SET THEORY 2021-11-08

- STATIONARY SETS
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BLOCH [1953]
A SUBSET A OF ω_1 IS STATIONARY IF FOR EVERY REGRESSIVE FUNCTION $f: A \to \omega_1$ THERE IS AN UNCOUNTABLE SUBSET OF A ON WHICH f IS CONSTANT
WHERE f IS REGRESSIVE IF $f(x) \leq x$ FOR ALL $x \in A$ WITH EQUALITY EXCLUDED EXCEPT FOR $x = 0$, IF $0 \in A$.

THEOREM: A SET IS STATIONARY IFF EVERY CLOSED SUBSET OF ITS COMPLEMENT IS COUNTABLE.

FIRST INSTANCE: ALEXANDROFF-URYSOHN (1929)
IF $f: \omega_1 \to \omega_1$ IS REGRESSIVE THEN f IS CONSTANT ON AN UNCOUNTABLE SET.

LET κ BE A REGULAR UNCOUNTABLE CARDINAL.
WE CALL $C \subseteq \kappa$ CLOSED IF FOR EVERY $\alpha < \kappa$: IF $C \cap \alpha \neq \emptyset$ THEN $\sup(C \cap \alpha) \in C$.

EXERCISE [IF YOU KNOW SOME TOPOLOGY]
DEFINE $O \subseteq \kappa$ TO BE OPEN IF FOR ALL $\alpha \in O$ IF $\alpha > 0$ THEN THERE IS A $\beta < \alpha$ SUCH THAT $(\beta, \alpha] \subseteq O$.

a) THIS DEFINES A TOPOLOGY ON κ; THE ORDER TOPOLOGY
b) 0 AND EVERY SUCCESSOR ORDINAL IS ISOLATED
c) OUR DEFINITION OF CLOSED IS EQUIVALENT TO "CLOSED IN THE ORDER TOPOLOGY".

THIS IS THE CLOSED IN BLOCH'S THEOREM

- WE CALL $C \subseteq \kappa$ CUB [CLOSED UNBOUNDED] IF C IS CLOSED AND COFINAL IN κ.
- BLOCH'S CHARACTERIZATION BECOMES OUR DEFINITION:

WE CALL $S \subseteq \kappa$ STATIONARY IF

$$S \cap C \neq \emptyset$$

FOR EVERY CUB SET C.

SO, EVERY CUB SET IS STATIONARY

PROPERTIES OF CUB SETS

- IF C AND D ARE CUB THEN SO IS $C \cap D$
- IF $\langle C_\xi : \xi < \delta \rangle$ IS A SEQUENCE OF CUB SETS AND $\delta < \kappa$ THEN $\bigcap_{\xi<\delta} C_\xi$ IS CUB.

PROOF

- $C \cap D$ IS CLOSED: ASSUME $\alpha \cap (C \cap D) \neq \emptyset$ AND LET $\beta = \sup(\alpha \cap C \cap D)$

SO $\beta \leq \alpha$ AND $\alpha \cap C \cap D = \beta \cap C \cap D$

NOW $\beta \cap C \supseteq \beta \cap C \cap D$ SO $\beta = \sup(\beta \cap C)$

AND $\beta \cap D \supseteq \beta \cap C \cap D$ SO $\beta = \sup(\beta \cap D)$

WE FIND $\beta \in C \cap D$.

- $C \cap D$ IS COFINAL: LET $\alpha < \kappa$ BE GIVEN

RECURSIVELY: $\gamma_0 = \min\{\gamma \in C : \gamma > \alpha\}$

$\delta_0 = \min\{\delta \in D : \delta > \gamma_0\}$

$\gamma_{n+1} = \min\{\gamma \in C : \gamma > \delta_n\}$; $\delta_{n+1} = \min\{\delta \in D : \delta > \gamma_{n+1}\}$

SO $\alpha < \gamma_0 < \delta_0 < \cdots < \delta_n < \gamma_{n+1} < \delta_{n+1} < \cdots$

