SET THEORY 2020-11-23

WHAT ARE LARGE CARDINALS?
PAGE 58:

LARGE CARD: NORMAL CARD.

= $\omega / \aleph_0$: NATURAL NUMBERS

$\aleph_0$ IS REGULAR AND
STRONG LIMIT

$\aleph_0$ HAS AN $\aleph_0$ κ-COMPLETE
κ ULTRAFILTER.
NONPRINCIPAL!

$\aleph_0 \longrightarrow (\aleph_0)^{n}_{k}$ ← FALSE FOR ω
$k \in <\omega$

W.C. W.C. W.C.

A PROPERTY IS LARGE IF IT PROVES THE CONSISTENCY OF ITS NON-EXISTENCE

IF κ IS THE FIRST WITH THE PROPERTY THEN

$V_\kappa \models$ "NO CARDINALS WITH THE PROPERTY + ZFC"

IF κ IS INACCESSIBLE THE

$V_\kappa \models$ "ZFC + NO INACCESSIBLES"

LET κ BE THE FIRST INACCESSIBLE

LOOK AT $(V_\kappa, \in, =)$

$V_\kappa \models$ PAIRING :

$\forall x \in V_\kappa \ \forall y \in V_\kappa \ \exists z \in V_\kappa$: "$z = \{x, y\}$"

RANK $\{x, y\} = \max\{\text{RANK}(x), \text{RANK}(y)\} + 1$

$V_\kappa \models$ POWER SET :

$x \in V_{\alpha+1}$: $x \subseteq V_\alpha$, $y \subseteq x : y \subseteq V_\alpha$

SO $\mathcal{P}(x) \subseteq V_{\alpha+1}$

$\mathcal{P}(x) \in V_{\alpha+2}$

RANK$\left(\mathcal{P}(x)\right) =$ RANK$(x) + 1$

EXERCISE :

CALCULATE THE RANKS OF $\bigcup x$, $\langle x, y \rangle$, $x \times y$, x^y, ---
IN TERMS OF THE RANKS OF x AND y.

$V_\kappa \models$ "REPLACEMENT"

$X \in V_\kappa$, $\varphi(x, y, p)$ "FUNCTIONAL"

$\forall p : \forall x \ \exists! \, y \ \varphi^{V_\kappa}(x, y, p)$ ASSUMPTION

FOR EVERY A THERE IS A B
SUCH THAT

$\forall x \in A \ \exists y \in B \ \varphi(x, y, p)$

WE NEED $Y \in V_\kappa$ SUCH THAT

$$\forall x \in X \ \exists y \in Y \ \varphi^{V_\kappa}(x, y, p)$$

GIVEN x TAKE ITS y AND THE FIRST $\alpha < \kappa$ SUCH THAT $y \in V_{\alpha+1}$: α_x

$X \in V_\kappa$ SO $|X| < \kappa$.

LET $\alpha = \sup_{x \in X} \alpha_x$

SO $\alpha < \kappa$

SO FOR ALL $x \in X$ THERE IS $y \in V_{\alpha+1}$ SUCH THAT $\varphi^{V_\kappa}(x, y, p)$

$\underline{Y = V_{\alpha+1}}$ WORKS.

$V_\kappa \models ZFC$

IF $X \in V_\kappa$ THEN EVERY CHOICE FUNCTION FOR X (SUBSET OF $X \times \bigcup X$) IS IN V_κ.

$\lambda \in V_\kappa$: λ IS A CARDINAL IFF $V_\kappa \models \lambda$ IS A CARDINAL

- λ IS REGULAR IFF $V_\kappa \models \lambda$ IS REGULAR
- λ IS STRONG LIMIT IFF $V_\kappa \models \lambda$ IS A STRONG LIMIT

IF $x \in V_\kappa$ THEN $\mathcal{O}(x) \in V_\kappa$.

