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WE ASSUME $Z = \{z : |W_z| = 1\}$ IS INFINITE
NOTE THAT VIA THE BIJECTIONS AND INJECTION

$$X \longleftrightarrow C(X) \longleftrightarrow W \longleftarrow Z$$

WE CAN (AND WILL) TREAT Z AS AN INFINITE SUBSET OF A.

(1) FOR $\alpha \in On$ PUT $Z_\alpha = \{e \in J : \langle e, \alpha\rangle \in Z\}$

- $\alpha \longmapsto Z_\alpha$ IS INJECTIVE FROM A SET OF ORDINALS INTO $\mathcal{P}(A)$.

WE KNOW $\mathcal{P}(A)$ IS DEDEKIND-FINITE, SO $Y = \{\alpha : Z_\alpha \neq \emptyset\}$ IS FINITE

(2) FIX $\alpha \in Y$ AND PUT $Z_{\alpha,i} = \{e \in Z_\alpha : |e| = i\}$

$i \longmapsto Z_{\alpha,i}$ IS INJECTIVE FROM A SUBSET OF ω INTO $\mathcal{P}(A)$, HENCE $\{i : Z_{\alpha,i} \neq \emptyset\}$ IS FINITE.

SO FAR: $\{\langle\alpha, i\rangle : Z_{\alpha,i} \neq \emptyset\}$ IS FINITE

(3) IF $\langle\alpha, i\rangle$ IS SUCH THAT $Z_{\alpha,i} \neq \emptyset$ THEN $Z_{\alpha,i} \subseteq [U]^i$ AND VIA MONOTONE ENUMERATIONS WE ASSUME $Z_{\alpha,i} \subseteq U^i$.

THE FOLLOWING LEMMA SUPPLIES THE DESIRED CONTRADICTION:

LEMMA: LET $i \geq 1$. THEN THERE IS NO BIJECTION <u>IN N</u> BETWEEN AN INFINITE SUBSET OF U^i AND AN INFINITE SUBSET OF A.

PROOF, INDUCTION ON i

$i = 1$: ALREADY KNOWN

$i \to i+1$: LET $P \subseteq U^{i+1}$ AND $Q \subseteq A$ BE MEMBERS OF N AND SUCH THAT THERE IS A BIJECTION $f : P \to Q$ IN N.

LET, FOR $u \in U$, $P_u = \{v \in U^i : \langle u, v\rangle \in P\}$

- $R = \{u : |P_u| = 1\}$ IS FINITE BECAUSE $f \restriction R : R \to f[R]$ IS A BIJECTION

BETWEEN A SUBSET OF U AND
A SUBSET OF A.

- $T = \{u \in U : |P_u| \geq 2\}$ IS ALSO FINITE
BECAUSE $\{f[P_u] : u \in U, P_u \neq \emptyset\}$ IS
A PARTITION, IN N, OF Q.
- BY OUR INDUCTIVE ASSUMPTION
THE SET P_u IS FINITE WHENEVER $u \in T$.

THUS WE SEE THAT P IS FINITE.

CONCLUSION

WE HAVE A FINITE SET $Z \subseteq] \times On$
SUCH THAT $W = \bigcup_{z \in Z} W_z$
VIA THE BIJECTIONS WE HAVE
$|X| = \sum_{z \in Z} |W_z|$

CONSIDER A z SUCH THAT $|W_z| \geq 2$.
FOR $i \in \omega$ LET $W_{z,i} = \{E \in W_z : |E| = i\}$
THE SET $I_z = \{i : W_{z,i} \neq \emptyset\}$ IS FINITE
FIX ONE $i \in I_z$
AS ABOVE ASSUME $W_{z,i} \subseteq A^i$, WRITE $S = W_{z,i}$.
IF $i = 1$ THEN S IS BASICALLY A SUBSET OF A
IF $i > 1$ WRITE $S_a = \{b \in A^{i-1} : \langle a, b \rangle \in S\}$
LET $P_0 = \{a : |S_a| = 1\}$ -- A SUBSET OF A
THE SET $Q_0 = \{a : |S_a| \geq 2\}$ IS FINITE
FOR $a \in Q_0$ WORK IN S_a TO GET
$P_{a,0} = \{b : |\{c \in A^{i-2} : \langle b, c \rangle \in S_a\}| = 1\}$
$Q_{a,0} = \{b : |\{c \in A^{i-2} : \langle b, c \rangle \in S_a\}| \geq 2\}$
}