LET $\beta = \sup\{\gamma_n : n \in \omega\} = \sup\{\delta_n : n \in \omega\}$

THEN $\beta < \kappa$ AND $\beta \in C \cap D$

BECAUSE $\beta = \sup(\beta \cap C)$ AND $\beta = \sup(\beta \cap D)$

- $\bigcap_{\xi<\delta} C_\xi$ IS CLOSED: AS ABOVE
  IF $\beta = \sup(\beta \cap \bigcap_{\xi<\delta} C_\xi)$
  THEN $\beta = \sup(\beta \cap C_\xi)$ FOR ALL $\xi < \delta$
- $\bigcap_{\xi<\delta} C_\xi$ IS COFINAL: LET α BE GIVEN
  RECURSIVELY BUILD A MATRIX:

$$\begin{array}{l} \alpha < \beta_{0,0} < \beta_{0,1} < \dots < \beta_{0,\xi} < \dots \qquad \xi < \delta \\ < \beta_{1,0} < \beta_{1,1} < \dots < \beta_{1,\xi} < \dots \\ < \beta_{2,0} < \beta_{2,1} < \dots < \beta_{2,\xi} < \dots \end{array}$$

FOR $m \in \omega$:

- $\beta_{0,0} = \min\{\gamma \in C_0 : \gamma > \alpha\}$
- $\beta_{m,\xi} = \min\{\gamma \in C_\xi : (\forall \eta < \xi)(\gamma > \beta_{m,\eta})\}$
- $\beta_{m+1,0} = \min\{\gamma \in C_0 : (\forall \xi < \delta)(\gamma > \beta_{m,\xi})\}$

ALL THESE EXIST AND ARE SMALLER THAN κ BECAUSE κ IS <u>REGULAR</u>

LET $\beta = \sup\{\beta_{m,\xi} : m \in \omega, \xi < \delta\}$
THEN $\beta = \sup\{\beta_{m,\xi} : m \in \omega\}$ FOR ALL $\xi < \delta$
SO $\beta \in \bigcap_{\xi<\delta} C_\xi$.

SO THE CUB SETS GENERATE A <u>FILTER</u>!

$\mathrm{Cub}(\kappa) = \{A \subseteq \kappa : \text{THERE IS A CUB } C \text{ WITH } C \subseteq A\}$

- $\emptyset \notin \mathrm{Cub}(\kappa)$; $\kappa \in \mathrm{Cub}(\kappa)$
- IF $A_0, A_1 \in \mathrm{Cub}(\kappa)$ THEN $A_0 \cap A_1 \in \mathrm{Cub}(\kappa)$
- IF $A \in \mathrm{Cub}(\kappa)$ AND $A \subseteq B$ THEN $B \in \mathrm{Cub}(\kappa)$

THIS FILTER IS <u>κ-COMPLETE</u>:
IF $\mathcal{A} \subseteq \mathrm{Cub}(\kappa)$ AND $|\mathcal{A}| < \kappa$
THEN $\bigcap \mathcal{A} \in \mathrm{Cub}(\kappa)$

CAN WE DO EVEN BETTER?
MAYBE NOT: $\bigcap_{\alpha<\kappa} [\alpha, \kappa) = \emptyset$ BUT...

EXAMPLES:

WHAT IS $\Delta_{\alpha<\kappa} [\alpha, \kappa)$?

WHAT IS $\Delta_{\alpha<\kappa} [\alpha+1, \kappa)$?

WHAT IS $\Delta_{\alpha<\kappa} [\alpha+\omega, \kappa)$?

