SET THEORY 2020-11-30

ULTRAPOWERS OF THE UNIVERSE

QUIZ: (1) INACCESSIBLE STRONG / WEAK !

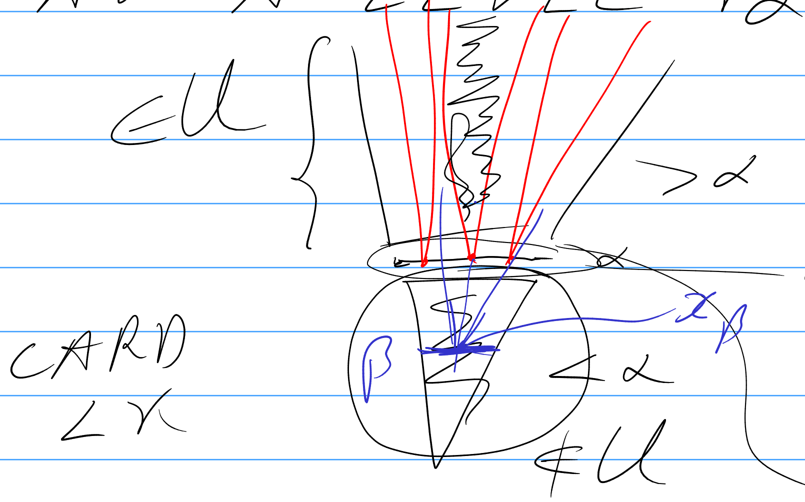
(2) $\kappa \to (\kappa)^2_2$ IMPLIES L HAS A κ-SEQUENCE INCR. OR DECR.

$\{(\alpha, \alpha+2) : \alpha < \kappa, \alpha \text{ EVEN}\}$

(3) TAKE THE UF ON κ PRETEND IT LIVES ON κ^+

MEASURABLE $\longrightarrow$ WEAKLY COMPACT
(LAST WEEK: MEASURABLE IMPLIES RAMSEY)

GIVEN A TREE T LOOK AT A LEVEL T_α



$x \in T_\alpha$:
$A_x = \{y : y > x\}$

$\in U$ UF ON T
κ-COMPLETE

THERE IS x_α WITH $A_{x_\alpha} \in U$

$\{x_\alpha : \alpha < \kappa\}$ IS A BRANCH

GROUP INT // EX(3)

MEASURABLE $\rightarrow$ MAHLO

SET OF REG. CARDS BELOW κ IS STAT.

κ MEAS'LE $\quad D$ NORMAL MEASURE

- $\{\alpha < \kappa : \operatorname{cf}\alpha = \alpha\} \in D$
- SUPPOSE NOT : $\{\alpha : \operatorname{cf}\alpha < \alpha\} \in D$

FODOR : THERE IS A $\lambda < \kappa$ SUCH THAT $E_\lambda = \{\alpha < \kappa : \operatorname{cf}\alpha = \lambda\} \in D$

FOR $\alpha \in E_\lambda$ TAKE $\langle x_{\alpha,\eta} : \eta < \lambda \rangle$ INCR. COFINAL IN α.

APPLY FODOR λ TIMES TO GET $D \ni A_\eta \subseteq E_\lambda$ AND x_η SUCH THAT $\alpha \in A_\eta \rightarrow x_{\alpha,\eta} = x_\eta$

D IS κ-COMPLETE!

$$A = \bigcap_{\eta<\lambda} A_\eta \in D$$

COMPARE HW or Ex (32)

NOW

$$\alpha \in A \rightarrow \langle x_{\alpha,\eta} : \eta < \lambda \rangle = \langle x_\eta : \eta < \lambda \rangle$$

BUT NOW $\alpha = \sup_{\eta<\lambda} x_\eta$ FOR ALL $\alpha \in A$ ...... CONTRADICTION!

ULTRAPOWERS OF THE UNIVERSE

$$V^\omega_\omega / U = \text{"EQ. CLASSES"}$$

$$V_\omega \hookrightarrow V^\omega_\omega / U$$

S A SET, ULTRAFILTER $\mathcal{U}$ ON S

V^S IS THE CLASS
$\{f : f$ IS A FUNCTION, $\mathrm{DOM}\, f = S\}$

$f =_u g$ MEANS $\{x \in S : f(x) = g(x)\} \in \mathcal{U}$
$f \in_u g$ MEANS $\{x \in S : f(x) \in g(x)\} \in \mathcal{U}$

SETS PLEASE $[f]_u = \{g : g =_u f\}$

[$[f]_u$ IS A PROPER CLASS! GT]

