Group Interaction #10

MasterMath: Set Theory
2021/22: 1st Semester
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Every week, there will be one group interaction of roughly one hour. The group interactions take place remotely via Zoom. A group interaction consists of two students who work together on a work sheet in the presence of one of the two teaching assistants (Steef Hegeman or Robert Paßmann). A group does not have to cover the entire work sheet. If you do not finish the work sheet, feel free to return to it later or in the preparation of the exam.

This group interaction is devoted to absoluteness for transitive sets. The excerpts on this sheet are from Jech's book.

(1) Remind yourself of the definition of a Δ_0-formula:

> **Definition 12.8.** A formula of set theory is a Δ_0-formula if
>
> (i) it has no quantifiers, or
> (ii) it is $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$, $\varphi \to \psi$ or $\varphi \leftrightarrow \psi$ where φ and ψ are Δ_0-formulas, or
> (iii) it is $(\exists x \in y)\,\varphi$ or $(\forall x \in y)\,\varphi$ where φ is a Δ_0-formula.

(2) Consider the following formula:

$$\varphi(z) :\iff \exists x \forall y (z = z).$$

Is this formula a Δ_0-formula? Is it equivalent to a Δ_0-formula?

(3) Consider the following definitions from Chapter 1 in Jech's book:

$$C \subset D \text{ if and only if for all } x,\ x \in C \text{ implies } x \in D,$$

$$\begin{aligned} C \cap D &= \{x : x \in C \text{ and } x \in D\}, \\ C \cup D &= \{x : x \in C \text{ or } x \in D\}, \\ C - D &= \{x : x \in C \text{ and } x \notin D\}, \\ \bigcup C &= \{x : x \in S \text{ for some } S \in C\} = \bigcup\{S : S \in C\}. \end{aligned}$$

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

$$\emptyset = \{u : u \neq u\}$$

Write down the formulas describing these definitions and check that they are all Δ_0-formulas. Make sure that your formulas are really formulas in the language of set theory, not using any abbreviations.

(4) Write a Δ_0-formula in for "z is transitive" (again, do not use abbreviations).

(5) By the theorem about absoluteness of Δ_0-formulas for transitive sets, the concepts from (3) are absolute between transitive sets.
Take the formula for the last concept as an example, i.e., the formula $\psi(z)$ that expresses $z = \varnothing$. If now $M \subseteq N$ are transitive sets and $m \in M$, then

$$(M, \in) \models \psi(m) \iff (N, \in) \models \psi(m).$$

Check that you really understand why this is the case for the concrete formula ψ. In particular, contemplate why the transitivity of M and N matters for this.

(6) Lemma 12.10 in Jech's book contains a lot of examples of things that can be expressed as Δ_0-formulas:

> **Lemma 12.10.** *The following expressions can be written as Δ_0-formulas and thus are absolute for all transitive models.*
>
> (i) $x = \{u, v\}$, $x = (u, v)$, *x is empty*, $x \subset y$, *x is transitive*, *x is an ordinal*, *x is a limit ordinal*, *x is a natural number*, $x = \omega$.
>
> (ii) $Z = X \times Y$, $Z = X - Y$, $Z = X \cap Y$, $Z = \bigcup X$, $Z = \operatorname{dom} X$, $Z = \operatorname{ran} X$.
>
> (iii) *X is a relation*, *f is a function*, $y = f(x)$, $g = f \restriction X$.
>
> *Proof.*
>
> (i) $x = \{u, v\} \leftrightarrow u \in x \wedge v \in x \wedge (\forall w \in x)(w = u \vee w = v)$.
> $x = (u, v) \leftrightarrow (\exists w \in x)(\exists z \in x)(w = \{u\} \wedge z = \{u, v\})$
> $\wedge\, (\forall w \in x)(w = \{u\} \vee w = \{u, v\})$.
> x is empty $\leftrightarrow (\forall u \in x)\, u \neq u$.
> $x \subset y \leftrightarrow (\forall u \in x)\, u \in y$.
> x is transitive $\leftrightarrow (\forall u \in x)\, u \subset x$.
> x is an ordinal $\leftrightarrow x$ is transitive $\wedge (\forall u \in x)(\forall v \in x)(u \in v \vee v \in u \vee u = v)$
> $\wedge\, (\forall u \in x)(\forall v \in x)(\forall w \in x)(u \in v \in w \rightarrow u \in w)$.
> x is a limit ordinal $\leftrightarrow x$ is an ordinal $\wedge\, (\forall u \in x)(\exists v \in x)\, u \in v$.
> x is a natural number $\leftrightarrow x$ is an ordinal $\wedge$ (x is not a limit $\vee\, x = 0$)
> $\wedge\, (\forall u \in x)(u = 0 \vee u$ is not a limit$)$.
> $x = \omega \leftrightarrow x$ is a limit ordinal $\wedge\, x \neq 0 \wedge (\forall u \in x)\, x$ is a natural number.
>
> (ii) $Z = X \times Y \leftrightarrow (\forall z \in Z)(\exists x \in X)(\exists y \in Y)\, z = (x, y)$
> $\wedge\, (\forall x \in X)(\forall y \in Y)(\exists z \in Z)\, z = (x, y)$.
> $Z = X - Y \leftrightarrow (\forall z \in Z)(z \in X \wedge z \notin Y) \wedge (\forall z \in X)(z \notin Y \rightarrow z \in Z)$.
> $Z = X \cap Y$ ... similar.
> $Z = \bigcup X \leftrightarrow (\forall z \in Z)(\exists x \in X)\, z \in x \wedge (\forall x \in X)(\forall z \in x)\, z \in Z$.
> $Z = \operatorname{dom}(X) \leftrightarrow (\forall z \in Z)\, z \in \operatorname{dom} X \wedge (\forall z \in \operatorname{dom} X)\, z \in Z$,

Check that you understand these and check whether they are all complete. (Some of the given formulas are not quite complete.)

(7) Look at the formula for "x is an ordinal" in the excerpt from Jech's book in (6) and compare it to our official definition of ordinal:

> **Definition 2.10.** A set is an *ordinal number* (an *ordinal*) if it is transitive and well-ordered by $\in$.

Compare the two and say why (and under which conditions) they are equivalent.

(8) Write a Δ_0-formula expressing "$x = \omega$" (here, you may use the abbreviations $\varnothing$, $\{.\}$, $\{.,.\}$, and $\cup$).

(9) Prove that for every natural number n there is a Δ_0-formula that expresses "$x = n$". Can you give lower and upper bounds for the size of your formula in the case that $n = 10^{100}$?

(10) The results in class imply that if we take enough axioms in our list then we obtain a countable set A that has $\mathcal{P}(\omega)$ as an element such that also $[\mathcal{P}(\omega)$ is uncountable$]^A$ holds.
Let, as in class, $\pi : A \rightarrow M$ be the Mostowski collapse, defined recursively by $\pi(x) = \{\pi(y) : y \in A \cap x\}$.
Is it possible that $\pi(\mathcal{P}(\omega)) = \mathcal{P}(\omega)$? If yes, how would you show this? If no, what is the relationship between the two sets?