EXERCISES SET THEORY (08)

2022/23

A direct proof that $|L_\alpha| = |\alpha|$, when $\alpha \geqslant \omega$. And some exercises on elementarity.

1. We assume we have the well-orders $<_\alpha$ of the L_α as constructed in class and Jech's book; we denote the order-type of $(L_\alpha, < \alpha)$ by λ_α. Fix an $\alpha \geqslant \omega$ and let β_n be the order-type of W_n^α for each n.

a. Show that $\beta_{n+1} \leqslant \beta_n^2 \cdot 10$ for all n (ordinal arithmetic).

b. Deduce that $\lambda_{\alpha+1} \leqslant \lambda_\alpha^\omega$ (ordinal arithmetic).

c. Deduce that $|L_{\alpha+1}| = |L_\alpha|$.

2. Continuing the previous exercise, but making α variable again.

a. Prove that for cardinals κ we have $\lambda_\kappa = \kappa$.

b. Prove that $\alpha \mapsto \lambda_\alpha$ is a normal function.

c. Prove: if κ is regular then $\{\alpha \in \kappa : \lambda_\alpha = \alpha\}$ is closed and unbounded.

3. This exercise fleshes out the comments made in class about elementary substructures of the $H(\kappa)$. We work in $H(\aleph_2)$ and we let M be a countable elementary substructure of $H(\aleph_2)$. We let $\delta = M \cap \omega_1$.

a. Show that $\emptyset \in M$. *Hint*: We have $(\exists x \in H(\aleph_2))((\forall y \in x)(y \neq y))^{H(\aleph_2)}$, so we also have $(\exists x \in M)((\forall y \in x)(y \neq y))^{H(\aleph_2)}$. But $\emptyset$ is the *only* member of $H(\aleph_2)$ that satisfies the formula.

b. Show: if $\alpha \in M$ then $\alpha + 1 \in M$. *Hint*: Again: there is only one element of $H(\aleph_2)$ that satisfies the defining formula of $\alpha + 1$.

c. Show: $\omega \in M$ and $\omega_1 \in M$ (so M is definitely not transitive). *Hint*: ω and ω_1 are uniquely definable.

d. Prove that $\delta \subseteq M$ and $\delta \notin M$.

e. Let $C \in M$ be a cub subset of ω_1. Prove that $\delta \in C$. *Hint*: Prove that $\delta = \sup(C \cap \delta)$, via $(\exists \gamma \in H(\aleph_2))(\gamma \in C \wedge \gamma > \alpha)$, where $\alpha < \delta$.

f. Let $S \in M$ be such that $\delta \in S$. Prove that S is stationary in M. *Hint*: Assume there is a cub set C such that $C \cap S = \emptyset$. Show that $C \in H(\aleph_2)$. Deduce that there is (another) cub set D such that $D \in M$ and $D \cap S = \emptyset$.

Date: dinsdag 13-12-2022 at 11:50:05 (cet).