SET THEORY 2020-12-07

- ULTRAPOWERS OF THE UNIVERSE II
- WHY 'WEAKLY COMPACT'?
- BUT FIRST: 7-12-1873

So glaube ich schliesslich zum Grunde gekommen zu sein, weshalb sich der in meinen früheren Briefen mit (x) bezeichnete Inbegriff *nicht* dem mit (n) bezeichneten eindeutig zuordnen lässt.

· PREVIOUSLY
STARTED WITH κ MEAS'LE
$\mathcal{U}$ κ-COMPLETE UF (NON-TRIVIAL)
FROM THAT

$=_{\mathcal{U}}$ ON V^{κ}
$\in_{\mathcal{U}}$ ON V^{κ}
$[f]_{\mathcal{U}}$ EQ CLASS

$(\mathrm{Ult}_{\mathcal{U}}(V), \in_{\mathcal{U}})$ ULTRAPOWER OF V BY $\mathcal{U}$.

$x \in V \longrightarrow \underline{x} : \kappa \longrightarrow V$ CONSTANT
$\alpha \longmapsto x$

$$j_{\mathcal{U}} : V \longrightarrow \mathrm{Ult}_{\mathcal{U}}(V)$$
$$x \longmapsto [\underline{x}]_{\mathcal{U}}$$

Łoś: THIS IS AN ELEMENTARY EMBEDDING.

$$(\mathrm{Ult}_{\mathcal{U}} V, \in_{\mathcal{U}}) \models \varphi([f_1]_{\mathcal{U}}, \ldots, [f_k]_{\mathcal{U}})$$
IFF
$$\{\alpha : (V, \in) \models \varphi(f_1(\alpha), \ldots, f_k(\alpha))\} \in \mathcal{U}$$

$d(\alpha) = \alpha$

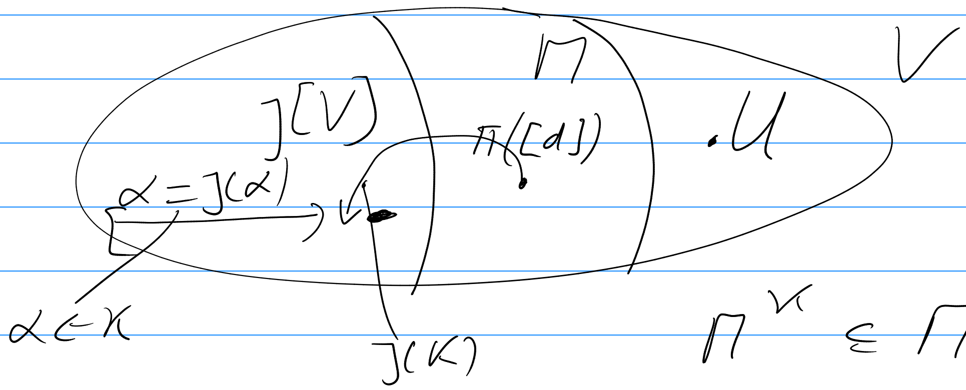
$[\underline{\alpha}] \in_u [d] \in_u [\underline{X}]$

$J_u(\kappa)$ IS NO LONGER
$\sup\{J_u(\alpha) : \alpha < \kappa\}$

IF $x \in V_\kappa$ AND $[f]_u \in_u [x]_u$
THEN $[f]_u =_u [y]_u$ FOR
SOME $y \in x$.
USES κ-COMPLETENESS.

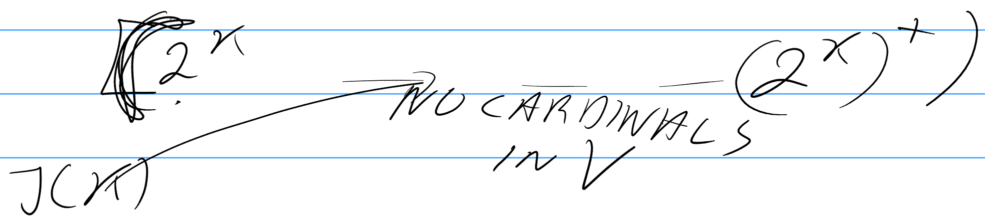
$(\mathrm{Ult}_u(V), \in_u)$ IS WELL-FOUNDED
MOSTOWSKI-COLLAPSE
$\pi : \mathrm{Ult}_u(V) \longrightarrow M$
$[f]_u \longmapsto \{\pi([g]_u) : g \in_u f\}$