SCOTT'S TRICK

REDEFINE

$[f]_u = \{g : g =_u f$ AND $(\forall h)(h =_u f \to \mathrm{RANK}\, g \le \mathrm{RANK}\, h)\}$

[$f \notin [f]_u$ IS NOW POSSIBLE]

ULTRAPOWER V^S/u IS THE CLASS $\{[f]_u : f \in V^S\}$

JECH'S NOTATION $\mathrm{ULT}_u(V)$
(JUST ULT)

$[f]_u \in_u [g]_u$ MEANS $f \in_u g$
INDEPENDENT OF REPRESENTATIVES

$(V, \in)$ $\qquad$ $(ULT, \in_U)$

- ELEMENTARILY EQUIVALENT
  IF φ IS A SENTENCE
  $(V, \in) \models \varphi$ IFF $(ULT, \in_U) \models \varphi$
  ŁOS'S THEOREM

- $[f_1], \dots, [f_n]$
  $(ULT, \in_U) \models \varphi([f_1], \dots, [f_n])$ –
  IFF $\{x : (V, \in) \models \varphi(f_1(x), \dots, f_n(x))\} \in U$

- $$J_U : V \longrightarrow ULT_U(V)$$
  $$x \longmapsto [\underline{x}]_U$$
  IS AN ELEMENTARY EMBEDDING.

$(V, \in) \models \varphi(a_1, \dots, a_k)$ IFF
$(ULT, \in_U) \models \varphi(J(a_1), \dots, J(a_k))$

IF $\varphi(y, x_1, \dots, x_k)$ IS A FORMULA
FOR ALL $a_1, \dots, a_n$ IN V

IF THERE IS $[f]$ IN ULT
SUCH THAT $\varphi([f], J(a_1), \dots, J(a_k))$ HOLDS IN ULT
THEN THERE ALREADY $b \in V$
SUCH THAT $\varphi(b, a_1, \dots, a_k)$ HOLDS
$\varphi(J(b), J(a_1), \dots, J(a_k))$ IN ULT.

"EVERY EQUATION WITH PARAMETERS IN V THAT HAS A SOLUTION IN ULT ALREADY HAS A SOLUTION IN V"

$x \in \mathrm{Ult}_U(S)$ IFF $\varphi(x, S, U)$

$\exists f$:
- f IS A FUNCTION
- $\mathrm{dom}\, f = S$
- $\forall y \in x$
  - y IS FUNCTION
  - $\mathrm{dom}\, y = S$
  - $y =_U f$

AND $\forall z$ RANK $y \le$ RANK z

ULT ⊨ REGULARITY

POSSIBLY, THERE IS $\langle f_n : n \in \omega \rangle$ SUCH THAT

$$f_{n+1} \in_U f_n \quad \text{(ALL } n\text{)}$$

$\langle f_n : n \in \omega \rangle$ OR $\langle [f_n]_U : n \in \omega \rangle$ IS NOT A MEMBER OF ULT.

IS $\mathrm{EXT}(f) = \{ [g]_U : g \in_U f \}$ A SET?
ANS: YES [G.I.]

NO DECR. SEQUENCES?
TAKE U σ-COMPLETE!
NO DECR. SEQ: WELL-FOUNDED
BECAUSE ~~THE~~ EVERY NON-EMPTY SUBSET HAS AN $\in_U$-MINIMAL ELEMENT

IF $f_{n+1} \in_U f_n$ FOR ALL n

LOOK AT $A_n = \{x : f_{n+1}(x) \in f_n(x)\}$

$A_n \in U$

U σ-COMPLETE: $A = \bigcap_n A_n \in U$

TAKE $x \in A$

THEN $f_{n+1}(x) \in f_n(x)$ FOR ALL n.

CONTRADICTING REGULARITY

IN V

- U IS PRINCIPAL
  BUT THEN $ULT_U(V) = V$

- U NON-PRINCIPAL?

  WE NEED MEAS'LE CARDINALS!