- IF $\langle C_\alpha : \alpha < \kappa \rangle$ IS A SEQUENCE OF CUB SETS THEN

$$\Delta_{\alpha<\kappa} C_\alpha = \{ \delta : (\forall \alpha < \delta)(\delta \in C_\alpha) \}$$

IS ALSO CUB. [DIAGONAL INTERSECTION]

- CLOSED NEEDS PROOF [NOT TOPOLOGICAL]

  ASSUME $\delta = \sup(\delta \cap \Delta_{\alpha<\kappa} C_\alpha)$

  LET $\alpha < \delta$ THEN WE ALSO HAVE

  $\delta = \sup((\alpha, \delta) \cap \delta \cap \Delta_{\gamma<\kappa} C_\gamma)$

  BUT IF $\beta \in \Delta_{\gamma<\kappa} C_\gamma$ AND $\alpha < \beta$

  THEN $\beta \in C_\alpha$

  SO $(\alpha, \delta) \cap \Delta_{\gamma<\kappa} C_\gamma \subseteq C_\alpha$

  AND SO $\delta = \sup(\delta \cap C_\alpha)$ HENCE $\delta \in C_\alpha$.

- COFINAL LET $\gamma < \kappa$ BE GIVEN

  LET $\delta_0 = \min\{ \delta \in \bigcap_{\tau \leq \gamma} C_\tau : \delta > \gamma \}$

  AND, RECURSIVELY,

  $\delta_{n+1} = \min\{ \delta \in \bigcap_{\xi \leq \delta_n} C_\xi : \delta > \delta_n \}$

  NOW LET $\delta = \sup_{n\in\omega} \delta_n$

  THEN $\delta > \gamma$ AND $\delta \in \Delta_{\alpha<\kappa} C_\alpha$

  FOR IF $\alpha < \delta$ THEN $\alpha \leq \delta_m$ FOR SOME m AND THEN

  $\{ \delta_n : n \geq m \} \subseteq C_\alpha$.

FILTERS WITH THIS PROPERTY ARE CALLED <u>NORMAL</u>

THESE ARE USEFUL FOR MANY REASONS BUT FOR NOW:

THEOREM 8.7 FODOR'S PRESSING-DOWN LEMMA

IF $S \subseteq \kappa$ IS STATIONARY AND $f : S \to \kappa$ IS REGRESSIVE THEN f IS CONSTANT ON A STATIONARY SET.

MAJOR APPLICATION: SILVER'S THEOREM

SEE pg 6 FF:

IF κ IS SINGULAR WITH $cf\,\kappa > \aleph_0$
AND $2^\lambda = \lambda^+$ FOR ALL $\lambda < \kappa$
THEN $2^\kappa = \kappa^+$.

PROOF: Suppose not.

So let, for each $\alpha \in \kappa$, C_α be cub such that $C_\alpha \cap \{\sigma \in S : f(\sigma) = \alpha\} = \emptyset$.

Consider $\Delta_{\alpha<\kappa} C_\alpha$ and take $\sigma \in S \cap \Delta_{\alpha<\kappa} C_\alpha$.

Then $f(\sigma) < \sigma$ but $\sigma \in C_{f(\sigma)}$ and so $\sigma \notin \{\tau \in S : f(\tau) = f(\sigma)\}$ ......

The converse also holds:

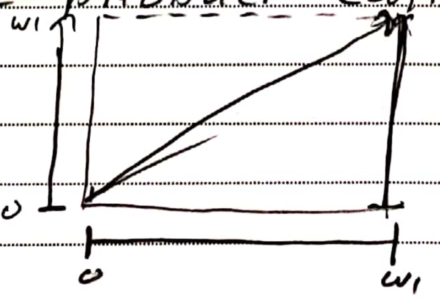
If $A \cap C = \emptyset$ where C is cub then $f(\alpha) = \max\{\gamma \in C : \gamma < \alpha\}$ is regressive and every set $\{\alpha : f(\alpha) = \gamma\}$ is even bounded.

Some Applications STATIONARY

(1) If $f : \kappa \to \mathbb{R}$ is continuous then there is an $\alpha < \kappa$ such that f is constant on $[\alpha, \kappa)$.
[Veeeery useful in General Topology]
[Holds for stationary subsets also]

(2) Let $f : \kappa \to [\kappa]^{<\aleph_0}$ be a function such that $\alpha \notin f(\alpha)$ for all α.
There is a stationary set S such that $\beta \notin f(\alpha)$ whenever $\alpha, \beta \in S$ and $\alpha \neq \beta$.
[Homework]

(3) Classic Counterexample in General Topology
The topological product $(\omega_1 + 1) \times \omega_1$ is not normal



SEE ALSO CHAPTER 10 p131-132
FOR ULAM MATRICES.
(PROVES SOLOVAY'S THEOREM
FOR SUCCESSOR CARDINALS)

QUESTION: IS EVERY STATIONARY SET
IN THE CUB(κ)?