J OR J_u USUALLY DENOTES $\pi \circ J_u$



$M^\kappa \in M$

$2^\kappa \le (2^\kappa)^M < J(\kappa) < (2^\kappa)^+$



IF THERE IS A MEAS'LE CARDINAL
THEN THERE ARE $M \subseteq V$
AND AN ELEM $J : V \longrightarrow M$
WITH $J(\kappa) > \kappa \leftarrow \kappa$ THE FIRST

IF D IS A NORMAL MEASURE
THEN THIS MACHINERY
HELPED US SHOW:

$\{\lambda < \kappa : \lambda$ IS WEAKLY COMPACT$\}$
IS IN D

$V_\kappa \models$ "THERE IS NO MEAS'LE CARDINAL + THERE ARE MANY WEAKLY COMPACT CARDINALS"

1ST MEAS'LE

CONVERSE:
SUPPOSE WE HAVE $M \subseteq V$
- M MODELS ZFC
- $j : V \hookrightarrow M$ ELEM. EMB.
  NOT THE IDENTITY

SO THERE IS A FIRST ORDINAL
WITH $j(\delta) > \delta$.

[EX. IF $j(\alpha) = \alpha$ $\quad \alpha \in \mathrm{ORD}$
THEN $j = \mathrm{ID}_V$
$j(V_\alpha) \in V_{j(\alpha)}$]

δ IS UNCOUNTABLE:
$j(\emptyset) = \emptyset \qquad j(\alpha+1) = j(\alpha)+1$
$j(\omega) = \omega$

CLAIM δ IS A MEASURABLE CARDINAL.

$$\delta \in j(\delta)$$

$D = \{ X \subseteq \delta : \delta \in j(X) \}$

- $j(\emptyset) = \emptyset$: $\emptyset \notin D$
- $\delta \in j(\delta)$: $\delta \in D$
- $j(\delta \setminus X) = j(\delta) \setminus j(X)$

  ALWAYS: $X \in D$ OR $\delta \setminus j(X) \in D$
- $j(X \cap Y) = j(X) \cap j(Y)$

  D CLOSED UNDER $\cap$
- $X \subseteq Y \to j(X) \subseteq j(Y)$

$\therefore$ D IS AN ULTRAFILTER

D IS κ-COMPLETE

LET $\lambda < \delta$ AND $\langle X_\alpha : \alpha < \lambda \rangle = S$

SUCH THAT $X_\alpha \in D$ FOR ALL α.

WHY $\delta \in j\left(\bigcap_{\alpha<\lambda} X_\alpha\right)$?

- $j(\alpha) = \alpha$ $\quad \underline{\alpha \leq \lambda}$
- S IS A λ-SEQUENCE SO

$j(S)$ IS A $\underbrace{j(\lambda)}_{\lambda}$-SEQUENCE

$j(S) = \langle j(X_\alpha) : \alpha < \lambda \rangle$

$\langle \alpha, X_\alpha \rangle \in S$ SO

$\langle j(\alpha), j(X_\alpha) \rangle \in j(S)$

SO $\langle \alpha, j(X_\alpha) \rangle \in j(S)$

$$\bigcap_{\alpha<\lambda} X_\alpha = \{ x : (\forall \alpha < \lambda)(x \in X_\alpha) \}$$

$$j\left(\bigcap_{\alpha<\lambda} X_\alpha\right) = \{ x : (\forall \alpha < j(\lambda))(x \in j(X_\alpha)) \}$$

$$j\left(\bigcap_{\alpha<\lambda} X_\alpha\right) = \bigcap_{\alpha<\lambda} j(X_\alpha) \ni \delta$$

HW: D IS EVEN NORMAL

JECH HAS

$$\begin{tikzcd} V \arrow[rr, hook, "j"] \arrow[dr, "j_D"'] & & M \\ & \mathrm{Ult}_D(V) \arrow[ur] & \end{tikzcd}$$

LAST QUIZ. $V \hookrightarrow \mathrm{Ult}_D(V) \xrightarrow{\pi} M_D$

$D \notin M_D$

$D \in V_{\kappa+2} \setminus (V_{\kappa+2})^M$

TEMPLATE $j: V \hookrightarrow M$

WITH $V_{\kappa+2} \subseteq M$

NO DISCONNECT HERE

WHO SAYS M IS AN ULTRAPOWER?

