SET THEORY 2020-11-09 ①

① THE CUB FILTER IS NOT AN ULTRAFILTER.

$\kappa > \aleph_1$:

$E_0 = \{\alpha \in \kappa : \mathrm{cf}\,\alpha = \aleph_0\}$

$E_1 = \{\alpha \in \kappa : \mathrm{cf}\,\alpha = \aleph_1\}$

EXERCISE: BOTH ARE STAT'RY.

② $\kappa = \aleph_1$

AC THE CUB FILTER IS NOT ULTRA

THERE IS AN INJECTION $f : \omega_1 \to \mathbb{R}$

THERE IS NO σ-COMPLETE UF ON $\mathbb{R}$.

CAPITA SELECTA THIS YEAR: HAS A MODEL OF ZF IN WHICH $\mathcal{C}_{\aleph_1}$ <u>IS</u> AN ULTRAFILTER

SOLOVAY: EVERY STAT. SUBSET OF κ CAN BE SPLIT IN κ MANY STAT. SETS

② $\langle C_\alpha : \alpha < \kappa \rangle$ CUB SETS:

$\Delta_{\alpha < \kappa} C_\alpha$ IS CUB

$\to \{\delta : \delta \in \bigcap_{\alpha < \delta} C_\alpha\}$

$\mathcal{C}$ IS A <u>NORMAL</u> FILTER

JECH: 8.11 IF $\mathcal{F}$ IS NORMAL THEN $C_\kappa \subseteq \mathcal{F}$

NORMAL ULTRAFILTERS LEAD TO LARGE CARDINALS AGAIN.

RAMSEY'S THEOREM

THEOREM A. *Let Γ be an infinite class, and μ and r positive integers; and let all those sub-classes of Γ which have exactly r members, or, as we may say, let all r-combinations of the members of Γ be divided in any manner into μ mutually exclusive classes C_i ($i = 1, 2, \ldots, \mu$), so that every r-combination is a member of one and only one C_i; then, assuming the axiom of selections, Γ must contain an infinite sub-class Δ such that all the r-combinations of the members of Δ belong to the same C_i.*

GIVEN n, k AND

$$F : [\omega]^n \longrightarrow k$$

THERE IS AN INFINITE $H \subseteq \omega$ SUCH THAT F IS CONSTANT ON $[H]^n$.
- F IS CALLED A COLOURING.
- H IS CALLED HOMOGENEOUS (FOR F).

"COMPLETE DISORDER IS IMPOSSIBLE"

(1) $n = 1$: F IS CONSTANT ON AN INFINITE SET.

(2) $n = 2$: $F : [\omega]^2 \longrightarrow k$

LET $\mathcal{U}$ BE A FREE ULTRAFILTER ON ω.

FOR $a \in \omega$ LOOK AT $\underline{\{b \in \omega : b > a\}}$

IT IS THE UNION OF

$C(a,i) = \underline{\{b : b > a \ , \ F(\{a,b\}) = i\}}$

$i \in k$

U IS ULTRA: THERE IS ONE i_a SUCH THAT $C(a, i_a) \in U$.

$a_0 = 0$

$a_1 = \min C(a_0, i_{a_0})$

$a_2 = \min(C(a_0, i_0) \cap C(a_1, i_{a_1})) \in U$

$\vdots$

$a_\ell = \min \bigcap_{j<\ell} C(a_j, i_{a_j}) \qquad \in U$

NOTE - $\ell < m \longrightarrow a_\ell < a_m$

AND

$F(\{a_\ell, a_m\}) = \underline{i_{a_\ell}}$

TAKE $i < k$ SUCH THAT

$I = \{\ell : i_{a_\ell} = i\}$ IS IN U

LET $H = \{a_\ell : \ell \in I\}$

$F(\{a_\ell, a_m\}) = i$ FOR $\ell, m \in I$

H IS INFINITE AND HOMOGENEOUS.

