Homework Sheet #9

MasterMath: Set Theory
2021/22: 1st Semester
K. P. Hart, Steef Hegeman, Benedikt Löwe & Robert Paßmann

Deadline for Homework Set #9: Monday, 15 November 2021, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(31) Let $f : \omega_1 \to \mathbb{R}$ be an injective map. For $q \in \mathbb{Q}$ put $A_q = \{\alpha : f(\alpha) < q\}$ and $B_q = \{\alpha : f(\alpha) > q\}$. Let $I = \{q : A_q \text{ contains a cub set}\}$ and $J = \{q : B_q \text{ contains a cub set}\}$.

a. Prove: if $p \in I$ and $q \in J$ then $q < p$.

b. Prove: $I \neq \mathbb{Q}$ and $J \neq \mathbb{Q}$.

c. Prove: $\sup J < \inf I$ (by convention: $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$).

d. Prove: there is a $q \in \mathbb{Q}$ such that both A_q and B_q are stationary.

(32) Let S be a stationary subset of ω_1. Prove that for every $\alpha \in \omega_1$ there is a closed subset of ω_1 of order type $\alpha + 1$ that is a subset of S. *Hint*: Prove the following statement by induction on α: "for every stationary subset S of ω_1 there is closed subset of order type $\alpha+1$ that is contained in S". For the limit case let $\langle \alpha_n : n \in \omega \rangle$ be increasing and cofinal in α. Show that there is a sequence $\langle C_\gamma : \gamma \in \omega_1 \rangle$ of countable closed sets such that $C_\gamma \subseteq S$ for all γ; $\max C_\gamma < \min C_\delta$ whenever $\gamma < \delta$ and if $\gamma = \omega \cdot \delta + n$ then C_γ has order type $\alpha_n + 1$. Consider the set of limit points of $\{\max C_\gamma : \gamma \in \omega_1\}$.

(33) In class the proof of the Δ-system lemma for $\aleph_2$ many countable sets failed because of, as we shall see later, the possiblity that $2^{\aleph_0} \geqslant \aleph_2$.

a. Study the proof and extract from it a proof of the following statement: if $\langle C_\alpha : \alpha \in \omega_2 \rangle$ is a sequence of countable subsets of ω_2 such that $a \notin C_\alpha$ for all α then there is a subset F of ω_2 of cardinality $\aleph_2$ such that $\alpha \notin C_\beta$ whenever α and β are different elements of F.

A set set like F is called *free* for the *set mapping* $\alpha \mapsto C_\alpha$, and the statement you just proved is a special case of what is known as the *Free Set Lemma*, which states:

> if κ and λ are cardinals with $\lambda < \kappa$ and $S : \kappa \to [\kappa]^{<\lambda}$ is a map such that $\alpha \notin S(\alpha)$ for all α then there is s free set F for S of cardinality κ.

b. Verify that there are counterexamples to this statement if we only require that $|S(\alpha)| \leqslant \lambda$ for all α.

c. Prove the Free Set Lemma in case both κ and λ are regular, and κ is uncountable. *Hint*: Consider E^κ_λ.

d. Prove the Free Set Lemma in case κ is regular and uncountable, and λ is singular. *Hint*: Prove first that there is a regular cardinal μ below λ such that $\{\alpha : |S(\alpha)| < \mu\}$ has cardinality κ.

e. Prove the Free Set Lemma in case $\kappa = \aleph_0$. *Hint*: Induction on λ.

Note: the proof of the Free Set Lemma for singular κ (and arbitrary λ) is much longer than the proof for the regular case given here and needs new ideas. The Free Set Lemma illustrates a common occurrence in Set Theory: proofs for regular cardinals tend to be much shorter than proofs for singular cardinals. We shall see another instance of this in Lecture 10.