KUNEN: THERE IS NO ELEMENTARY EMBEDDING

$j: V \hookrightarrow V$

NON-TRIVIAL OF COURSE

M IS ALWAYS ASSUMED TRANSITIVE

BUT $j[V]$ NEED NOT BE

IS NOT ACTUALLY

$\kappa < j(\kappa)$

IF $j : V \hookrightarrow M$ IS ELEMENTARY
TRANSITIVE
AND j IS NON-TRIVIAL
THEN $M \neq V$

LET $j : V \longrightarrow M$ BE ELEMENTARY
AND κ MINIMAL WITH $j(\kappa) > \kappa$.
SUPPOSE $C \subseteq \kappa$ IS CUB
THEN $\kappa \in j(C)$
$V \models$ "C IS CUB IN κ"
$M \models$ "$j(C)$ IS CUB IN $j(\kappa)$"
• $j(\alpha) = \alpha \quad \alpha < \kappa$
SO $j(C) \cap \kappa = C$
$\alpha < \kappa$: $\alpha \in C$ IFF $j(\alpha) \in j(C)$
IFF $\alpha \in j(C)$
– $\sup(j(C) \cap \kappa) \in j(C)$
– $\sup(j(C) \cap \kappa) = \sup C = \kappa$
CONTINU THIS MAGIC:

κ IS REGULAR IN M
SUPPOSE $\lambda < \kappa$ AND
$f : \lambda \longrightarrow \kappa$ IS GIVEN
THEN $j(f) = f$
WHY $j(\alpha) = \alpha$ FOR $\alpha < \kappa$
SO $j(\lambda) = \lambda$
$\rightarrow \langle \alpha, \beta \rangle \in f$ IFF
$\langle j(\alpha), j(\beta) \rangle \in j(f)$
SO $\langle \alpha, \beta \rangle \in j(f)$
DOM $j(f) = j(\lambda) = \lambda$
NOW $\forall \alpha < \lambda \; f(\alpha) < \kappa$

M ⊨ $\exists \delta$ such that ← κ itself!

$$(\forall \alpha < j(\lambda))(j(f)(\alpha) < \delta < j(\kappa))$$

V ⊨ $\exists \delta$ such that

$$(\forall \alpha < \lambda)(f(\alpha) < \delta < \kappa)$$

κ regular in V hence in M

If $\lambda < \kappa$ and $f \in M$ is a cofinal map: $\lambda \to \kappa$

$\kappa \in j(R)$ $\quad R = \{\alpha : cf\alpha = \alpha\}$

But then

$$C \cap \{\alpha < \kappa : cf\alpha = \alpha\} \neq \emptyset$$

for all cub $C \subseteq \kappa$.

$j(C) \cap j(R) \ni \kappa$

so $C \cap R \neq \emptyset$

The regular cardinals are stationary below κ.

Let κ be measurable

D a normal measure on κ

so $j([d]) = \kappa$

If $\{\lambda < \kappa : 2^\lambda = \lambda^+\} \in D$

then $2^\kappa = \kappa^+$

$[d]_D$ SATISFIES ιT IN $ULT_D(V)$

$\kappa = [d]_D$

$\Downarrow$

THAT MEANS $M \models \underline{2^\kappa = \kappa^+}$

WHAT ABOUT V?

IN M $b : P^M(\kappa) \longrightarrow (\kappa^+)^M$

- $P^M(\kappa) = P(\kappa)$
- $(\kappa^+)^M = \kappa^+$

EVERY WELL-ORDER OF κ IS AN ELEMENT OF M

THERE WE ARE

$b : P(\kappa) \longrightarrow \kappa^+$

WHY 'WEAKLY COMPACT'?