U IS σ-COMPLETE NON-PRINCIPAL

SO $ULT_U(V)$ IS WELL-FOUNDED

MOSTOWSKI SEE LECTURE 4

$$\boxed{\text{MOSTOWSKI COLLAPSE}}$$

$\pi : (ULT, \in_U) \longrightarrow (M, \in)$

- M IS TRANSITIVE: $y \in x,\ x \in M$
  $\Rightarrow y \in M$

π IS AN ISOMORPHISM

$$[f] \in_u [g] \iff \pi([f]) \in \pi([g])$$

(ALSO BIJECTIVE)

$$\pi([f]) = \{ \pi([g]) : [g] \in_u [f] \}$$

π IS INJECTIVE BY <u>EXTENSIONALITY</u>

$$\emptyset \longrightarrow j(\emptyset) = [\emptyset] \mapsto \pi(j(\emptyset)) = \emptyset$$

$$\omega \longrightarrow j(\omega) = [\omega] \mapsto \{ j(i) : i \in \omega \} \; ? \; = \omega$$

IF $[f] \in_u [\omega]$

THEN $S = \bigcup_i \{x : f(x) = i\}$

σ-compl: $\{x : f(x) = i\} \in U$ FOR ANY i

$$[f] =_u [i]$$

GJ [IF U IS <u>NOT</u> σ-COMPLETE (ON ω)

$d : \omega \to \omega \quad d(n) = n$

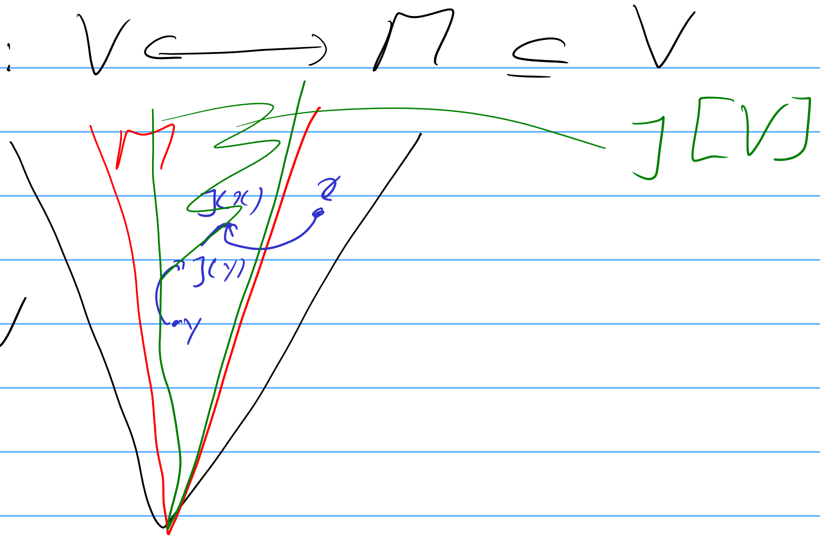
$i \in_u d \in_u \omega \quad [i] \in_u [d] \in_u [\omega]$]

WE SUPPRESS MENTION OF π

$$\begin{tikzcd} V \arrow[r, "j_u"] \arrow[rr, bend right, "j = j_u"'] & \mathrm{Ult}_u(V) \arrow[r, "\pi"] & M \end{tikzcd}$$

WE WRITE $j(x)$ FOR $\pi(j(x))$

$$j : V \hookrightarrow M \subseteq V$$



ELEMENTARITY: α ORDINAL $\longrightarrow j(\alpha)$ ORDINAL

$j(\alpha) < j(\beta)$ IF $\alpha < \beta$

SO $\alpha \leq j(\alpha)$

$j(\alpha+1) = j(\alpha)+1$

THERE WILL BE α WITH $\alpha < j(\alpha)$

κ MEASURABLE, $\mathcal{U}$ κ-COMPLETE NON PRINCIPAL UF ON κ

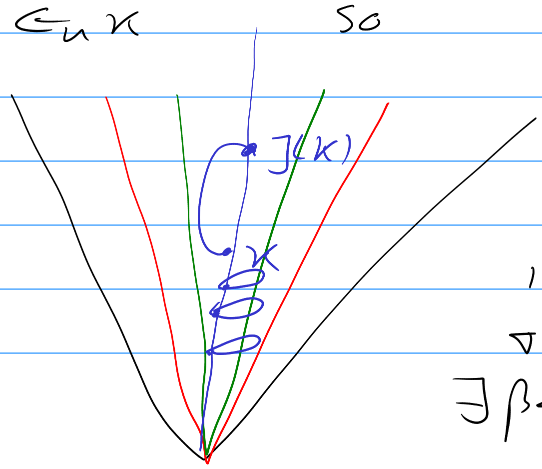
OUR $\mathrm{ULT}_{\mathcal{U}}(V) = V^{\kappa}/\mathcal{U}$

$d : \kappa \longrightarrow \kappa$

$\alpha \longmapsto \alpha$

$\alpha < \kappa$: $[\underline{\alpha}] \in_{\mathcal{U}} [d]$: $\alpha < \pi([d])$