$n = 2$ TALKS ABOUT GRAPHS.

IF YOU COLOUR THE LINES IN THE COMPLETE GRAPH ON ω, WITH FINITELY MANY COLOURS THEN THERE IS AN INFINITE SET H

SUCH ALL LINES BETWEEN MEMBERS OF H HAVE THE SAME COLOUR

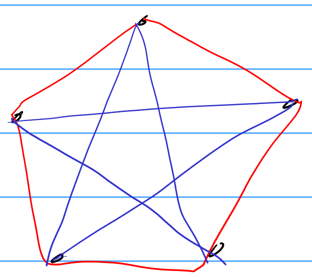
THEOREM B. *Given any r, n, and μ we can find an m_0 such that, if $m \geqslant m_0$ and the r-combinations of any Γ_m are divided in any manner into μ mutually exclusive classes C_i $(i = 1, 2, \ldots, \mu)$, then Γ_m must contain a sub-class Δ_n such that all the r-combinations of members of Δ_n belong to the same C_i.*

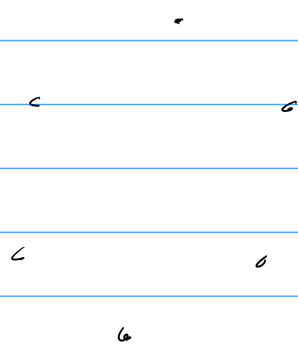
GIVEN A NUMBER k
THERE IS A NUMBER N
SUCH THAT WHENEVER
$M \geq N$ AND THE LINES
IN K_M THE COMPLETE GRAPH
ON M POINTS ARE COLOURED
RED AND BLUE
THEN THERE IS A SUBSET
OF k POINTS SUCH THAT
LINES BETWEEN THOSE
HAVE THE SAME COLOUR

FANCY PROOF: COMPACTNESS
NO INFO ABOUT N

HARD WORK: $N = \binom{2k-2}{k-1}$
WORKS.

$k = 3$ $\qquad N = 6$





GAME:

SIM

$n \to n+1$

$F : [\omega]^{n+1} \to k$ GIVEN

TAKE A FREE UF $\mathcal{U}$ AGAIN

IF $x \in [\omega]^n$ LOOK AT

$A_x = \{b \in \omega : b > \max x\}$

$b \mapsto F(x \cup \{b\})$

WE GET i_x SUCH THAT

$A_{x, i_x} \in \mathcal{U}$

$\{b \in A_x : F(x \cup \{b\}) = i_x\}$

! THIS GIVES $G : [\omega]^n \to k$

$x \mapsto i_x$

- $a_j = j \quad : \quad j < n$
- $a_n : \quad x = \{a_j : j < n\} \in [\omega]^n$

  $a_n = \min A_{x, i_x}$

- $m > n : \quad a_m = \min \bigcap \{A_{x, i_x} : x \in [\{a_j : j < m\}]^n\} \in \mathcal{U}$

APPLY INDUCTIVE ASSUMPTION TO G AND $\{a_m : m \in \omega\}$

WE GET i AND I SUCH THAT $G(x) = i$ IF $x \in [\{a_m : m \in I\}]^n$

NOW $F(x) = i$ IF $x \in [\{a_m : m \in I\}]^{n+1}$

$x = \{a_{m_0}, a_{m_1}, \dots, a_{m_{n-1}}, a_{m_n}\}$ $\quad F(x) = G(\{a_{m_0}, \dots, a_{m_{n-1}}\})$

$n=2$: CAN WE GET $H \in \mathcal{U}$?

WE HAD

$$\underbrace{C_0 \supseteq C_0 \cap C_1 \supseteq \dots \supseteq \bigcap_{i \leq m} C_i \supseteq \dots}_{\text{IN } \mathcal{U}}$$

$H \in \mathcal{U}$ IF WE CAN GET $\{a_m : m \in \omega\} \in \mathcal{U}$

THE PROOF OFFERS NO GUARANTEE FOR THIS WHATSOEVER.

