# Group Interaction #8

**MasterMath: Set Theory**
*2021/22: 1st Semester*
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Every week, there will be one group interaction of roughly one hour. The group interactions take place remotely via Zoom. A group interaction consists of two students who work together on a work sheet in the presence of one of the two teaching assistants (Steef Hegeman or Robert Paßmann). A group does not have to cover the entire work sheet. If you do not finish the work sheet, feel free to return to it later or in the preparation of the exam.

For this group interaction $\kappa$ will always denote a regular uncountable cardinal.

(1) Revisit the definition of closed and unbounded subsets of $\kappa$. Consider some examples of sets that are (or are not) closed and unbounded. The set $\kappa$ itself, the empty set, an initial segment, a final segment, the set of successor ordinals, the set of limit ordinals, ...

(2) Show: if $f : \kappa \to \kappa$ is an arbitrary function then $\{\alpha : f[\alpha] \subseteq \alpha\}$ is closed and unbounded. *Hint*: think of a diagonal intersection.

(3) Let $f : \kappa^n \to \kappa$ be an arbitrary function for some $n \in \mathbb{N}$. Show that $C_f = \{\alpha : f[\alpha^n] \subseteq \alpha\}$ is closed and unbounded.

(4) Remember from Lecture V that an ordinal fuction $F$ is *normal* if it is increasing ($\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$) and continuous ($F(\alpha) = \sup_{\beta<\alpha} F(\beta)$ whenever $\alpha$ is a limit ordinal). Also recall that normal functions have many fixed points (points $\alpha$ such that $F(\alpha = \alpha)$).

(5) Show: if $F : \kappa \to \kappa$ is a normal function then $F[\kappa]$ is closed and unbounded.

(6) Show: if $F : \kappa \to \kappa$ is a normal function then $\{\alpha : F(\alpha) = \alpha\}$ is closed and unbounded.

(7) Show: if $C$ is closed and unbounded in $\kappa$ then there is a normal function $F$ such that $C = F[\kappa]$.

(8) If $C$ is closed and unbounded is there then a normal function $G$ such that $C = \{\alpha : G(\alpha) = \alpha\}$?

(9) Remember that a cardinal is *weakly inaccessible* if it is a regular limit cardinal.

(10) Show that is $\kappa$ is weakly inaccessible then the set of limit cardinals below $\kappa$ is closed and unbounded in $\kappa$.

(11) Recall te definition of a stationary subset of $\kappa$. Consider some examples of sets that are (or are not) closed and unbounded. The set $\kappa$ itself, the empty set, an initial segment, a final segment, the set of successor ordinals, the set of limit ordinals, arbitrary closed and unbounded subsets, ...

(12) Show that is $S$ is a stationary subset of $\kappa$ and $T$ is a subset of $\kappa$ such that $S \subseteq T$ then $T$ is also stationary. Deduce that not every stationary subset is closed. *Hint*: Try $\kappa \setminus \{\omega\}$

(13) Let $R(\kappa)$ denote the set of regular cardinals in $\kappa$. Remember that in the proof of Solovay's theorem we had to consider the case where $R(\kappa)$ is stationary in $\kappa$. We call a cardinal with this property a *Mahlo cardinal.*

(14) Show that every Mahlo cardinal is weakly inaccessible.

(15) Assuming Mahlo cardinals exist, what is the relation between the first Mahlo cardinal and the weakly inaccessible cardinals below it (if any)? *Hint*: Apply Exercise (10).