Set Theory

2022/2023 1st Semester

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Name:

University:

Student ID:

General comments.

1. **The time for this exam is 3 hours (180 minutes).**
2. **There are 115 points in the exam: if a student obtains x points, the exam grade will be $\frac{x+15}{13}$.**
3. **Please mark the answers to the questions in Question I on this sheet by crosses.**
4. **Make sure that you have your name, university and student ID on each of the sheets you are handing in.**
5. **If you have any questions, please indicate this silently and someone will come to you. Answers to questions that are relevant for everyone will be announced publicly.**
6. **No talking during the exam.**
7. **Cell phones must be switched off and stowed.**

Question I.	(45 points)	**Question IV.**	(20 points)
Question II.	(20 points)	**Question V.**	(15 points)
Question III.	(15 points)	***TOTAL***	(115 points)
		GRADE	

(45) Question I

Every multiple-choice question below has exactly one correct answer (3 points each):

Week 1. Consider Zermelo's set Z_0 (the smallest set that contains $\varnothing$ and that is closed under $a \mapsto \{a\}$). Which of the following properties is ***not*** shared by Z_0 and ω.

☐ **A**: There is a well-order $\prec$ on the set such that "$x \in y$" implies $x \prec y$".

☐ **B**: The set is infinite.

☐ **C**: There is no well-order $\prec$ on the set such that "$x \in y$" implies $y \prec x$".

☐ **D**: There are elements x and y in the set such that $x \neq y$ and there is a bijection $f : x \to y$.

Week 2. Our definition of the ordered pair is $\langle x, y\rangle = \big\{\{x\}, \{x, y\}\big\}$. What is the relationship between the ordered pair $\langle 0, 1\rangle$ and the ordinal 3?

☐ **A**: They are incomparable.

☐ **B**: $3 = \langle 0, 1\rangle$.

☐ **C**: $3 \subset \langle 0, 1\rangle$ (proper subset).

☐ **D**: $\langle 0, 1\rangle \subset 3$ (proper subset).

Week 3. The definition of $\operatorname{ran} R$ makes sense (formally) for arbitrary sets. Using this formal definition $\operatorname{ran}\omega$ is equal to

☐ **A**: $\varnothing$.

☐ **B**: $\{1, 2\}$.

☐ **C**: $\omega \setminus \{0\}$.

☐ **D**: ω.

Week 4. Zermelo's proof of the well-ordering theorem starts, given a set X, with a choice function γ for the family $\mathcal{P}(X) \setminus \{\varnothing\}$ and produces a well-order $\prec_\gamma$. The map $\gamma \mapsto \prec_\gamma$ from the set of choice functions to the well-orders of X is

☐ **A**: surjective and injective.

☐ **B**: neither injective nor surjective.

☐ **C**: injective but not surjective.

☐ **D**: surjective but not injective.

Week 5. What is $(\omega^{2022} + 2023) \cdot (\omega^{2023} + 2022)$ (ordinal arithmetic)?

☐ **A**: $\omega^{4045} + 4090506$.

☐ **B**: $\omega^{4045} + \omega^{2023} \cdot 2023 + 2022$.

☐ **C**: $\omega^{4045} + \omega^{2022} \cdot 2022 + 2023$.

☐ **D**: $\omega^{4045} + \omega^{2023} + \omega^{2022} \cdot 2022 + 4090506$.

Week 6. One of the following statements ***can*** be proved in ZF without the Axiom of Choice. Which one?

☐ **A**: For every pair of sets X and Y, every surjective map $f : X \to Y$ has a right-inverse: a map $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$.

☐ **B**: For every pair of sets X and Y there is an injective map $f : X \to Y$ or an injective map $f : Y \to X$.

☐ **C**: $\aleph_{2023}$ is a regular cardinal.

☐ **D**: For every pair of sets X and Y, every injective map $f : X \to Y$ has a left-inverse: a map $g : Y \to X$ such that $g \circ f = \mathrm{Id}_X$.

Week 7. Which of the following statements is ***incompatible*** with the Axiom of Foundation?

☐ **A**: There is a sequence $\langle x_n : n \in \omega\rangle$ of sets such that $x_{2n+2} \in x_{2n}$ for all n.