- $\kappa = \omega_2$: NO $E^{\kappa}_{\lambda} = \{\alpha \in \kappa : \mathrm{cf}\,\alpha = \lambda\}$
  $E^{\omega_2}_{\omega_0}$ AND $E^{\omega_2}_{\omega_1}$ ARE STATIONARY
  AND DISJOINT.
- $\kappa = \omega_1$: NO
  BLOCH: EVERY STATIONARY SUBSET IS
  THE UNION OF TWO DISJOINT STATIONARY
  SUBSETS.
  EVERY CUB SET IS THE UNION
  OF $\aleph_1$ MANY PAIRWISE DISJOINT
  STATIONARY SETS.
  THIS USES AC A LOT
  IT IS CONSISTENT WITH ZF THAT THE ANSWER
  IS YES FOR ω_1.

THEOREM 8.10 (SOLOVAY)
EVERY STATIONARY SUBSET OF κ CAN
BE WRITTEN AS THE UNION OF κ MANY
PAIRWISE DISJOINT STATIONARY SETS.

PROOF [A FEW STEPS] [SEE ALSO BLOCH!!]

- EVERY STATIONARY SUBSET OF E^{κ}_{ω} HAS
  THIS PROPERTY: LET $S \subseteq E^{\kappa}_{\omega}$ BE STATIONARY
  CHOOSE SEQUENCES $S_\alpha = \langle \beta_{\alpha,m} : m \in \omega \rangle$ $(\alpha \in S)$
  SUCH THAT S_α IS INCREASING AND
  COFINAL IN α.

THERE IS AN $m \in \omega$ SUCH THAT FOR ALL $\eta < \kappa$
THE SET $\{\alpha \in S : \beta_{\alpha,m} \geq \eta\}$ IS STATIONARY.
IF NOT: TAKE FOR EVERY $m \in \omega$ A CUB SET C_m
AND $\eta_m < \kappa$ SUCH THAT
$$\beta_{\alpha,m} < \eta_m \quad \text{FOR } \alpha \in C_m \cap S.$$

BUT LET $\alpha \in S \cap \bigcap_{m \in \omega} C_m$

THEN $\beta_{\alpha, m} < \eta_m$ FOR ALL m

IN PARTICULAR

$$\alpha = \sup_{m \in \omega} \beta_{\alpha, m} \leq \sup_{m \in \omega} \eta_m$$

So $S \cap \bigcap_{m \in \omega} C_m$ WOULD BE BOUNDED.

TAKE OUR m AND LET $f : S \to \kappa$

BE GIVEN BY $f(\alpha) = \beta_{\alpha, m}$

- f IS REGRESSIVE SO THERE IS $\gamma_0 < \kappa$ SUCH THAT $T_0 = \{\alpha \in S : f(\alpha) = \gamma_0\}$ IS STATIONARY
- BUT $S_0 = \{\alpha \in S : f(\alpha) > \gamma_0\}$ IS STATIONARY SO WE HAVE $\gamma_1 < \kappa$ SUCH THAT $T_1 = \{\alpha \in S_0 : f(\alpha) = \gamma_1\}$ IS STATIONARY

  NOTE: $\gamma_0 < \gamma_1$ AND $T_0 \cap T_1 = \emptyset$.
- GIVEN $\beta < \kappa$ AND $\langle \gamma_\eta : \eta < \beta \rangle$ WITH $\delta = \sup_{\eta < \beta} \gamma_\eta$

  THE SET $S_\beta = \{\alpha \in S : f(\alpha) > \delta\}$ IS STATIONARY

  THERE HAS $\gamma_\beta < \kappa$ SUCH THAT

  $$T_\beta = \{\alpha \in S_\beta : f(\alpha) = \gamma_\beta\}$$

  IS STATIONARY ..
- EVERY STATIONARY SUBSET OF E^κ_λ HAS THIS PROPERTY WHENEVER $\lambda < \kappa$.

