EXERCISES SET THEORY (03)

2022/23

These exercise are there to fill in the details in the arguments that I gave in class for the properties of ordinal arithmetic.

1. Prove: if $\boldsymbol{F} : \boldsymbol{On} \to \boldsymbol{On}$ is a normal function then

a. $\boldsymbol{F}(\beta) \geqslant \beta$ for all β

b. For every α there is a $\beta \geqslant \alpha$ such that $\boldsymbol{F}(\beta) = \beta$.

2. Verify that the following are normal functions.

a. $\beta \mapsto \alpha + \beta$

b. $\beta \mapsto \alpha \cdot \beta$ (if $\alpha > 0$)

c. $\beta \mapsto \alpha^\beta$ (if $\alpha > 1$)

3. Let α be an ordinal.

a. Prove: there is a β such that $\alpha + \gamma = \gamma$ whenever $\gamma \geqslant \beta$.

b. Express the minimum β with this property in terms of α.

4. Subtraction and division with remainder.

a. Prove: if $\alpha \leqslant \beta$ then there is exactly one γ such that $\beta = \alpha + \gamma$.

b. Prove: if $\alpha \neq 0$ and $\alpha \leqslant \beta$ then there are unique γ and δ such that $\beta = \alpha \cdot \delta + \gamma$ and $\gamma < \alpha$.

5. Prove that the following are equivalent for an ordinal α.

(1) $(\exists\delta)(\alpha = \omega^\delta)$,

(2) $(\forall\beta < \alpha)(\beta + \alpha = \alpha)$,

(3) $(\forall\beta, \gamma < \alpha)(\beta + \gamma < \alpha)$.

Such ordinals are called indecomposable.

Date: maandag 10-10-2022 at 22:37:38 (cest).