Homework Sheet #12

MasterMath: Set Theory
2020/21: 1st Semester
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Deadline for Homework Set #12: Monday, 30 November 2020, 2pm.

(40) Let $Q = \omega_1^{<\omega}$ be the tree of finite sequences of countable ordinals. Define an order $\prec$ on Q by

$$s \prec t \text{ if } \begin{cases} s \subset t & \text{that is: } s \text{ is a proper initial segment of } t\text{, or} \\ s(i) < t(i) & \text{where } i \text{ is minimal such that } s(i) \neq t(i) \end{cases}$$

a. Prove that $\prec$ is a linear order on Q.
b. Prove: if $s \in Q$ then $\{t : s \subset t\}$ is an interval in $(Q, \prec)$ that is order-isomorphic to Q.
c. Prove: for every $\alpha < \omega_2$ and every nonempty interval I of $(Q, \prec)$ there is an isomorphic copy of α in I. *Hint*: induction on α.

Note: this idea can be used on any tree to turn it into a linearly ordered set. Many interesting examples can be obtained in this way.

(41) [Exercise 28.5 in Jech] Assume the Continuum Hypothesis and construct an $\aleph_2$-Aronszajn tree. *Hint*: Take the linear order $(Q, \prec)$ from the previous exercise and mimic the construction of an $\aleph_1$-Aronszajn tree, but now inside the subtree I of $Q^{<\omega_2}$ that consists of all increasing sequences.

(42) [Exercise 8.8 in Jech] Let κ be regular uncountable and let $\mathcal{F}$ be a κ-complete filter on κ that contains the family $\{\kappa \setminus \alpha : \alpha \in \kappa\}$. Let $\mathcal{F}^+$ denote the family of sets that are *positive* with respect to $\mathcal{F}$, where A is positive if $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. (For example if $\mathcal{C}_\kappa$ is the cub filter then $\mathcal{C}_\kappa^+$ is the family of stationary sets.)
Prove that $\mathcal{F}$ is a normal filter if and only if it satisfies Fodor's Pressing-Down Lemma: if $A \in \mathcal{F}^+$ and is $f : A \to \kappa$ is regressive then f is constant on a set in $\mathcal{F}^+$.