SELECTIVE ULTRAFILTERS ARE NOT CALLED RAMSEY ULTRAFILTERS FOR NOTHING.

IF $F : [\omega]^n \longrightarrow k$ IS GIVEN THEN THERE IS A HOMOGENEOUS SET H IN $\mathcal{U}$

" SELECTIVE UF'S HAVE THE RAMSEY PROPERTY "

RAMSEY prop $\longrightarrow$ SELECTIVE

TAKE $f : \omega \longrightarrow \omega$

$F : [\omega]^2 \longrightarrow \{0, 1\}$

$$\{x, y\} \longmapsto \begin{cases} 0 & f(x) = f(y) \\ 1 & f(x) \neq f(y) \end{cases}$$

IF H IS HOMOGENEOUS FOR F THEN f IS CONSTANT OR INJECTIVE ON H.

LEMMA

LET $\mathcal{U}$ BE SELECTIVE

LET $\langle X_n : n \in \omega \rangle$ BE A SEQUENCE IN $\mathcal{U}$ SUCH THAT $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$.

THEN THERE IS A SEQUENCE $\langle x_n : n \in \omega \rangle$ IN ω SUCH THAT

- $\{x_n : n \in \omega\} \in \mathcal{U}$
- $x_0 \in X_0$
- $x_{n+1} \in X_{x_n}$ $(n \in \omega)$

- $\omega \setminus X_0, \; X_0 \setminus X_1, \; X_1 \setminus X_2, \; \cdots$
  $\bigcap_{n \in \omega} X_n$.
- $\omega \setminus X_0, X_0 \setminus X_1, X_1 \setminus X_2, \cdots \notin \mathcal{U}$
- $\bigcap_{n \in \omega} X_n \in \mathcal{U}$

  LET $\langle x_n : n \in \omega \rangle$ ENUMERATE THE INTERSECTION.

  [CHECK THAT THIS WORKS]

- $\bigcap_{n \in \omega} X_n \notin \mathcal{U}$

  WE TAKE $Y \in \mathcal{U}$ WITH
  - $Y \cap \bigcap_{n \in \omega} X_n = \emptyset$
  - $Y \cap (\omega \setminus X_0), \; Y \cap (X_n \setminus X_{n+1}) \cdots$

    ALL AT MOST ONE POINT.

- DEFINE $y_0 < y_1 < y_2 < \cdots$ IN Y

$y_0 = \min\{y \in Y : \{z \in Y : z > y\} \subseteq X_0\}$

$y_1 = \min\{y \in Y : \{z \in Y : z > y\} \subseteq X_{y_0}, \; y > y_0\}$

$y_{n+1} = \min\{y \in Y : y > y_n, \; \{z \in Y : z > y\} \subseteq X_{y_n}\}$

DIVIDE Y INTO INTERVALS:

$[0, y_0] \cap Y$, $(y_0, y_1] \cap Y$,

$\dots, (y_n, y_{n+1}] \cap Y, \dots$

THERE IS A $Z \in \mathcal{U}$ THAT MEETS EVERY INTERVAL IN ONE POINT: $Z = \{z_0, z_1, z_2, \dots\}$

$\le y_0 < z_0 \le y_1 < z_1 \le y_2 < \dots$

$\dots < z_{n-1} \le y_n < z_n \le y_{n+1} < z_{n+1} \le y_{n+2} \dots$

$< z_{n+2}$

$z_{n+2} > y_{n+2}$ SO $z_{n+2} \in Y_{z_n}$

$Y_{y_{n+1}}$

$$\boxed{z_{n+2} \in Y_{z_n}}$$

EITHER $\{z_{2n} : n \in \omega\} \in \mathcal{U}$

OR $\{z_{2n+1} : n \in \omega\} \in \mathcal{U}$

LET $x_n = z_{2n}$ DONE!