$[d] \in_{\mathcal{U}} \kappa$ SO $\pi([d]) < j(\kappa)$



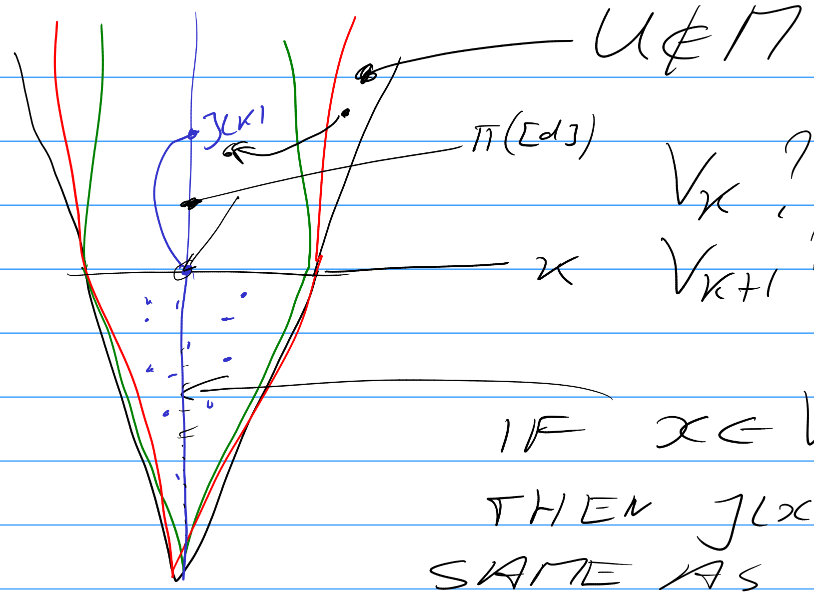
$\alpha < \kappa \rightarrow$

$j(\alpha) = \alpha$

IF $[f] \in_{\mathcal{U}} [\underline{\alpha}]$

THEN κ-COMPLETENESS

$\exists \beta < \alpha : [f] =_{\mathcal{U}} [\underline{\beta}]$



V_κ ?

$V_{\kappa+1}$?

IF $x \in V_\kappa$ THEN $j(x) = x$ SAME AS FOR $\alpha < \kappa$

TECHNICAL STUFF

- $M^\kappa \subseteq M$: IF $\langle a_\xi : \xi < \kappa \rangle$ IS A SEQUENCE OF MEMBERS OF M THEN $\langle a_\xi : \xi < \kappa \rangle \in M$.

THERE IS $f : \kappa \longrightarrow V$ SUCH THAT

$$\pi([f]) = \langle a_\xi : \xi < \kappa \rangle$$

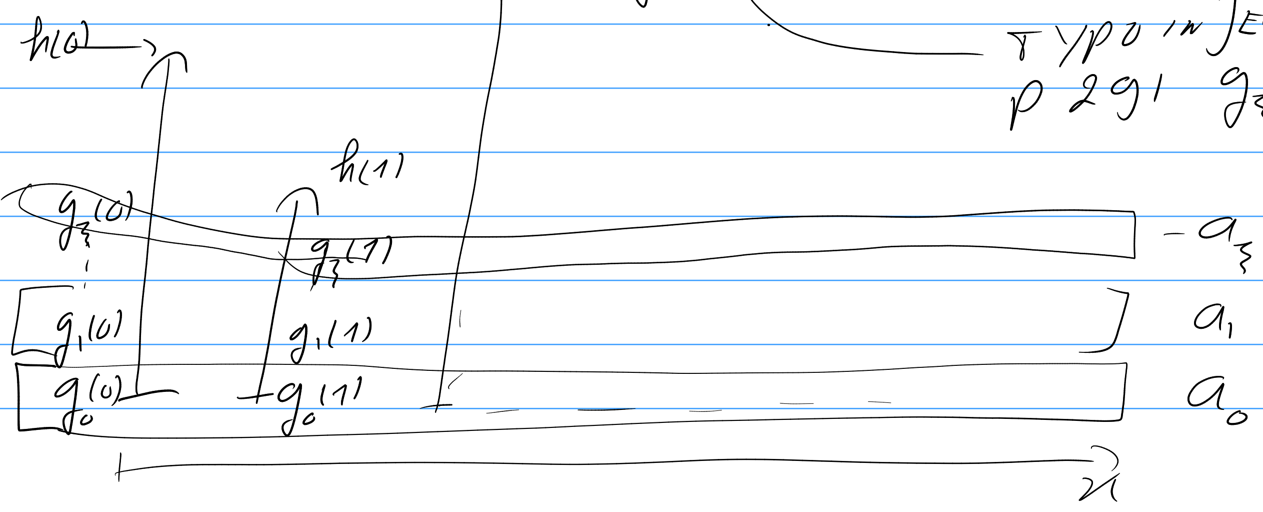
- TAKE g_ξ SUCH THAT $a_\xi = \pi([g_\xi])$

