF A IN $\mathcal{V}$, AND SO OF $\tilde{A}$ IN N, THAT WE USE ARE

- A HAS NO COUNTABLE SUBSET (A IS DEDEKIND-FINITE)
- IF $X \subseteq A$ IS IN $\mathcal{V}$ THEN X IS A FINITE UNION OF INTERVALS (POINTS COUNT AS INTERVALS).
- IF $\mathcal{P}$ IS A PARTITION OF A IN $\mathcal{V}$ THEN $\{P \in \mathcal{P} : |P| \geq 2\}$ IS FINITE
- IF $X, Y \subseteq A$ AND $|X| = |Y|$ THEN $X \Delta Y$ IS FINITE
- IF $X, Y_1, \ldots, Y_n \subseteq A$ AND $|X| = |Y_1| + \cdots + |Y_n|$ THEN $X \Delta (Y_1 \cup \cdots \cup Y_n)$ IS FINITE
- THERE IS NO FUNCTION $f : \omega \to [A]^{<\omega}$ SUCH THAT $|f(n)| \geq n$ FOR ALL n.

IF X AND Y ARE INFINITE SETS IN N SUCH THAT $X \subseteq U$ AND $Y \subseteq A$ THEN THERE IS NO BIJECTION IN N BETWEEN X AND Y.
[THIS USES FORCING.]

ASSUME THERE IS A SYMMETRIC CLASS FUNCTION C ON N SUCH THAT

- $|C(X)| = |X|$ FOR ALL X
- $C(X) = C(Y)$ IFF $|X| = |Y|$ FOR ALL X, Y.

WE DERIVE A CONTRADICTION

WE ONLY NEED $C \upharpoonright P(A)^N$

THE CONTRADICTION: WE BUILD $f: \omega \to [A]^{<\omega}$ SUCH THAT $|f(n)| \geq n$ FOR ALL n.

AS IN THE MOSTOWSKI-MODEL ITSELF WE NOW HAVE AN INJECTIVE CLASS FUNCTION

$$F : N \longrightarrow I \times J \times On$$

WHERE $I = [A]^{<\omega}$ AND $J = [U]^{<\omega}$

LET $n \in \omega$. TAKE $X \subseteq A$ WITH $|A \setminus X| = n$ AND CONSIDER $C(X)$.

[$C(X)$ DEPENDS ON n ONLY]

LET $W = F[C(X)]$

- FOR NOW $A = X \cup \{a_1\} \cup \cdots \cup \{a_n\}$ (PARTITION)
- SPLIT W INTO SUBSETS FOR $z \in J \times On$ LET $W_z = \{E \in I : \langle E, z \rangle \in W\}$
- VIA A BIJECTION $X \leftrightarrow C(X) \overset{F}{\leftrightarrow} W$ WE GET $|X| = \Sigma_z |W_z|$ AND A PARTITION $\{W_z : z \in J \times On\} \cup \{\{a_i\} : i \leq n\}$ OF A IN $\underline{N}$

  SO ALL BUT FINITELY MANY W_z ARE SINGLETON SETS.
- THERE ARE ALSO (JUST) FINITELY MANY W_z'S THAT ARE SINGLETONS, HENCE ONLY FINITELY MANY W_z ARE NONEMPT.