OR $x_n = z_{2n+1}$ DONE!

SELECTIVE $\Longrightarrow$ RAMSEY PROP.

INDUCTION ON n

$n = 1$ CLEAR

$n \to n+1$ LET $F : [\omega]^{n+1} \to k$ BE GIVEN

FOR $m \in \omega$ DEFINE

$$F_m : [\omega \setminus \{m\}]^n \longrightarrow k$$

BY $F_m(x) = F(\{m\} \cup x)$

FOR EVERY $m \in \omega$ THERE IS $H_m \in \mathcal{U}$ THAT IS HOMOGENEOUS FOR F_m. VALUE $\underline{\underline{i_m}}$

WLOG $H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_m \supseteq \cdots$

$$\left[H_m \sim \bigcap_{i \leq m} H_i \right]$$

TAKE $\langle x_m : m \in \omega \rangle$ AS IN THE LEMMA.

- $\{x_m : m \in \omega\} \in \mathcal{U}$
- $x_0 \in H_0$, $x_{m+1} \in H_{x_m}$

IF $K \in [\omega]^{n+1}$ $m = \min K$

$$\{x_\ell : \ell \in K \setminus \{m\}\} \in \underline{\underline{[H_{x_m}]^n}}$$

$$F(\underline{\{x_\ell : \ell \in K\}}) = \underline{\underline{i_{x_m}}}$$

FIND $\underline{I} \subseteq \omega$ AND ONE i SUCH THAT $\{x_m : m \in I\} \in \mathcal{U}$ AND $i_{x_m} = i$ $m \in I$

THEN $H = \{x_m : m \in I\}$ IS HOMOGENEOUS AND IN $\mathcal{U}$.

THE EXPRESSION

$$\kappa \longrightarrow (\lambda)^{\nu}_{\mu}$$

"κ ARROWS λ-ν-μ"

MEANS

IF $F : [\kappa]^{\nu} \longrightarrow \mu$ IS A COLOURING
THEN THERE IS $H \in [\kappa]^{\lambda}$ THAT
IS HOMOGENEOUS: F IS CONSTANT
ON $[H]^{\nu}$

RAMSEY: $\aleph_0 \longrightarrow (\aleph_0)^{n}_{k}$ $(n, k \in \omega)$

$5 \not\longrightarrow (3)^2_2$ $\qquad$ $6 \longrightarrow (3)^2_2$

$$\binom{2n-2}{n-1} \longrightarrow (n)^2_2$$

? $\aleph_1 \not\longrightarrow (\aleph_1)^2_2$

Existe-t-il une relation symétrique R, dont le champ E est non dénombrable, telle que dans tout sous-ensemble non dénombrable de E existent deux éléments différents α et β, tels que αRβ, et deux éléments différents γ et δ, tels que γ non Rδ.

Nous prouverons (à l'aide de l'axiome du choix) que *la réponse y est affirmative.*

SIERPIŃSKI
1933

$f : \omega_1 \xrightarrow{1-1} \mathbb{R}$ $\qquad$ $R \subseteq \omega_1 \times \omega_1$ SYMM

$\updownarrow$

$A \subseteq [\omega_1]^2$

xRy MEANS
$x \in y$ AND $f(x) < f(y)$
ARE EQUIVALENT.

Il me semble difficile à résoudre le problème de M. KNASTER pour un champ E dont la puissance est $> \aleph_1$ (par exemple pour $\overline{\overline{E}} = \aleph_2$).

Or, il est à remarquer que:

Il n'existe aucune relation symétrique R, dont le champ E est infini, telle que dans tout sous-ensemble infini de E il existe deux éléments différents a et b, tels que aRb et deux éléments différents c et d, tel que c non *Rd* [1].