COMPACT AS IN COMPACTNESS IN MODEL THEORY

COMPACTNESS THEOREM FOR FIRST-ORDER LOGIC:

IF Σ IS A SET OF FORMULAS SUCH THAT EVERY FINITE SUBSET HAS A MODEL

THEN Σ HAS A MODEL.

MORE EXPRESSIVE LANGUAGES

JECH: THE LANGUAGES $\mathcal{L}_{\kappa,\omega}$ AND $\mathcal{L}_{\kappa,\kappa}$

BETTER

LANGUAGES OF TYPE $\mathcal{L}_{\kappa,\omega}$ AND $\mathcal{L}_{\kappa,\kappa}$.

TYPE: $\mathcal{L}_{\kappa,\omega}$, $\mathcal{L}_{\kappa,\kappa}$

- κ MANY VARIABLES
- RELATION SYMBOLS } n-ARY FOR $n<\omega$
  FUNCTION SYMBOLS
  CONSTANTS
- LOGICAL CONNECTIVES $\neg, \to$ $\vee, \wedge$

$\kappa \to$ • INFINITARY CONNECTIVES

$\bigvee_{\xi<\alpha} \varphi_\xi$, $\bigwedge_{\xi<\alpha} \varphi_\xi$ $(\alpha < \kappa)$

$\omega \to$ • $\exists v$, $\forall v$ — $\mathcal{L}_{\kappa,\omega}$

$\kappa \to$ • $\exists_{\xi<\alpha} v_\xi$, $\forall_{\xi<\alpha} v_\xi \leftarrow \mathcal{L}_{\kappa,\kappa}$

$\exists_{n<\omega} v_n$ LETS YOU SAY "THERE IS A SEQUENCE"

$\forall_{n<\omega} v_n$ "FOR ALL SEQUENCES"

$\mathcal{L}_{\omega,\omega}$ IS JUST FIRST ORDER

$\exists_{i<n} v_i = \exists v_0 \cdots \exists v_{n-1}$

PROVE KÖNIG'S INFINITY LEMMA USING COMPACTNESS OF $\mathcal{L}_{\omega,\omega}$

TAKE A TREE $(T, <_T)$ INFINITE FINITE LEVELS

- c_t ; FOR $t \in T$
- B UNARY PREDICATE

$\uparrow$ BRANCH

FORMULAS:

- IF $\boxed{s \not\leq_T t \text{ AND } t \not\leq_T s}$ THEN

$\varphi_{s,t}$ $\quad \neg(B(c_t) \wedge B(c_s))$ IS ONE OF OUR FORMULAS

- FOR EACH n:

σ_n IS $\bigvee_{t \in T_n} B(c_t)$

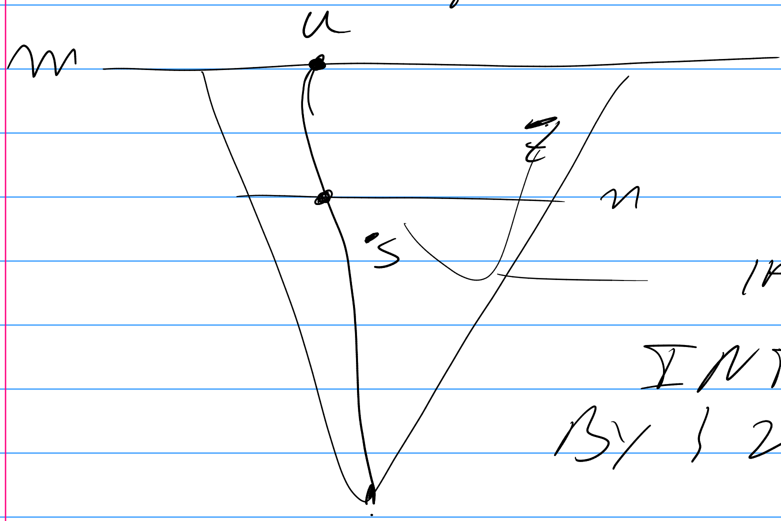
$\boxed{\text{FOR EACH } \alpha < \omega_1 \quad \sigma_\alpha \text{ IS } \bigvee_{t \in T_\alpha} B(c_t)}$

