SOUSLIN'S PROBLEM*

By S. Tennenbaum

UNIVERSITY OF ROCHESTER, ROCHESTER, NEW YORK

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In this note we apply the method of P. J. Cohen[1, 2] to the following problem of Souslin[3]: Let S be a linearly ordered continuous set without first and last elements in which every family of disjoint intervals is countable. Is S isomorphic to the real line? We obtain models of the contemporary axioms for set theory in which the answer is *negative*. Moreover, in one of these the continuum hypothesis holds, and in others it fails. In a later paper,[4] written with R. Solovay, Cohen's method is extended to define models in which the answer is *affirmative*. Thus the current axioms do not suffice to settle Souslin's problem.

For convenience, the following version of the problem will be considered: Let T be a tree[5] such that every chain and antichain[6] in T has cardinality less than $\aleph_\alpha$ but the field of T has cardinality $\aleph_\alpha$. Call such a T an $\aleph_\alpha$-*Souslin tree.* It is known that a negative answer to Souslin's problem is equivalent to the existence of an $\aleph_1$-Souslin tree.[7]

THEOREM 1. *There exists a model* $\mathfrak{N}$ *of set theory such that* $\mathfrak{N}$ *contains an* $\aleph_1$-*Souslin tree and satisfies* $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

The proof of this theorem assumes complete familiarity with references 1 and 2, hereafter referred to as (C).

Let $\mathfrak{M}$ be a countable transitive model for set theory satisfying $V = L$. We will obtain $\mathfrak{N}$ from $\mathfrak{M}$ by adjoining a generic tree $T \subset \aleph_1 \times \aleph_1$.

Definition 1: For α an ordinal in $\mathfrak{M}$ define F_α as follows:

(1) If $0 \leq \alpha \leq \aleph_1$, $F_\alpha = \alpha$.

(2) If $\aleph_1 < \alpha < 2\aleph_1$, let F_α enumerate all unordered pairs (α,β) for $\alpha, \beta < \aleph_1$.

(3) If $2\aleph_1 \leq \alpha < 3\aleph_1$, let F_α enumerate all ordered pairs $\langle\alpha,\beta\rangle$ for $\alpha, \beta < \aleph_1$.

(4) If $\alpha = 3\aleph_1$, $F_\alpha = \aleph_1 \times \aleph_1$.

(5) If $\alpha = 3\aleph_1 + 1$, $F_\alpha = T$.

(6) Suppose $\alpha > 3\aleph_1 + 1$. If $N(\alpha) = 0$, $F_\alpha = \{F_{\alpha'} | \alpha' < \alpha\}$. If $N(\alpha) = i$, $0 < i < 9$, then $F_\alpha = \mathfrak{F}_i(F_\beta,F_\gamma)$ where $\beta = K_1(\alpha)$, $\gamma = K_2(\alpha)$.[8] If $N(\alpha) = 9$, $F_\alpha = \{F_{\alpha'} | \alpha' < \alpha, N(\alpha') = 9\}$.

This definition replaces Definition 1 of (C). Definition 5 of (C) is replaced by

Definition 2: Let Σ be the set of all characteristic functions of trees whose fields are finite subsets of $\aleph_1$. Thus $Q \in \Sigma$ if and only if there is a tree t whose field t^* is a finite subset of $\aleph_1$ such that $Q(\alpha,\beta) = 1$ if $\alpha = \beta$ or if α precedes β in t, $Q(\alpha,\beta) = 0$ if $\alpha, \beta \in t^*$ but α doesn't precede β in t, and $Q(\alpha,\beta)$ is undefined otherwise. Define a partial order in Σ as follows: $Q_1 \geq Q_2$ if the function Q_1 extends the function Q_2. Let $\Sigma' = \{Q | Q \in \Sigma, Q(\alpha,\beta) = 1 \Longrightarrow \alpha \leq \beta\}$. In the rest of this note P, P', P_0, P_1, etc., shall always designate elements of Σ' or, when there is no chance of confusion, the associated trees. Finally P_1 and P_2 are called *compatible* if there is a P such that $P \geq P_1$, $P \geq P_2$.