IN THIS WAY WE WRITE $W_{z,i}$ AS A DISJOINT
UNION OF SETS $\{P_{s,0} : s \in R_{z,i}\}$ FINITELY MANY.
EACH OF WHICH HAS A NATURAL
BIJECTION WITH A SUBSET $Y_{s,0}$ OF A.

WE DO THIS FOR EVERY $W_{z,i}$

SO

$$|W_{z,i}| = \Sigma\{|Y_{s,i}| : s \in R_{z,i}\}$$

AND

$$|W_z| = \Sigma\{|Y_{s,i}| : s \in R_{z,i}; i \in I_z\}$$

AND ULTIMATELY

$$|X| = |W| = \Sigma\{|Y_{s,i}| : s \in R_{z,i}; i \in I_z; z \in Z\}$$

REWRITE THIS AS

$$|X| = |Y_1| + \dots + |Y_k| + m$$

BY SPLITTING THE FAMILY INTO TWO:

- MEMBERS WITH ONE POINT
- MEMBERS WITH MORE THAN ONE POINT

NOTE: POTENTIALLY THE Y_i INTERSECT

BUT STILL

$$X \Delta \bigcup_{i=1}^{k} Y_i$$

IS FINITE.

LET $Y = \bigcup_{i=1}^{k} Y_i$ AND $E = A \setminus Y$

- E IS FINITE
- $|Y| \leq |Y_1| + \dots + |Y_m| \leq |X|$

SO $|E| \geq m$.

NOW E DEPENDS ONLY ON W, WHICH DEPENDS ONLY ON n.

SO $f(n) = |E|$ IS WELL-DEFINED

AND WE HAVE OUR FINAL CONTRADICTION.

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SUMMARY

WE HAVE $F : N \longrightarrow I \times J \times On$ INJECTIVE

$I = [A]^{<\omega}, \quad J = [U]^{<\omega}$

ASSUME C IS A CARDINALITY FUNCTION

LET $n \in \omega$ TAKE $X \subseteq A$ WITH $|A \setminus X| = n$

LET $W = F[C(X)]$

- $Z = \{ z \in J \times On : W_z \neq \emptyset \}$

  WHERE $W_z = \{ E \in I : \langle E, z \rangle \in W \}$

- $Z_a = \{ z : |W_z| \geq 2 \}$

  $Z_b = \{ z : |W_z| = 1 \}$

  - Z_a IS FINITE (GENERAL FACT ON PARTITIONS)
  - Z_b IS FINITE

    VIA $Z_\alpha = \{ e \in J : \langle e, \alpha \rangle \in Z_b \}$

    - FINITELY MANY α WITH $Z_\alpha \neq \emptyset$
    - EACH Z_α IS FINITE

- $\{ W_z : z \in Z \}$ IS A FINITE PARTITION OF W
- EACH W_z SATISIES

  $|W_z| = |Y_1^z| + \dots + |Y_{k_z}^z|$ FOR SOME SUBSETS OF A.

- $|W| = |Y_1| + \dots + |Y_k|$ FOR SOME SUBSETS OF A

- $|W| = |Y_1| + \dots + |Y_l| + m$ FOR SOME $m \in \omega$

  COLLECT ALL ONE-POINT Y_i

- $E_m = A \setminus \bigcup_{i=1}^{l} Y_i$ IS FINITE

  BECAUSE $X \triangle \bigcup_{i=1}^{l} Y_i$ IS FINITE

- THIS DEFINITION DEPENDS ON m ONLY

SO $m \longmapsto |E_m|$ IS A WELL-DEFINED FUNCTION.

ALSO $|E_m| \geq n + m \geq n$ FOR ALL m.