IF "ZFC + THERE IS AN INACC."
IS CONSISTENT THEN
SO IS "ZFC + THERE IS NO INACC"

- $\kappa \longrightarrow (\kappa)^2_2$ ⟵ WEAKLY COMPACT

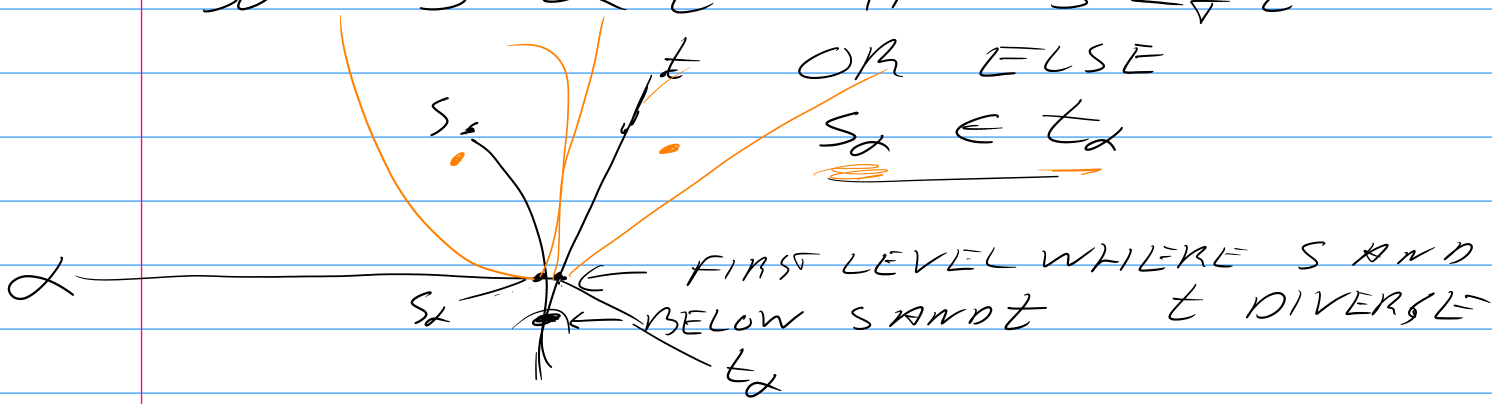
κ IS NOT SINGULAR
NOT A SUCCESSOR
A STRONG LIMIT

- WEAKLY COMPACT → TREE PROPERTY.

TAKE A TREE $(T, <_T)$
- LEVELS CARDINALITY $< \kappa$
- $|T| = \kappa$.
- WE WANT A BRANCH OF LENGTH κ.

WE MAY ASSUME $T = \kappa$.
WE TAKE $<_T$ AND EXTEND IT.
WE MAKE A LINEAR ORDER $\prec$
SUCH THAT $s <_T t \rightarrow s \prec t$

SO $s \prec t$ IF $s <_T t$
OR ELSE
$s_\alpha \in t_\alpha$



EXERCISE: THIS IS TRANSITIVE

$F : [\kappa]^2 \longrightarrow 2$

$$F(\{s,t\}) = \begin{cases} 1 & \text{IF } s < t \Leftrightarrow s \in t \\ 0 & \text{IF } s < t \Leftrightarrow t \in s \end{cases}$$

LET H BE HOMOGENEOUS OF CARDINALITY κ.

LET $B = \{x \in \kappa : |\{\alpha \in H : x <_T \alpha\}| = \kappa\}$

- IF $x \in B$ AND $y <_T x$ THEN $y \in B$
- $B \cap T_\alpha$ FOR ALL α

$\{x \in H : o(x) > \alpha\} = \kappa$

$\{x \in H : o(x) \leq \alpha\} < \kappa$

$|T_\alpha| < \kappa$

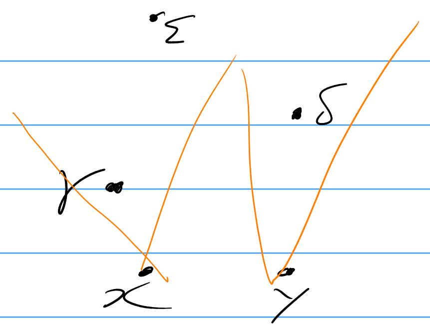
THERE IS $x \in B \cap T_\alpha$

- IF $x, y \in B \cap T_\alpha$ AND $x \neq y$ THEN WE'LL GET A CONTRADICTION.

SAY $x \in y$

TAKE $\gamma \in H : x <_T \gamma$

TAKE $\delta \in H : y <_T \delta$
$\gamma \in \delta$



TAKE $\varepsilon \in H : x <_T \varepsilon$
$\delta \in \varepsilon$

WE HAVE: $x \in y$

SO $\gamma \not< \delta$, $\varepsilon \not< \delta$
$\gamma \in \delta$, $\delta \in \varepsilon$

$F(\{\gamma, \delta\}) = 1 \quad F(\{\delta, \varepsilon\}) = 0$

CONTRADICTION
H WAS HOMOGENEOUS.