- $\Sigma = \{\sigma_n : n \in \omega\} \cup \{\varphi_{s,t} : s \not\leq t,\ t \not\leq s\}$

α

IF $S \subseteq \Sigma$ IS FINITE COUNTABLE

TAKE m α SO LARGE THAT IF $s \in T$ AND c_s OCCURS IN A FORMULA IN S THEN $o(s) < m$ α



IF $\sigma_n \in S$

IF $\varphi_{s,t} \in S$

INTERPRET B BY $\{v : v < u\}$

THIS IS A MODEL FOR S

SO Σ HAS A MODEL

TAKE $t \in T$ FOR WHICH $B(c_t)$ IS TRUE IN THE MODEL

BUT Σ HAS NO MODEL

FOR EVERY n ONE t IS IN T_n

IF s AND t ARE SUCH

THEN $s < t$ OR $t < s$

THESE t'S FORM A BRANCH.

= THE WEAK COMPACTNESS THEOREM FOR $\mathcal{L}_{\kappa,\omega}$ OR $\mathcal{L}_{\kappa,\kappa}$

IF Σ IS A SET OF κ MANY FORMULAS SUCH THAT EVERY SUBSET S OF CARD. $< \kappa$ HAS A MODEL

THEN Σ HAS A MODEL.

GI $\mathcal{L}_{\omega_1,\omega}$, $\mathcal{L}_{\omega_1,\omega_1}$

AN ARONSZAJN TREE SHOWS THAT WEAK COMPACTNESS FAILS FOR TYPE $\mathcal{L}_{\omega_1,\omega}$ AND $\mathcal{L}_{\omega_1,\omega_1}$

THEOREM:

1. IF κ IS WEAKLY COMPACT THEN $\mathcal{L}_{\kappa,\kappa}$ SATISFIES THE WEAK COMPACTNESS THEOREM.

2. IF κ IS INACCESSIBLE AND IF $\mathcal{L}_{\kappa,\omega}$ SATISFIES WCT THEN κ IS WEAKLY COMPACT.

WEAK : $|\Sigma| = \kappa$

STRONG $(-\phi)$ $|\Sigma|$ ARBITRARY

PROOF

(2) WE DID THAT ALREADY

κ HAS THE TREE PROPERTY

IF $(T, <)$ HAS CARD κ

WITH LEVELS SIZE $< \kappa$

THEN REPEAT THE ABOVE ARGUMENT

B

$\{c_t : t \in T\}$

$$\Sigma \left| \begin{array}{lcl} \varphi_{s,t} & = & \neg(B(c_s) \wedge B(c_t)) \\ & & s,t \text{ INCOMPATIBLE} \\ \sigma_\alpha & = & \bigvee_{t \in T_\alpha} B(c_t) \end{array} \right.$$

IF $S \in [\Sigma]^{<\kappa}$ THEN

TAKE α ABOVE

ALL $O(S)$ WHERE

c_s OCCURS IN S

TAKE $u \in T_\alpha$

AND $B \sim \{v : v < u\}$

A MODEL FOR Σ GIVES US

A BRANCH OF TYPE κ.

(1) IN A LANGUAGE OF TYPE $\mathcal{L}_{\kappa,\kappa}$

TAKE A SET OF FORMULAS Σ

SUCH THAT - $|\Sigma| = \kappa$

$S \in [\Sigma]^{<\kappa} \rightarrow S$ HAS A MODEL.

MODEL. INTERPRET
PREDICATES
CONSTANTS
FUNCTIONS

$\varphi \wedge \psi$

$\bigvee_{\xi<\alpha} \psi_\xi \qquad \bigwedge_{\xi<\alpha} \psi_\xi$

$\mathbb{A}$: MODEL $\underline{a} = \langle a_\alpha : \alpha < \kappa \rangle$ IN $\mathbb{A}$

$\forall_{\xi<\alpha} v_\xi \ \varphi(\text{---})$

φ IS TRUE FOR ALL SEQUENCES THAT DIFFER FROM $\underline{a}$ BEFORE α.

WE CAN ASSUME OUR LANGUAGE HAS CARDINALITY κ. USE ONLY THE PREDICATES, CONSTANTS AND FUNCTIONS THAT OCCUR IN Σ