SIERPIŃSKI ASKED

$$\circledast \quad \aleph_2 \longrightarrow (\aleph_1)^2_2 \quad ?$$

$$2^{\aleph_0} \not\longrightarrow (\aleph_1)^2_2$$

$\neg$CH : NO TO $\circledast$

CH : YES TO $\circledast$

$$2^{\kappa} \not\longrightarrow (\kappa^+)^2_2$$

$$2^{\kappa} \not\longrightarrow (3)^2_{\kappa}$$

$$\kappa \not\longrightarrow (\aleph_0)^{\aleph_0}_2$$

Theorem I : Let a and b be infinite cardinals such that $b > a^a$. If we split the complete graph of power b into a sum of a subgraphs at least one of them contains a complete graph of power $> a$.

In particular : If $b > c$ (the power of the continuum) and we split the complete graph of power b into a countable sum of subgraphs, at least one subgraph contains a non denumerable complete graph.

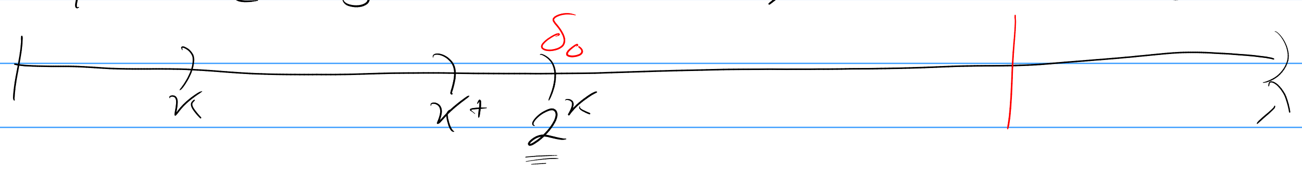
ERDŐS–RADO THEOREM.

$$(2^{\kappa})^+ \longrightarrow (\cancel{2}\kappa^+)^2_{\kappa}$$

TREES, LIKE RAMSEY'S THM,
AN ELEMENTARY VERSION OF THE LATTER.
USES MODEL THEORY AND LÖWENHEIM–SKOLEM.

PUT $\lambda = (2^\kappa)^+$

$F : [\lambda]^2 \longrightarrow \kappa$ A COLOURING.



BUILD A SEQUENCE $\langle \delta_\alpha : \alpha \leq \kappa^+ \rangle$

- $\delta_0 = 2^\kappa$
- α LIMIT $\delta_\alpha = \sup_{\beta<\alpha} \delta_\beta$
- $\delta_\alpha \longrightarrow \delta_{\alpha+1}$??

FOR $\eta < \lambda$ WE HAVE

$$F_\eta : \lambda \setminus \{\eta\} \longrightarrow \kappa$$

$$F_\eta(\xi) = F(\{\xi, \eta\})$$

WE ONLY NEED $F_\eta : \eta \longrightarrow \kappa$.

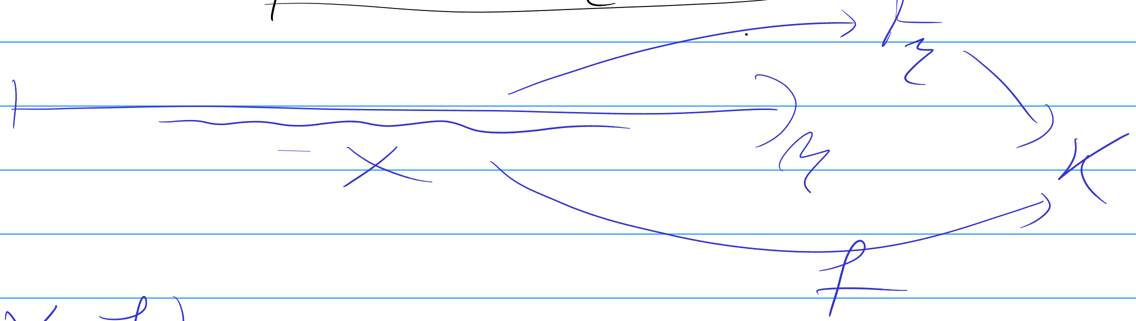
IF $X \subseteq \lambda$ OF CARDINALITY $\leq \kappa$

IF $f : X \longrightarrow \kappa$

THERE MAY BE (OR NOT)

AN η SUCH THAT

- $X \subseteq \eta$
- $f = F_\eta \upharpoonright X$



$\eta(X, f)$ IS THE FIRST SUCH η.