☐ **B**: For every sequence $\langle x_n : n \in \omega\rangle$ of sets there are m and n such that $m < n$ and $x_n \notin x_m$.

☐ **C**: There is a sequence $\langle x_n : n \in \omega\rangle$ of sets such that $x_{3n+2} \in x_{3n+1} \in x_{3n}$ for all n.

☐ **D**: There is a sequence $\langle x_n : n \in \omega\rangle$ of such that $x_n \in x_{2^n}$ for all n.

Week 8. Which ***one*** of the following alephs represents a possible value of $2^{\aleph_{2022}}$?

☐ **A**: $\aleph_{\omega_{2023}+\omega_{2022}}$.

☐ **B**: $\aleph_{\omega_{2023}+\omega_{2023}}$.

☐ **C**: $\aleph_{\omega_{2023}+\omega_{2020}}$.

☐ **D**: $\aleph_{\omega_{2023}+\omega_{2021}}$.

Week 9. Let $S = \{\alpha + 1 : \alpha \in \omega_1\}$ be the set of successor ordinals in ω_1. Using the Pressing-Down Lemma we can prove

☐ **A**: If $f : S \to \omega_1$ is regressive then f is constant on an uncountable subset of S.

☐ **B**: Nothing, because S is not stationary.

☐ **C**: If $f : S \to S$ is regressive then f is constant on an uncountable subset of S.

☐ **D**: S has subsets that are stationary in ω_1.

Week 10. Let $\langle q_n : n \in \omega\rangle$ be an enumeration of $\mathbb{Q}$, the set of rational numbers. Define $F : [\mathbb{R}]^2 \to \omega$ by $F\bigl(\{x, y\}\bigr) = \min\{n : x < q_n < y \text{ or } y < q_n < x\}$ (in words: q_n is the first rational in the enumeration that lies between x and y).
This colouring provides a counterexample to which partition relation?

☐ **A**: $2^{\aleph_0} \to (\aleph_0)^2_2$.

☐ **B**: $\aleph_0 \to (3)^2_3$.

☐ **C**: $\aleph_2 \to (\aleph_1, \aleph_0)^2$.

☐ **D**: $2^{\aleph_0} \to (3)^2_{\aleph_0}$.

Week 11. Let $T = \bigl\{s \in \omega^{<\omega_1} : \bigl|\{i \in \operatorname{dom} s : s(i) \neq 0\}\bigr| < \aleph_0\bigr\}$. The cardinality of the set of branches of length ω_1 of the tree T is

☐ **A**: 0.

☐ **B**: $\aleph_0$.

☐ **C**: $\aleph_1$.

☐ **D**: $2^{\aleph_0}$.

Week 12. The informal definition of $\boldsymbol{L}$ in week 12 was made formal by

☐ **A**: Fixing an enumeration of the formulas of Set Theory.

☐ **B**: Working with Δ_0-formulas only.

☐ **C**: Showing that $\alpha \to L_\alpha$ is absolute.

☐ **D**: Redefining $\operatorname{Def}(M)$ as the closure of a set under some functions.

Week 13. Let M be a countable transitive model of $\mathsf{ZF} - \mathsf{P}$. Which of the following notions ***is not*** downward absolute between M and $\boldsymbol{V}$?

☐ **A**: x is a natural number

☐ **B**: x is countable.

☐ **C**: x is an ordinal

☐ **D**: x is an ordered pair.

Week 14. Let $M \prec L_{\omega_1}$ be a countable elementary substructure. Among the statements below, which is the ***strongest*** that we can prove about M?

☐ **A**: M is isomorphic to L_β for some successor ordinal β.

☐ **B**: M is transitive.

☐ **C**: M is isomorphic to L_β for some limit ordinal β.

☐ **D**: $M = L_\beta$ for some limit ordinal β.

Week 15. Which of the following statements about ω_1-trees in $\boldsymbol{L}$ is provable?

☐ **A**: Every Aronszajn tree is a Souslin tree.

☐ **B**: The $<_{\boldsymbol{L}}$-first Aronszajn tree is also Souslin.

☐ **C**: There is a definable Aronszajn tree that is not Souslin.

☐ **D**: A tree is either an Aronszajn tree or a Kurepa tree.

(20) Question II

In this problem we do not assume the Axiom of Choice.