  SAME PROOF.
- $S \subseteq \{\alpha : \mathrm{cf}\,\alpha < \alpha\}$

  THERE IS A STATIONARY $T \subseteq S$ ON WHICH CF IS CONSTANT, SO WE ARE IN ONE OF THE PREVIOUS CASES
- $S \subseteq \{\alpha : \mathrm{cf}\,\alpha = \alpha\}$ [κ IS QUITE LARGE!]

  WE USE THE SAME METHOD AS ABOVE BUT WITH A TWIST.

GROUP INTERACTION: κ IS WAY LARGER THAN WEAKLY INACCESSIBLE

FIRST: $T = \{\alpha \in S : S \cap \alpha$ NOT STATIONARY IN $\alpha\}$ IS STATIONARY

LET $C \subseteq \kappa$ BE CUB; LET C' BE THE SET OF LIMIT POINTS OF C:

$\gamma \in C'$ IFF $\gamma = \sup(C \cap \gamma) \wedge \gamma \in C$

THE SET C' IS CUB; TAKE $\alpha = \min S \cap C'$

- $\alpha \in C'$ SO $C \cap \alpha$ IS CUB IN α
- $C' \cap S \cap \alpha = \emptyset$ SO S IS NOT STATIONARY IN α AS $C' \cap \alpha$ IS CUB IN α TOO

SECOND WORK WITH T:

– T IS STATIONARY

– $\alpha \in T$ IMPLIES α IS A REGULAR CARDINAL AND $T \cap \alpha$ IS NOT STATIONARY IN α.

LET $C_\alpha \subseteq \alpha$ BE CUB WITH $C_\alpha \cap T = \emptyset$

ENUMERATE C_α AS $\langle \beta_{\alpha,\xi} : \xi < \alpha \rangle$

SO – THE SEQUENCE IS INCREASING

– η LIMIT IMPLIES $\beta_{\alpha,\eta} = \sup_{\xi<\eta} \beta_{\alpha,\xi}$

– $\alpha = \sup_{\xi<\alpha} \beta_{\alpha,\xi}$.

- THERE IS A $\xi < \kappa$ SUCH THAT FOR ALL $\gamma < \kappa$ THE SET $\{\alpha \in T : \beta_{\alpha,\xi} \geq \gamma\}$ IS STATIONARY.

IF NOT TAKE FOR EACH $\xi < \kappa$ AN ORDINAL γ_ξ AND A CUB SET C_ξ SUCH THAT

$C_\xi \cap T \subseteq \{\alpha \in T : \beta_{\alpha,\xi} < \gamma_\xi\}$.

LET $C = \Delta_{\xi<\kappa} C_\xi$ AND $D = \{\delta : \xi < \delta \to \gamma_\xi < \delta\}$

THEN C AND D ARE CUB [EXERCISE!]

LET $\alpha < \delta$ IN $T \cap C \cap D$

IF $\xi < \alpha$ THEN $\beta_{\alpha,\xi} < \gamma_\xi < \alpha$

AND $\beta_{\delta,\xi} < \gamma_\xi < \alpha$

WE SEE $\beta_{\delta,\alpha} = \alpha$

BUT $\alpha \in T$ AND $\beta_{\delta,\alpha} \notin T$

NOW FINISH AS BEFORE

FIND $\langle \gamma_\eta : \eta < \varkappa \rangle$ AND $\langle S_\eta : \eta < \varkappa \rangle$

WITH EACH S_η STATIONARY

AND $\beta_{\alpha,1} = \gamma_\eta$ IF $\alpha \in S_\eta$.

