EXERCISES SET THEORY (06)

2022/23

Some exercises about trees.

1. Go through the construction of an Aroszajn tree carefully.

a. Show that we can arrange that for all $t \in T$ we have $\sup \operatorname{ran} t \in \mathbb{Q}$.

b. Show that in that new situation: if $\sup \operatorname{ran} s = \sup \operatorname{ran} t$ then $s = t$ or s and t are incomparable.

c. Deduce that in that case our tree is the union of countably many antichains.

2. Let $(T, <_T)$ be the Aronszajn tree constructed in class. Let L be the set of all maximal chains in T.

a. Let $C \in L$. Show that the order type of C is a limit ordinal, call it α_C, and verify that $\bigcup C$ is a strictly increasing function from α_C to $\mathbb{Q}$, call it f_C.

Define an order on L by $C \prec D$ iff $f_C(\gamma) < f_D(\gamma)$, where $\gamma = \min\{\beta : f_C(\beta) \neq f_D(\beta)\}$, and $<$ is the normal order of $\mathbb{Q}$.

b. Prove that $\prec$ is a linear order.

c. Prove that there is no strictly increasing ω_1-sequence in $(L, \prec)$. *Hint*: Let $\langle C_\xi : \xi < \omega_1 \rangle$ is strictly increasing. Prove that for every α there are a $t_\alpha \in T_\alpha$ and an $\eta < \omega_1$ such that for all $\xi > \eta$ the order type of C_ξ is larger than α and $t_\alpha \in C_\xi$. Deduce that $\langle t_\alpha : \alpha < \omega_1 \rangle$ would be an uncountable branch in T.

Date: donderdag 24-11-2022 at 15:50:56 (cet).