# SET THEORY 2021-11-22

## FORCING I.

HOW TO PROVE THAT WE CANNOT PROVE CH?

ANSWER FROM MATHEMATICAL LOGIC:

- BUILD A MODEL FOR ZFC IN WHICH $\neg$CH HOLDS

TWO DIFFICULTIES

① BUILDING MODELS IS DONE WITH SETS, HENCE HAPPENS IN SET THEORY, BUT ZF(C) CANNOT PROVE ITS OWN CONSISTENCY.

A WAY OUT OF THIS DIFFICULTY.

A PROOF OF CH (OR $\neg$CH, OR ...) FROM ZFC INVOLVES ONLY FINITELY MANY AXIOMS: WE APPLY SEPARATION AND REPLACEMENT TO ONLY FINITELY MANY FORMULAS.

SO LET $\varphi_1, \ldots, \varphi_k$ BE THE (INSTANCES OF) AXIOMS THAT WE USE IN OUR PROOF.

OUR PROOF WILL THEN BE A FORMAL DEDUCTION OF $(\varphi_1 \wedge \ldots \wedge \varphi_k) \to$ CH

BY THE RULES OF FORMAL DEDUCTIONS AND OF SATISFACTION OF FORMULAS THE FORMULA $(\varphi_1 \wedge \ldots \wedge \varphi_k) \to \neg$CH IS VALID IN EVERY INTERPRETATION OF THE LANGUAGE OF SET THEORY.

SO: ALL WE NEED TO DO IS SHOW HOW TO MAKE, GIVEN (INSTANCES OF) AXIOMS $\varphi_1, \ldots, \varphi_k$, AN INTERPRETATION IN WHICH $\varphi_1 \wedge \ldots \wedge \varphi_n$ IS VALID AND CH IS NOT, I.E., $\neg$CH IS VALID.

WHAT DOES THIS DO?

- IT DOES NOT SHOW THAT ZFC PROVES THAT CH IS NOT PROVABLE
- IT DOES TELL US THAT ANY PROOF OF CH THAT WE MAKE CANNOT BE TRANSLATED INTO A FORMAL DEDUCTION OF CH FROM ZFC. (AND THAT IS THE BEST WE CAN EXPECT)

FORCING IS A SYSTEMATIC WAY OF DOING THIS: GIVEN $\varphi_1, \ldots, \varphi_k$ IT SHOWS US HOW TO BUILD AN INTERPRETATION $M$ SUCH THAT $\varphi_1, \ldots, \varphi_k$ ARE VALID IN $M$

AND $\neg$CH IS VALID IN $M$

OR CH IS VALID IN $M$

OR --- IS VALID IN $M$

(2) HOW DOES THAT WORK?

BUILDING THOSE INTERPRETATIONS WILL BE "JUST SET THEORY": WE WORK WITH SETS, IN PARTICULAR PARTIALLY ORDERED SETS, FILTERS ON THESE SETS, AND THIS WORK TENDS TO CONSUME LOTS OF INFINITE COMBINATORICS.

STEP 1 BUILDING INTERPRETATIONS THAT SATISFY (FINITELY MANY) AXIOMS.

WE WORK IN SET THEORY AND WE JUST USE SETS AS INTERPRETATIONS.

LET $M$ BE A SET / CLASS

WE INTERPRET FORMULAS IN $M$ BY RELATIVATION:

WITH EVERY FORMULA $\varphi$ WE ASSOCIATE A FORMULA $\varphi^M$:

$(x = y)^M$ IS $x = y$

$(x \in y)^M$ IS $x \in y$

$(\varphi \wedge \psi)^M$ IS $\varphi^M \wedge \psi^M$

$(\neg\varphi)^M$ IS $\neg(\varphi^M)$

$(\exists x \varphi)^M$ IS $(\exists x)(x \in M \wedge \varphi^M)$ IN SHORT $[(\exists x \in M)(\varphi^M)]$

WE DO $\vee, \rightarrow, \leftrightarrow, \ldots$ VIA EQUIVALENT EXPRESSIONS THAT USE $\wedge$ AND $\neg$

AND $\forall x\, \varphi$ GOES VIA $\neg(\exists x)(\neg\varphi)$

IN WORDS: RESTRICT ALL QUANTIFIERS TO $M$.

