

HOMWORK SHEET #10

MasterMath: Set Theory

2020/21: 1st Semester

K. P. Hart, Benedikt Löwe, Ezra Schoen, & Ned Wontner

Deadline for Homework Set #10: Monday, 16 November 2020, 2pm.

- (33) Some applications of Ramsey's theorem.
- Let $\langle L, < \rangle$ be an infinite linearly ordered set. Prove that L has an infinite subset X that is well-ordered by $<$ or an infinite subset Y that is well-ordered by $>$.
 - Prove that every bounded sequence of real numbers has a convergent subsequence (the Bolzano-Weierstraß theorem). *Hint:* Find a monotone subsequence.
 - Let $\langle P, < \rangle$ be an infinite partially ordered set. Prove that P has an infinite subset C that is linearly ordered by $<$ (a chain) or an infinite subset U that is unordered by $<$, which means that if x and y in U are distinct then neither $x < y$ nor $y < x$.
- (34) Check the details of Erdős' example that shows $2^{\aleph_0} \not\rightarrow (3)_{\aleph_0}^2$: take the open unit interval $(0, 1)$ as the set of cardinality 2^{\aleph_0} and define $F(\{x, y\}) = k$ if $2^{-k+1} > |x - y| \geq 2^{-k}$. Show there is no three-element homogeneous set.
- (35) Prove that $2^\kappa \not\rightarrow (3)_\kappa^2$ for every cardinal κ . *Hint:* Take the set 2^κ of functions from κ to 2 and define $F : [2^\kappa]^2 \rightarrow \kappa$ by $F(\{x, y\}) = \min\{\alpha : x(\alpha) \neq y(\alpha)\}$.
- (36) Prove that $2^\kappa \not\rightarrow (\kappa^+)_2^2$ for every cardinal κ . *Hint:* Take the set 2^κ of functions from κ to 2. Let $<$ be the lexicographic order of 2^κ , defined by $x < y$ iff $x(\delta) < y(\delta)$ where $\delta = \min\{\alpha : x(\alpha) \neq y(\alpha)\}$ and let \prec be some well-order of 2^κ . $F : [2^\kappa]^2 \rightarrow 2$ by $F(\{x, y\}) = 1$ if \prec and $<$ agree on $\{x, y\}$ and let $F(\{x, y\}) = 0$ in the opposite case. The crux is to prove that any set well-ordered by $<$ has cardinality at most κ .