h SUCH THAT $\kappa = \pi([h])$

- MAKE f OUT OF THIS

$$f(\alpha) = \langle g_\xi(\alpha) : \xi < h(\alpha) \rangle$$

TYPO IN JECH p 291 g_ξ



$\{\alpha : \text{"}f(\alpha) \text{ IS AN } h(\alpha)\text{-SEQUENCE}\} = \kappa \in U$

IN ULT SO "[f] IS AN [h]-SEQUENCE"
IN M SO "$\pi([f])$ IS A κ-SEQUENCE"

FOR $\xi < \kappa$ WE HAVE $[\xi] \in_U [h]$
SO $A_\xi = \{\alpha : \xi < h(\alpha)\} \in U$

IF $\alpha \in A_\xi$: $f(\alpha)_\xi = g_\xi(\alpha)$

$\pi([f]_{[\xi]}) = \pi([g_\xi]) = a_\xi$

- $U \notin M$ SUPPOSE $U \in M$
  - $\kappa^\kappa \in M$ WHY
    $(\kappa^\kappa)^M = \kappa^\kappa \cap M = \kappa^\kappa$
  - $e: f \longmapsto \pi([f])$ FROM κ^κ TO M
    IS IN M
  - WE GET AS ITS IMAGE: $j(\kappa)$
    $\alpha < j(\kappa) \iff \alpha = \pi([f])$
    FOR SOME $f \in \kappa^\kappa$
  - IN M $|j(\kappa)| \leq 2^\kappa$ FALSE IN M
  - ELEMENTARITY: $j(\kappa)$ IS
    STRONGLY INACCESSIBLE
    IN M
    THAT WOULD MEAN $2^\kappa < j(\kappa)$
    CONTRADICTION
    THIS IS TRUE IN M
- $2^\kappa \leq (2^\kappa)^M < j(\kappa) < (2^\kappa)^+$
  $P(\kappa) = P(\kappa)^M$ FEWER BIJECTIONS IN M?
  $j(\kappa)$ COMES FROM κ^κ

- λ LIMIT $\quad cf\lambda = \kappa$: $\quad j(\lambda) > \sup_{\alpha<\lambda} j(\alpha)$
  $\quad cf\lambda \neq \kappa \quad j(\lambda) = \sup_{\alpha<\lambda} j(\alpha)$
- $cf\lambda = \kappa$
  TAKE $\langle \lambda_\eta : \eta < \kappa \rangle$ INCR. COFINAL
  $j(\alpha) < \underline{[\langle \lambda_\eta : \eta < \kappa \rangle]} < j(\lambda)$

- $cf\lambda > \kappa$ EVERY $f: \kappa \to \lambda$ IS BOUNDED
  SO $\{ j(\alpha) : \alpha < \lambda \}$ COFINAL IN $j(\lambda)$
- $cf\lambda < \kappa$ κ-COMPLETENESS:
  $f: \kappa \to \lambda$ THEN
  $[f] < j(\alpha)$ FOR SOME $\alpha < \lambda$

THEOREM
LET κ BE MEASURABLE, D A NORMAL MEASURE
THEN NOT ONLY IS κ WEAKLY COMPACT
BUT EVEN $\{ \alpha < \kappa : \alpha \text{ WEAKLY COMPACT} \}$
BELONGS TO D

PROOF [DEPRESSINGLY EASY]
WE KNOW $\boxed{P^M(\kappa) = P(\kappa)}$

- LET $F : [\kappa]^2 \to 2$ BE A PARTITION / COLOURING. $\underline{F \in M}$.
  IN V THERE IS $H \subseteq \kappa$ OF CARDINALITY κ SUCH THAT F IS CONSTANT ON $[H]^2$
  BUT $H \in M$

$M \models$ " κ IS WEAKLY COMPACT "

$$\kappa = \pi([d])$$

$Ult \models$ "$[d]$ IS WEAKLY COMPACT"

ŁOS' THEOREM

$\{\alpha : V \models$ "α IS WEAKLY COMPACT"$\} \in D$