Homework Sheet #12

MasterMath: Set Theory
2021/22: 1st Semester
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Deadline for Homework Set #12: Monday, 6 December 2021, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(40) In contrast with Chapter 2 of Jech's book we define a partial order on a set $\mathbb{P}$ to be a relation $\leqslant$ that is
(1) reflexive: $x \leqslant x$
(2) antisymmetric: $x \leqslant y$ and $y \leqslant x$ implies $x = y$
(3) transitive: $x \leqslant y$ and $y \leqslant z$ implies $x \leqslant z$

We call the pair $\langle \mathbb{P}, \leqslant \rangle$ a partially ordered set.
As in class we call a subset D of $\mathbb{P}$ *dense* if for every $p \in \mathbb{P}$ there is a $q \in D$ such that $q \leqslant p$.
We call two elements p and q *compatible* if there is an r such that $r \leqslant p$ and $r \leqslant q$; otherwise they are *incompatible* we write $p \perp q$ if p and q are incompatible.
An *antichain* in $\mathbb{P}$ is a subset A such that $p \perp q$ whenever p and q in A are distinct.

a. Prove that every antichain is contained in a *maximal* antichain.

b. Assume D is dense in $\mathbb{P}$ and let A be a antichain that is maximal as an antichain in D. Prove that A is also maximal as an antichain in $\mathbb{P}$.

We denote the partial order use in class by $\mathrm{Fn}(\omega_2 \times \omega, 2)$.

(41) Prove that a filter G on $\mathrm{Fn}(\omega_2^M \times \omega, 2)$ is M-generic if and only if it intersects every maximal antichain of $\mathrm{Fn}(\omega_2 \times \omega, 2)$ that is in M.

(42) Analogous to the partial order used in class we define a partial order consisting of *countable* partial functions: $\mathrm{Fn}(X, Y, \aleph_1)$ is the set of functions p that satisfy $\operatorname{dom} p \subseteq X$, $\operatorname{ran} p \subseteq Y$ and $|p| < \aleph_1$.
For this exercise we assume that X is uncountable and $Y = 2$.

a. Construct an antichain in $\mathrm{Fn}(X, 2, \aleph_1)$ that is of cardinality $2^{\aleph_0}$.

b. Revisit the material from Lecture 9 on Δ-systems and prove that if κ is regular and satifies $\lambda^{\aleph_0} < \kappa$ for all $\lambda < \kappa$ then every family of κ many countable sets has a subfamily of cardinality κ that is a Δ-system.

c. Now modify the proof in class to show that every antichain in $\mathrm{Fn}(X, 2, \aleph_1)$ has cardinality at most $2^{\aleph_0}$.