AND $\langle \gamma_\eta : \eta < \varkappa \rangle$ INCREASING. □

IMPORTANT APPLICATION FOR LATER

THEOREM [SHANIN]

LET $\mathcal{A}$ BE AN UNCOUNTABLE FAMILY OF FINITE SETS

THEN THERE ARE A FIXED FINITE SET R AND AN UNCOUNTABLE SUBFAMILY $\mathcal{B}$ OF $\mathcal{A}$ SUCH THAT: IF $B_0, B_1 \in \mathcal{B}$ AND $B_0 \neq B_1$ THEN $B_0 \cap B_1 = R$.

" $\mathcal{B}$ IS A Δ-SYSTEM WITH ROOT R "

PROOF

WE CAN ASSUME $|\mathcal{A}| = \aleph_1$ AND $\mathcal{A} \in [\omega_1]^{<\aleph_0}$; WE ENUMERATE $\mathcal{A}$ AS $\langle a_\alpha : \alpha \in \omega_1 \rangle$

DEFINE $f : \omega_1 \to \omega_1$ BY

$$f(\alpha) = \begin{cases} 0 & \text{IF } a_\alpha \cap \alpha = \emptyset \\ \max(a_\alpha \cap \alpha) & \text{IF } a_\alpha \cap \alpha \neq \emptyset \end{cases}$$

SO f IS REGRESSIVE.

TAKE $\beta \in \omega_1$ SUCH THAT $S = \{\alpha : f(\alpha) = \beta\}$ IS STATIONARY.

SO, IF $\alpha \in S$ THEN $a_\alpha \cap \alpha \subseteq \beta + 1$.

THE FAMILY $[\beta+1]^{<\aleph_0}$ IS COUNTABLE

SO THERE IS AN $R \in [\beta+1]^{<\aleph_0}$ SUCH THAT $T = \{\alpha \in S : a_\alpha \cap \alpha = R\}$ IS STATIONARY

GROUP INTERACTION

LET $C = \{\alpha : \beta < \alpha \to \max a_\beta < \alpha\}$; C IS CUB

NOW $\{a_\alpha : \alpha \in T \cap C\}$ IS THE DESIRED FAMILY: $\alpha < \beta$: $a_\alpha \cap a_\beta \subseteq \alpha$ AND $a_\alpha \cap \alpha = a_\beta \cap \beta = R$.

NOTE THIS PROVES MORE

- IF $\{a_\alpha : \alpha < \kappa\} \subseteq [\kappa]^{<\aleph_0}$ THEN THERE IS A STATIONARY SET $S \subseteq \kappa$ SUCH THAT $\{a_\alpha : \alpha \in S\}$ IS A Δ-SYSTEM

HOW ABOUT COUNTABLE SETS?

NOT WITH ω_1 : LET $a_\alpha = \alpha$ ....

LET'S TRY ω_2 : LET $\{a_\alpha : \alpha < \omega_2\}$ BE GIVEN THEN $f(\alpha) = \sup(\alpha \cap a_\alpha)$ IS REGRESSIVE ON $E^{\omega_2}_{\omega_1}$, HENCE CONSTANT ON A STATIONARY SUBSET S OF $E^{\omega_2}_{\omega_1}$, SAY WITH VALUE γ.

BUT $[\gamma]^{\leq\aleph_0}$ HAS CARDINALITY $\aleph_1^{\aleph_0} = 2^{\aleph_0}$

SO, IF $\aleph_2 \leq 2^{\aleph_0}$ .... WE HAVE A PROBLEM.

! CH GIVES A POSITIVE ANSWER

IN GENERAL:

TAKE $\{a_\alpha : \alpha \in c^+\} \subseteq [c^+]^{\leq\aleph_0}$

WE GET $S \subseteq E^{c^+}_{\omega_1}$ STATIONARY AND $\gamma < c^+$ WITH

$$\sup(a_\alpha \cap \alpha) = \gamma \qquad \alpha \in S$$

BUT $|\gamma+1| \leq 2^{\aleph_0}$ SO $[\gamma+1]^{\leq\aleph_0} \leq 2^{\aleph_0}$

NOW OUR PROOF WORKS