IF THERE IS NO η

WE SET $\eta(X, f) = 0$.

WE KNOW! $|\delta_\alpha| = 2^\kappa$

WE KNOW $|[\delta_\alpha]^{\leq\kappa}| = 2^\kappa$

EVERY ONE OF THOSE SETS HAS AT MOST 2^κ FUNCTIONS TO κ.

So $\left|\left\{ (X, f) : \begin{array}{l} X \in [\delta_\alpha]^{\leq\kappa} \\ f: X \to \kappa \end{array} \right\}\right| = 2^\kappa$

LET $\delta_{\alpha+1} > \delta_\alpha$ BE SUCH THAT $\eta(X, f) < \delta_{\alpha+1} < \lambda$ FOR ALL THOSE PAIRS

NOW LOOK AT $\delta = \delta_{\kappa^+}$

SUPPOSE $X \in [\delta]^{\leq\kappa}$ AND $f: X \to \kappa$

THERE IS AN α SUCH THAT $X \subseteq \delta_\alpha$

SO $\eta(X, f) < \delta_{\alpha+1} < \delta$

"δ IS CLOSED UNDER THIS η-FUNCTION"

WE BUILD $\langle \beta_\alpha : \alpha < \kappa^+ \rangle$ BELOW δ.

$\beta_0 = 0$

GIVEN $\langle \beta_\alpha : \alpha < \gamma \rangle$

LET $X = \{\beta_\alpha : \alpha < \gamma\} \in [\delta]^{\leq \kappa}$

$$f = F_\delta \restriction X$$

THERE IS AN η FOR (X, f) ! NAMELY δ_γ

SO $\eta(X, f) < \delta$.

THAT BECOMES β_γ

IF $\alpha < \gamma$ THEN

$$\left[\begin{array}{ccc} F(\{\beta_\alpha, \beta_\gamma\}) & = & F(\{\beta_\alpha, \delta\}) \\ & & \| \\ F_{\beta_\gamma}(\beta_\alpha) & = & F_\delta(\beta_\alpha) \end{array} \right.$$

WE HAVE $\langle \beta_\alpha : \alpha < \kappa^+ \rangle$

LET $I \subseteq \kappa^+$ BE OF CARDINALITY κ^+ (STATIONARY!) SUCH THAT

$$\alpha \longmapsto F_\delta(\beta_\alpha)$$

IS CONSTANT. WITH VALUE χ

NOW:

IF $\alpha < \gamma < \kappa^+$

THEN $F(\{\beta_\alpha, \beta_\gamma\}) = F(\{\beta_\alpha, \delta\}) = \chi$

So $\{\beta_\alpha : \alpha < \kappa^+\} \cup \{\delta\}$

IS HOMOGENEOUS OF CARD κ^+

ORDER-TYPE $\underline{\kappa^+ + 1}$

($\kappa^+ + 2$ IS MUCH HARDER)

Theorem I is best possible. As a matter of fact, if $b = a^a = 2^a$ we can split the complete graph of power b into the sum of a subgraphs, such that no one of them contains a triangle. For the sake of simplicity we show this only in the case $b = c = 2^{\aleph_0}$. We write

$$G = \sum_{k=1}^{\infty} G_k$$

where G is a graph connecting every two points of the interval (0, 1), and the edges of G_k connect two points x and y if $\frac{1}{2^{k-1}} > y - x \geq \frac{1}{2^k}$.
Clearly none of the G_k's contains any triangles.