SO B MEETS EVERY T_α
IS ONE POINT
SO B IS A BRANCH

REMARK:
THIS SHOWS: RAMSEY $\Rightarrow$ KÖNIG.

- IF $\varkappa$ IS (STRONGLY) INACCESSIBLE AND IT HAS THE TREE PROPERTY THEN IT IS WEAKLY COMPACT.

WE SHOW $\varkappa \longrightarrow (\varkappa)^2_\lambda$ $(\lambda < \varkappa)$

$F : [\varkappa]^2 \longrightarrow \lambda$

WE CONSTRUCT $T \subseteq \bigcup_{\alpha < \varkappa} \lambda^\alpha$

$\varkappa$ INACC:

$$|T_\alpha| = |\lambda^\alpha| < \varkappa$$

- $t_0 = \emptyset$
- ASSUME $\langle t_\beta : \beta < \alpha \rangle$ IS FOUND
WE MAKE t_α BY RECURSION ON $\gamma \in \varkappa$ $(\gamma \leq \alpha)$

- $t_\alpha \restriction 0 = t_0$
NOTICE $t_\alpha \restriction 0$ IS IN $\{t_\beta : \beta < \alpha\}$
IT IS t_0 WE MAKE t_α (AT LEAST) ON ELEMENT LONGER:
$t_\alpha(0) = F(\{0, \alpha\})$
WE HAVE $t_\alpha \restriction 1$

- IF $t_\alpha \restriction \xi$ IS FOUND
AND $t_\alpha \restriction \xi \in \{t_\beta : \beta < \alpha\}$
WE PUT $t_\alpha(\xi) = F(\{\beta, \alpha\})$
WHERE $t_\alpha \restriction \xi = t_\beta$.

IF $t_\alpha \restriction \xi \notin \{t_\beta : \beta < \alpha\}$ STOP

$\alpha \longmapsto t_\alpha$ IS INJECTIVE
SO $\overline{T}$ HAS CARDINALITY κ
LET B BE A BRANCH
OF LENGTH κ.
FOR $\gamma < \lambda$ LET
$H_\gamma = \{\alpha : \underline{\underline{t_\alpha \in B}}$ AND $t_\alpha{}^\frown\gamma \in B\}$

t_α γ

IF $\beta < \alpha$ IN H_γ
$\delta = \mathrm{DOM}\, t_\beta$
$t_\alpha \restriction \delta = t_\beta$ WE PUT $t_\alpha(\delta) = F(\{\beta,\alpha\})$
BUT $t_\alpha(\delta) = \gamma$ — BECAUSE $\beta \in H_\gamma$
SO H_γ IS HOMOGENEOUS
WITH COLOUR γ.

REGULARITY SOME H_γ HAS CARD. κ.

WITH MORE COMPLICATED TREES
WE GET
$$\kappa \longrightarrow (\kappa)^n_\lambda \qquad n \in \omega \quad \lambda < \kappa$$
FOR WEAKLY COMPACT κ.

[CHAPTER 28 THEOREMS 28.23
AND 28.24
PLUS EXERCISE 28.5

Σ^1_2 HAS TREE PROP. IFF
THERE IS A WEAKLY
COMPACT CARDINAL SOMEWHERE

MEASURABLE CARDINALS.

IS THERE A κ WITH A σ-COMPLETE NON-PRINCIPAL ULTRAFILTER?

SUPPOSE THERE IS ONE AND LET κ BE THE FIRST
CALL THE ULTRAFILTER $\mathcal{U}$.
THEN $\mathcal{U}$ IS κ-COMPLETE.
PROOF
SUPPOSE NOT:
WE HAVE $\lambda < \kappa$
AND $\{X_\alpha : \alpha < \lambda\} \subseteq \mathcal{U}$
WITH $I = \bigcap_{\alpha<\lambda} X_\alpha \notin \mathcal{U}$.