A set X is Dedekind-finite if every injective map $f : X \to X$ is surjective. We call X S-finite if every surjective map $f : X \to X$ is injective.

(5) (i) Prove that every S-finite set is Dedekind-finite. *Hint*: The contrapositive is easier.

Assume $\langle P_n : n \in \omega\rangle$ is a sequence of two-element sets *without* a choice function, i.e., there is *no* map $f : \omega \to \bigcup_{n\in\omega} P_n$ such that $f(n) \in P_n$ for all n (this assumption is consistent with ZF).

For $n \in \omega$ let T_n be the set of functions s such that $\operatorname{dom} s = n$ and $s(i) \in P_i$ for all $i \in n$. Let $T = \bigcup_{n\in\omega} T_n$ and order T by inclusion.

(5) (ii) Prove that T is a tree and that $|T_n| = 2^n$ for every $n \in \omega$.

(5) (iii) Prove that T has no infinite branches and define a surjective map $f : T \to T$ that is not injective (so T is not S-finite).

(5) (iv) Prove that T is Dedekind-finite. *Hint*: You may use that X is Dedekind-infinite iff there is an injective map $f : \omega \to X$. Use such a map to define an infinite branch.

(15) Question III

Define a relation $\lhd$ on the class $\boldsymbol{On}^2 = \{\langle\alpha,\beta\rangle : \alpha,\beta \in \boldsymbol{On}\}$ of pairs of ordinal numbers by

$$\langle\alpha,\beta\rangle \lhd \langle\gamma,\delta\rangle \text{ if } \begin{cases} \alpha+\beta < \gamma+\delta & \text{or} \\ \alpha+\beta = \gamma+\delta & \text{and } \alpha < \gamma \end{cases}$$

(5) (i) Prove that $\lhd$ is a well-order of $\boldsymbol{On}^2$.

(5) (ii) Prove that for every ordinal δ we have: if $\alpha,\beta < \omega^\delta$ (ordinal arithmetic) then $\alpha+\beta < \omega^\delta$.

(5) (iii) Prove: for every infinite cardinal κ we have $\kappa^2 = \{\langle\alpha,\beta\rangle : \langle\alpha,\beta\rangle \lhd \langle 0,\kappa\rangle\}$ and the order type of $\langle\kappa^2,\lhd\rangle$ is equal to κ.

(20) Question IV

(10) (i) Prove Hausdorff's formula: $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}$.

(10) (ii) Prove the first case of the Erdős-Rado theorem: $(2^{\aleph_0})^+ \to (\aleph_1)^2_{\aleph_0}$.

(15) Question V

We consider the construction of a Souslin tree from the $\Diamond$-principle.

(5) (i) Prove that $\Diamond$ is equivalent to the statement: there is a sequence $\langle f_\alpha : \alpha \in \omega_1\rangle$ such that $f_\alpha : \alpha \to \alpha$ for all $\alpha \in \omega_1$ and such that for every $f : \omega_1 \to \omega_1$ the set $\{\alpha : f \restriction \alpha = f_\alpha\}$ is stationary.

An auto-isomorphism of a tree $\langle T,\lhd\rangle$ is a bijection $g : T \to T$ such that $s \lhd t$ iff $g(s) \lhd g(t)$ for all $s,t \in T$.

The construction of a Souslin tree from $\Diamond$ involved defining an order $\lhd$ on ω_1 such that $\langle S,\lhd\rangle$ was a Souslin tree and the interval $[\omega\cdot\alpha, \omega\cdot(\alpha+1))$ was the αth level S_α of S.

(5) (ii) Assume (for convenience) that $\alpha = \omega\cdot\alpha$ and that the ordering $\lhd$ has been constructed on the set α. Also assume $g : \alpha \to \alpha$ is an auto-isomorphism of $\langle\alpha,\lhd\rangle$ such that $g(\beta) \neq \beta$ for some $\beta < \alpha$. Show how to extend the order $\lhd$ to $\alpha+\omega$ in such a way that there is no extension of g to an auto-isomorphism of $\langle\alpha+\omega,\lhd\rangle$.

(5) (iii) Use the previous part to modify the construction of a Souslin tree so as to obtain one that is *rigid*: the only auto-isomorphism is the identity map.