WE SAY "$\varphi$ IS TRUE/VALID IN $M$" OR "$M$ SATISFIES $\varphi$" IF WE CAN PROVE $\varphi^M$.

LOOK AT EXTENSIONALITY$^M$

$\forall x \in M\, \forall y \in M\, (\forall z \in M\, (z \in x \leftrightarrow z \in y) \rightarrow x = y)$

WHAT DOES THAT ENTAIL?

$\forall x \in M\, \forall y \in M\, (x \cap M = y \cap M \rightarrow x = y)$

SO, IF $M$ IS TRANSITIVE THEN (EXTENSIONALITY)$^M$

BECAUSE $x = x \cap M$ IF $x \in M$

$\forall z \in M\, (z \in x \leftrightarrow z \in y)$ IS EQUIVALENT TO

$\forall z\, (z \in x \leftrightarrow z \in y)$

HOMEWORK / GROUP INTERACTION:

PAIRING, UNION, POWER SET, INFINITY.

FIND $M$ WHERE THESE FAIL.

Our goal: find $M$ such that
$\varphi_1^M, \ldots, \varphi_k^M$ hold and $(\neg CH)^M$

The first part is (relatively) easy.
Our assumption is that $ZF(C)$ is consistent and that the axioms hold in $V = \bigcup_\alpha V_\alpha$.

We shall see that there is always an $\alpha$ such that we can take $M = V_\alpha$ for our $\varphi_1, \ldots, \varphi_k$.

This will take some effort and a new notion: <u>absoluteness</u>

Let $\varphi$ be a formula with its free variables among $x_1, \ldots, x_k$

Let $M$ and $N$ be sets/classes

- if $M \subseteq N$ then $\varphi$ is <u>absolute</u> for $M, N$ if $\forall m_1, \ldots, m_k \in M \, (\varphi^M(m_1, \ldots, m_k) \leftrightarrow \varphi^N(m_1, \ldots, m_k))$
- <u>absolute for $M$</u> means <u>absolute for $M, V$</u> so $\forall m_1, \ldots, m_k \in M \, (\varphi^M(m_1, \ldots, m_k) \leftrightarrow \varphi(m_1, \ldots, m_k))$

We will look voor $\alpha$ such that our $\varphi_i$ are absolute for $V_\alpha, V$.

— if $\varphi$ and $\psi$ are absolute for $M, N$ then so are $\neg\varphi$ and $\varphi \wedge \psi$

— formulas without quantifiers are absolute voor all $M$

What about quantifiers?
$(\forall x)(\forall y)(\exists z)(\forall w)(w \in z \leftrightarrow (w = x \vee w = y))$
$x \subseteq y$ means $(\forall z)(z \in x \rightarrow z \in y)$

$M = \{\emptyset, a\}$ $\quad a = \{\{\emptyset\}\}$

$(a \subseteq \emptyset)^M$ BUT $a \not\subseteq \emptyset$

LEMMA ASSUME $M$ AND $N$ ARE TRANSITIVE AND $\varphi$ IS ABSOLUTE FOR $M, N$

THEN $(\exists x \in y)(\varphi)$ IS ABSOLUTE FOR $M, N$.

$\varphi(x, y, z_1, \ldots, z_n)$ FREE VARIABLES SHOWN

$[\exists x(x \in y \wedge \varphi(x, y, z_1, \ldots, z_n)]^M$ IS

$(\exists x \in M)(x \in y \wedge \varphi(x, y, z_1, \ldots, z_n))^M$

NOW ASSUME $y, z_1, \ldots, z_n \in M$

THEN BY TRANSITIVITY $(\exists x \in M)(x \in y)$ IS $(\exists x \in y)$

SO WE GET

$$\exists x(x \in y \wedge \varphi^M(x, y, z_1, \ldots, z_n))$$

BY ASSUMPTION: IF $x \in y$ THEN

$$\varphi^M(x, y, z_1, \ldots, z_n) \Leftrightarrow \varphi^N(x, y, z_1, \ldots, z_n)$$