NOTE $\kappa \setminus I \in \mathcal{U}$
$X_\alpha \setminus I \in \mathcal{U}$

WORK ON $\kappa \setminus I$: $\bigcap_{\alpha<\lambda} (X_\alpha \setminus I) = \emptyset$

ASSUME $I = \emptyset$
PUT $Y_\alpha = \kappa \setminus X_\alpha$
$Z_\alpha = Y_\alpha \setminus \bigcup_{\beta<\alpha} Y_\beta$

THEN $\kappa = \bigcup_{\alpha<\lambda} Z_\alpha$, $\left(Z_\alpha \cap Z_\beta = \emptyset \atop \alpha \neq \beta\right)$

V ULTRAFILTER ON λ
$V = \{A \subseteq \lambda : \bigcup_{\alpha\in A} Z_\alpha \in \mathcal{U}\}$

- V IS AN ULTRAFILTER
- NON PRINCIPAL: $Z_\alpha \notin \mathcal{U}$
- σ-COMPLETE

CONTRADICTION: $\lambda < \kappa$.

DEF: κ IS MEASURABLE IF IT CARRIES A κ-COMPLETE NONPRINCIPAL ULTRAFILTER $\mathcal{U}$

- INACCESSIBLE
- REGULAR:
  - IF $|X| < \kappa$ THEN $X \notin \mathcal{U}$
    BECAUSE $\bigcap_{x \in X} (X \setminus \{x\}) = \emptyset$
  - IF $\mathrm{cf}\,\kappa < \kappa$ AND $\langle \beta_\gamma : \gamma < \mathrm{cf}\,\kappa \rangle$ IS COFINAL
    THEN $\bigcap_{\gamma < \mathrm{cf}\,\kappa} \underbrace{[\beta_\gamma, \kappa)}_{\in \mathcal{U}} = \emptyset$
- STRONG LIMIT
  IF $\lambda < \kappa \leq 2^\lambda$
  THEN USE THAT
  2^λ DOES NOT HAVE A NONPRINCIPAL λ^+-COMPL. ULTRAFILTER.

LOOK AT 2^λ (SET OF FUNCTIONS)

FOR $\alpha \in \lambda$
$2^\lambda = \{f : f(\alpha) = 1\} \cup \{f : f(\alpha) = 0\}$

OUR U.F. $\mathcal{U}$ PICKS i_α WITH
$A_\alpha = \{f : f(\alpha) = i_\alpha\} \in \mathcal{U}$

$g(\alpha) = i_\alpha$: $\bigcap_{\alpha < \lambda} A_\alpha = \{g\}$

NORMAL FILTER:
CLOSED UNDER Δ-INTERSECTIONS

$\mathcal{F}$ FILTER:
$$\mathcal{F}^+ = \{A : (\forall F \in \mathcal{F})(F \cap A \neq \emptyset)\}$$
$\mathcal{C}_\kappa^+$ = STATIONARY SETS

$\mathcal{F}$ IS NORMAL IFF
$\mathcal{F}^+$ SATISFIES FODOR'S
PRESSING DOWN LEMMA

- IF κ IS MEASURABLE THEN
IT CARRIES A NORMAL
κ-COMPLETE NONPRINCIPAL
ULTRAFILTER $\mathcal{U}$
[A NORMAL MEASURE]

PROOF:
IF $\mathcal{U}$ IS κ-COMPLETE NONPRINCIPAL
THEN THERE IS $f : \kappa \to \kappa$
SUCH THAT
$$D = \{X : f^{-1}[X] \in \mathcal{U}\}$$
IS A NORMAL MEASURE.

LOOK AT κ^κ
$$f \equiv g \quad \text{IF } \{\alpha : f(\alpha) = g(\alpha)\} \in \mathcal{U}$$
[$\equiv$ IS AN EQUIVALENCE RELATION]
$[f]$: THE EQUIVALENCE CLASS
$$[f] < [g] \text{ MEANS } \{\alpha : f(\alpha) \in g(\alpha)\} \in \mathcal{U}$$
$$[f \equiv f',\ g \equiv g' : \ f < g \leftrightarrow f' < g']$$

$<$ IS A LINEAR ORDER OF $\kappa^\kappa/\equiv$

IT IS EVEN A WELL-ORDER

SUPPOSE $[f_0] > [f_1] > \cdots > [f_n] > [f_{n+1}] > \cdots$

$A_n = \{\alpha : f_n(\alpha) \ni f_{n+1}(\alpha)\} \in U$

LET $A = \bigcap_{n \in \omega} A_n$ (IN U)