We modify clause III(i) in Definition 6 of (C) by stipulating that P forces the statement $[\langle\alpha,\beta\rangle \in T]$ if $P(\alpha,\beta) = 1$, P forces $[\langle\alpha,\beta\rangle \not\in T]$ if $P(\alpha,\beta) = 0$, and P forces $[F_\gamma \not\in T]$ if $0 \leq \gamma < 2\aleph_1$ or if $\gamma = 3\aleph_1, 3\aleph_1 + 1$. The properties of forcing expressed in Lemmas 2, 3, and 4 of (C) are now easily proved.

Definition 3: Let $\{P_n\}$ be a sequence such that $P_{n+1} \geq P_n$ for every n and such that every statement or its negation is forced by some P_n. Then $\lim_{n\to\infty} P_n$ is the characteristic function of a tree T, $T \subset \aleph_1 \times \aleph_1$. Let $\mathfrak{N} = \{F_\alpha | \alpha \text{ in } \mathfrak{M}\}$.

That a statement is true in $\mathfrak{N}$ if and only if it is forced by some P_n is proved in the same way as Lemma 5 of (C).

LEMMA 1. *Let $W \subseteq \Sigma'$. Suppose φ is a function such that $\varphi(P) \in P^*$ for all $P \in W$ and* $\sup \varphi(P) = \aleph_1$. *Then there exists $Q \in \Sigma'$, $P_1, P_2 \in W$ such that $\varphi(P_1) < \varphi(P_2)$, $Q(\varphi(P_1), \varphi(P_2)) = 1$, and $Q \geq P_1$, $Q \geq P_2$.*

Proof: Let W_1 be an uncountable subset of W such that φ maps W_1 one to one into $\aleph_1$. For some integer k there is an uncountable $W_2 \subseteq W_1$ such that the cardinality of $P^* = k$ for all $P \in W_2$. Let A be a maximal set contained in uncountably many P^*, $P^* \in W_2$. We may assume that $\varphi(P) \in P^* - A$ for all $P \in W_2$. Let $C(P)$ be the set of elements of P^* which equal or precede $\varphi(P)$ in P. Call $P_1, P_2 \in W_2$ equivalent if they agree within A and if $C(P_1) \cap A = C(P_2) \cap A$. Since there are only finitely many equivalence classes, there is an uncountable one, W_3.

Now there is an uncountable $W_4 \subseteq W_3$ such that $\min(P^* - A) > \max A$ for all $P \in W_4$ Otherwise, for uncountably many $P \in W_3$, $P^* - A$ would contain an ordinal $\leq \max A$. Then, since $\max A$ is countable, there would be a ξ such that $\xi \in P^* - A$ for uncountably many $P \in W_3$, thereby contradicting the maximality of A. Let $P_1 \in W_4$. In the same way it follows that there is a $P_2 \in W_4$ such that $\min(P_2{}^* - A) > \max P_1{}^*$.

Let B be the branch of P_2 consisting of the elements which succeed or are equal to some α in $C(P_2) - A$ and define $Q = P_1 \cup P_2 \cup H$, where $H = (C(P_1) - A) \times B$. Since P_1 and P_2 agree within A, the common part of their fields, and $P_1 \cap P_2$ is an initial segment of both P_1 and P_2, it is evident that $P_1 \cup P_2 \in \Sigma$ and $P_1 \cup P_2 \geq P_1, P_2$. Note that by adding H we are sliding the branch B of $P_1 \cup P_2$ upward along the interval $C(P_1) - A$ from its point of origin, $\max(C(P_2) \cap A) = \max(C(P_1) \cap A)$, to $\varphi(P_1)$, its new point of origin. Therefore $Q \in \Sigma$. Moreover, since H contains no pair of elements of either $P_1{}^*$ or $P_2{}^*$, $Q \geq P_1$, $Q \geq P_2$. Finally, it is clear that $Q \in \Sigma'$, $\varphi(P_1) < \varphi(P_2)$, and $Q(\varphi(P_1), \varphi(P_2)) = 1$.

LEMMA 2. *Any set of mutually incompatible P's is countable.*

Proof: Suppose W is an uncountable set of such P's. Define $\varphi(P) = \max P^*$ for all P in W. Lemma 1 yields a contradiction.

LEMMA 3. *$\mathfrak{N}$ is a model of set theory (including the axiom of choice). The ordinals and cardinals of $\mathfrak{N}$ are the same as those of $\mathfrak{M}$.*

Proof: Same as (C) with our Lemma 2 replacing Lemma 11 of (C).