SO WE GET THE EQUIVALENT

$$\exists x(x \in y \wedge \varphi^N(x, y, z_1, \ldots, z_n))$$

WHICH IS JUST BY TRANSITIVITY AGAIN

$$[\exists x(x \in y \wedge \varphi(x, y, z_1, \ldots, z_n))]^N.$$

"$\exists x \in y$" IS A <u>BOUNDED</u> QUANTIFIER

WE CALL $\varphi$ A $\Delta_0$-FORMULA

IF ALL ITS QUANTIFIERS ARE BOUNDED:

- $x \in y$ AND $x = y$ ARE $\Delta_0$
- IF $\varphi$ AND $\psi$ ARE $\Delta_0$ THEN $\neg\varphi$ AND $\varphi \wedge \psi$ ARE $\Delta_0$
- IF $\varphi$ IS $\Delta_0$ THEN $(\exists x)(x \in y \wedge \varphi)$ AND $(\forall x)(x \in y \rightarrow \varphi)$ ARE $\Delta_0$

SO IF $M$ AND $N$ ARE TRANSITIVE THEN EVERY $\Delta_0$-FORMULA IS ABSOLUTE FOR $M, N$ AND SO ALSO FOR $M$.

GROUP INTERACTIONS: FIND $\Delta_0$-FORMULAS FOR MANY THINGS!

$x \in y$, $x = y$, $x \subseteq y$, $z = \{x, y\}$, $z = \{x\}$, $z = \langle x, y \rangle$

$z = \emptyset$, $z = x \cup y$, $z = x \cap y$, $z = x \setminus y$, $z = S(x) = x \cup \{x\}$

$x$ IS TRANSITIVE, $z = \bigcup x$, $z = \bigcap x$ $(\bigcap \emptyset = \emptyset)$

$\langle x, R \rangle$ IS A PARTIAL ORDER

$\langle x, R \rangle$ IS A LINEAR ORDER

$z$ IS AN ORDERED PAIR

$z = x \times y$

$R$ IS A RELATION

$z = \operatorname{DOM} R$ $\quad z = \operatorname{RAN} R$

$R$ IS A FUNCTION $\quad z = R(x)$

$R$ IS AN INJECTIVE FUNCTION

$z = \omega$ $\quad z$ IS

WE ASSUME THE <u>AXIOM OF FOUNDATION</u>

FIND $\Delta_0$-FORMULAS EQUIVALENT TO

- $z$ IS AN ORDINAL
- $z$ IS A LIMIT ORDINAL
- $z$ IS A SUCCESSOR ORDINAL
- $z$ IS A FINITE ORDINAL
- $z = \omega$
- $z = 0$, $z = 1$, $z = 2$, ..., $z = 10^{100}$, ..., $z = 10^{10^{(10^{100})}}$

A GENERAL CRITERION FOR ABSOLUTENESS

ASSUME $M \subseteq N$ AND LET $\varphi_1, \ldots, \varphi_n$ BE A LIST OF FORMULAS THAT IS <u>SUBFORMULA CLOSED</u>: EVERY SUBFORMULA OF EVERY $\varphi_i$ IS ALSO IN THE LIST.

THEN THE FOLLOWING STATEMENTS ARE EQUIVALENT:

# Criterion of Tarski and Vaught

a) The formulas $\varphi_1, \dots, \varphi_m$ are absolute for $M, N$

b) whenever $\varphi_i$ is $\exists x\, \varphi_j(x, y_1, \dots, y_\ell)$
(free variables shown) we have
$\forall y_1, \dots, y_\ell \in M\, [\exists x \in N\, \varphi_j^N(x, y_1, \dots, y_\ell) \to \exists x \in M\, \varphi_j^N(x, y_1, \dots, y_\ell)]$
Note the $N$ twice

a) $\Rightarrow$ b) Let $y_1, \dots, y_\ell \in M$
assume $\exists x \in N\, \varphi_j^N(x, y_1, \dots, y_\ell)$ so $\varphi_i^N(y_1, \dots, y_m)$ holds
absoluteness of $\varphi_i$: $\varphi_i^M(y_1, \dots, y_m)$ holds
so $\exists x \in M\, \varphi_j^M(x, y_1, \dots, y_\ell)$ holds
absoluteness of $\varphi_j$: $\exists x \in M\, \varphi_j^N(x, y_1, \dots, y_\ell)$ holds

b) $\Rightarrow$ a) Induction on complexity.
- Atomic formulas: $x \in y$ and $x = y$ clear
- $\varphi_i = \varphi_j \wedge \varphi_k$ absoluteness for $\varphi_j$ and $\varphi_k$
- $\varphi_i = \neg \varphi_j$ —— " —— $\varphi_j$
- $\varphi_i = (\exists x)\, \varphi_j(x, y_1, \dots, y_\ell)$

Let $y_1, \dots, y_m \in M$

$$\varphi_i^M(y_1, \dots, y_\ell) \overset{\text{def}}{\Leftrightarrow} (\exists x \in M)(\varphi_j^M(x, y_1, \dots, y_m))$$
$$\overset{\text{ind. hyp}}{\Longleftrightarrow} \exists x \in M\, (\varphi_j^N(x, y_1, \dots, y_m))$$
$$\underset{\leftarrow \text{ assumption b)}}{\overset{\rightarrow\, M \subseteq N}{\Longleftrightarrow}} \exists x \in N\, (\varphi_j^N(x, y_1, \dots, y_m))$$
$$\overset{\text{def}}{\Leftrightarrow} \varphi_i^N(y_1, \dots, y_m).$$

This works in all first-order languages not just Set Theory.

## Reflection

Take finitely many formulas $\varphi_1, \dots, \varphi_m$
then $(\forall \alpha)(\exists \beta \geq \alpha)(\varphi_1, \dots, \varphi_m$ are absolute for $V_\beta)$

In particular if $\varphi_1, \dots, \varphi_m$ are (instances of) axioms
$$(\forall \alpha)(\exists \beta > \alpha)(\varphi_1^{V_\beta} \wedge \dots \wedge \varphi_m^{V_\beta})$$

PROOF $\qquad \varphi_i^N = \varphi_i$

LET $N = V$ AND ASSUME $\varphi_1, \ldots, \varphi_m$ IS SUBFORMULA CLOSED

EACH $\varphi_i$ GIVES US A FUNCTION $F_i$

IF $\varphi_i$ IS $(\exists x)\varphi_j(x, y_1, \ldots, y_n)$

DEFINE $G_i(y_1, \ldots, y_n) = \begin{cases} 0 & \text{IF } \neg\exists x\, \varphi_j(x, y_1, \ldots, y_n) \\ \min\{\eta : (\exists x \in V_\eta)\varphi_j(x, y_1, \ldots, y_n)\} & \text{OTHERWISE} \end{cases}$

DEFINE $F_i(\xi) = \sup\{G_i(y_1, \ldots, y_n) : y_1, \ldots, y_n \in V_\xi\}$

IF $\varphi_i$ IS NOT OF THE FORM $(\exists x)\varphi_j$ $\quad F_i(\xi) = 0$

LET $\alpha$ BE GIVEN.

PUT $\beta_0 = \alpha$

GIVEN $\beta_k$ LET $\beta_{k+1} = \max\{\beta_k + 1, F_1(\beta_k), \ldots, F_m(\beta_k)\}$

FINALLY $\beta = \sup_{k \in \omega} \beta_k$

$\beta$ IS A LIMIT; $\beta > \alpha$

IF $\xi < \beta$ THEN $\xi < \beta_k$ FOR SOME $k$

AND $F_i(\xi) \le F_i(\beta_k) \le \beta_{k+1} < \beta$. □

ONE MORE STEP AND THEN WE CAN START WORRYING ABOUT ¬CH, CH, ETC...