TAKE $\alpha \in A$ THEN

$$f_0(\alpha) \ni f_1(\alpha) \ni \cdots \ni f_n(\alpha) \ni f_{n+1}(\alpha) \ni \cdots$$

SO $\langle \kappa^\kappa/\equiv, < \rangle$ IS A WELL-ORDER

FOR $\alpha \in \kappa$: $\underline{\alpha} : \kappa \longrightarrow \kappa$ IS THE CONSTANT FUNCTION WITH VALUE α

$\alpha \longmapsto [\underline{\alpha}]$ IS AN EMBEDDING OF κ INTO $\kappa^\kappa/\equiv$

IT GIVES US AN INITIAL SEGMENT

IF $[f] < [\underline{\alpha}]$

THEN THERE IS $\beta < \alpha$ SUCH THAT $f \equiv \underline{\beta}$ (κ-COMPLETENESS)

$d : \kappa \longrightarrow \kappa$ $\quad d(\alpha) = \alpha$

SATISFIES

$$[\underline{\alpha}] < [d]$$

FOR ALL α.

TAKE $f : \kappa \longrightarrow \kappa$ SUCH THAT $[f]$ IS THE SMALLEST EQ CLASS ABOVE ALL $[\underline{\alpha}]$.

THAT f WORKS:

- $f : \kappa \longrightarrow \kappa$
- $[f] > [\check{\alpha}]$ ALL α
- IF $[g] > [\check{\alpha}]$ FOR ALL α THEN $[g] \geq [f]$.

$\rightarrow \{\gamma : f(\gamma) > \alpha\} \in U$ $(*)$

$D = \{X : f^{-1}[X] \in U\}$

- D ULTRAFILTER [ALWAYS]
- $f^{-1}[\{\alpha\}] \notin U$ BECAUSE $(*)$
- D IS κ-COMPLETE
- D IS NORMAL

LET $X \in D$ AND $r : X \to \kappa$ REGR.

$g = r \circ f : f^{-1}[X] \longrightarrow \kappa$

$g(\alpha) = r(f(\alpha)) < f(\alpha)$ $\alpha \in f^{-1}[X]$

SO $[g] < [f]$

HENCE $[g] = [\check{\beta}]$ FOR SOME β

$\{\alpha : r(f(\alpha)) = \beta\} \in U$

$\{\gamma : r(\gamma) = \beta\} \in D$

D IS NORMAL!

- LET κ BE MEASURABLE, D A NORMAL MEASURE, $F : [\kappa]^{<\omega} \longrightarrow \lambda$ WITH $\lambda < \kappa$. THEN THERE IS $H \in D$ SUCH THAT F IS CONSTANT ON ALL $[H]^n$, FOR ALL $n \in \omega$ AT ONCE.

THE COLOURS OF F ON THE $[H]^n$
CAN BE DIFFERENT;

$$\left(F(x) = |x| \;! \right)$$

CONSTANT ON $[x]^n$

PROOF BY κ-COMPLETENESS
WE NEED TO PROVE IT
FOR INDIVIDUAL n ONLY

$n = 1$ CLEAR

$n \to n+1$ TAKE $F : [\kappa]^{n+1} \to \lambda$

FOR $\alpha \in \kappa$

$$F_\alpha : [\kappa \setminus \{\alpha\}]^n \longrightarrow \lambda$$

$$x \longmapsto F(x \cup \{\alpha\})$$

WE FIND $X_\alpha \in D$ HOMOGENEOUS
FOR F_α WITH COLOUR c_α

$\cdot$ $X = \Delta_{\alpha < \kappa} X_\alpha \in D$ NORMALITY

IF $\gamma < \alpha_1 < \alpha_2 \cdots < \alpha_n$ IN X
THEN $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \in [X_\gamma]^n$
SO $F(\{\gamma, \alpha_1, \alpha_n \ldots, \alpha_n\}) = c_\gamma$

κ-COMPLETENESS THERE
IS A $c < \lambda$ SUCH THAT

$$Y = \{ \gamma \in X : c_\gamma = c \} \in D$$

F IS CONSTANT ON $[Y]^{n+1}$

GO BACK TO THE PROOF THAT
SELECTIVE UF'S ON ω
HAVE THE RAMSEY PROPERTY