LEMMA 4. $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ *holds in* $\mathfrak{N}$.

Proof: Since this statement holds in $\mathfrak{M}$ and all sets in $\mathfrak{N}$ are constructible from $T \subseteq \aleph_1 \times \aleph_1$, the proof in (C) of the power set axiom can be extended to show the lemma.[9]

LEMMA 5. *The cardinality of* $T^* = \aleph_1$.

Proof: Immediate.

LEMMA 6. *Every antichain in T is countable.*

Proof: Suppose otherwise. Then some P_0 forces $[F_\delta$ is an uncountable antichain of $T]$. From the definition of forcing, it follows that for each $\alpha < \aleph_1$ there is a $\beta > \alpha$ and $P \geq P_0$ such that P forces $[\beta \in F_\delta]$. We may assume for each such β and P that $\beta \in P^*$. Now let W be the set of these P's. For $P \in W$, define $\varphi(P)$ to be the largest $\beta \in P^*$ for which P forces $[\beta \in F_\delta]$. We conclude by Lemma 1 that there exists $Q \in \Sigma'$, $P_1, P_2 \in W$ such that $Q \geq P_1$, $Q \geq P_2$, $\varphi(P_1) < \varphi(P_2)$, and $Q(\varphi(P_1), \varphi(P_2)) = 1$. But since Q extends P_1, P_2, Q forces $[\varphi(P_1) \in F_\delta]$, $[\varphi(P_2) \in F_\delta]$. Thus Q cannot force $[F_\delta$ is an antichain of $T]$. Since $Q \geq P_0$, this is a contradiction.

LEMMA 7. *Every chain in T is countable.*

Proof: Let F_α be an uncountable chain in T. Whenever some P forces $[\gamma \in F_\alpha]$, $[\delta \in F_\alpha]$, and $[\langle\gamma,\delta\rangle \in T]$ for $\gamma < \delta$, there is an $\omega > \gamma, \delta$ and a $P' \geq P$ such that P' forces $[\langle\gamma,\omega\rangle \in T]$, $[\langle\delta,\omega\rangle \notin T]$, and $[\langle\omega,\delta\rangle \notin T]$. An uncountable antichain in T results by applying this remark to the $\aleph_1$-increasing sequence of ordinals in F_α, thereby contradicting Lemma 6.

Lemmas 5, 6, and 7 imply that T is an $\aleph_1$-Souslin tree in $\mathfrak{N}$. Theorem 1 is therefore established.

THEOREM 2. *There exists a model* $\mathfrak{N}$ *of set theory such that* $\mathfrak{N}$ *contains an* $\aleph_1$-*Souslin tree but* $2^{\aleph_0} \neq \aleph_1$.

This theorem is proved by starting with a model $\mathfrak{M}$ in which $2^{\aleph_0} \neq \aleph_1$ and imitating the proof of Theorem 1.[10]

The results of Mary Ellen Rudin[11] and C. H. Dowker[12] show that if there is an $\aleph_1$-Souslin tree, then there is a normal Hausdorff space whose product with the closed unit interval is not normal. It follows that the nonexistence of such pathological spaces cannot be deduced from the current axioms for set theory.

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1 Cohen, P. J., "The independence of the continuum hypothesis," these PROCEEDINGS, **50,** 1143–1148 (1963).

2 *Ibid.*, **51,** 105–110 (1964).

3 Souslin, M., "Problème 3," *Fundamenta Mathematicae*, **1,** 223 (1920).

4 Solovay, R., and S. Tennenbaum, "Souslin's problem II," to be submitted to *Fundamenta Mathematicae.*

5 A *tree* is a partially ordered set in which the set of elements preceding any given element forms a chain.

6 An *antichain* is a set of pairwise incomparable elements.

7 Miller, Edwin W., "A note on Souslin's problem," *Am. J. Math.*, **65,** 673–678 (1943).

8 The functions K_1 and K_2 are defined in exact analogy to those of (C).

9 For a detailed proof, see ref. 4.

[10] Theorems 1 and 2 are easily seen to remain true when any of the ordinary strong axioms of infinity is adjoined to set theory.

[11] Rudin, Mary Ellen, "Countable paracompactness and Souslin's problem," *Can. J. Math.*, **7**, 543–547 (1955).

[12] Dowker, C. H., "On countably paracompact spaces," *Can. J. Math.*, **3**, 219–224 (1951).