LET $\varphi_1, \ldots, \varphi_n$ BE A SUBFORMULA-CLOSED SET OF FORMULAS AND LET $\alpha$ BE SUCH THAT $\varphi_1, \ldots, \varphi_n$ ARE ABSOLUTE FOR $V_\alpha$.

LET $\triangleleft$ BE A WELL-ORDER OF $V_\alpha$.

DEFINE FUNCTION $H_i$ AS FOLLOWS

IF $\varphi_i$ IS $(\exists x)\varphi_j(x, y_1, \ldots, y_{\ell_i})$ WE MAKE $H_i : V_\alpha^{\ell_i} \to V_\alpha$

$H_i(y_1, \ldots, y_{\ell_i}) = \begin{cases} \triangleleft\text{-}\min\{x : \varphi_j(x, y_1, \ldots, y_{\ell_i})\} & \text{IF NON-EMPTY} \\ 0 & \text{IF EMPTY} \end{cases}$

IF $\varphi_i$ IS NOT $(\exists x)$-- THEN $H_i = 0$

Let $X \subseteq V_\alpha$

Let $A_0 = X$

$A_{n+1} = A_n \cup \bigcup_{i=1}^{m} H_i[A_n^{\ell_i}]$

and $A = \bigcup_{n \in \omega} A_n$

- $A$ is closed under the $H_i$
- $|A| \le \max\{|X|, \aleph_0\}$
- all formulas $\varphi_1, \dots, \varphi_m$ are absolute for $A, V_\alpha$ and hence for $A$.

This is the Löwenheim-Skolem theorem.
The $H_i$ are Skolem functions for the $\varphi_i$.

Now assume $\varphi_m$ is the axiom of extensionality.
We can define the Mostowski-collapse of $A$: $G(x) = \{G(y) : y \in A \wedge y \in x\}$ Jech 6.15
Then $G$ is an isomorphism between $A$ and $M = G[A]$.
If the $X$ is transitive then $G \restriction X = id \restriction X$.
And
Our axioms are sentences and for them we have

$\varphi_i^M \Leftrightarrow \varphi_i^A$ by isomorphism

$\varphi_i^A \Leftrightarrow \varphi_i^{V_\alpha}$ by absoluteness

$\varphi_i^{V_\alpha} \Leftrightarrow \varphi_i$ by absoluteness

Final result: Given finitely many axioms $\varphi_1, \dots, \varphi_n$ there is a countable transitive set $M$ such that $\varphi_i^M$ holds $\varphi_i^M$ for all $i$.

# Summary

Goal: prove unprovability of some $\varphi$
(e.g. CH, $\neg$CH, ....)

Strategy: avoid Gödels incompleteness theorems by finding, given finitely many axioms $\varphi_1, \dots, \varphi_n$, a set $M$ such that $\varphi_1, \dots, \varphi_n$ and $\neg\varphi$ hold in $M$
[This uses Gödel's Completeness Theorem]

How: - define what "$\varphi$ holds in $M$" means by relativation of formulas: $\varphi \rightsquigarrow \varphi^M$
- define absoluteness of formulas

Then: between sets and between sets and $V$.
- many notions are absolute between transitive sets and $V$
- given $\varphi_1, \dots, \varphi_n$ and $\alpha$ there is $\beta > \alpha$ such that $\varphi_1, \dots, \varphi_n$ are absolute between $V_\beta$ and $V$
- Given $\varphi_1, \dots, \varphi_n$ and $V_\beta$ there is a countable $A \in V_\beta$ such that $\varphi_1, \dots, \varphi_n$ are absolute between $A$ and $V_\beta$ [Löwenheim-Skolem Theorem]
- There are a transitive set $M$ and an isomorphism $G: A \to M$.

Conclusion: we can make countable transitive $M$ in which our finitely many axioms $\varphi_1, \dots, \varphi_n$ hold

Next step: enlarge $M$ to $N$ so that $\varphi_1, \dots, \varphi_n$ hold in $N$ and also $\neg\varphi$