EXERCISES SET THEORY (09)

2022/23

On variations of $\diamondsuit$, and a direct construction of a Souslin line.

1. Use suitable bijections to show that the following statements are equivalent to $\diamondsuit$:

(1) There is a sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ of sets such that $A_\alpha \subseteq \alpha \times \alpha$ for all α, and for every subset A of $\omega_1 \times \omega_1$ the set $\{\alpha : A \cap (\alpha \times \alpha) = A_\alpha\}$ is stationary.

(2) There is a sequence $\langle f_\alpha : \alpha < \omega_1 \rangle$ of functions such that $f_\alpha : \alpha \to \alpha$ for all α, and for every function $f : \omega_1 \to \omega_1$ the set $\{\alpha : f \restriction \alpha = f_\alpha\}$ is stationary.

(3) There is a sequence $\langle \langle A_\alpha, B_\alpha \rangle : \alpha < \omega_1 \rangle$ of such that $A_\alpha, B_\alpha \subseteq \alpha$ for all α, and for every ordered pair $\langle A, B \rangle$ of subsets of ω_1 the set $\{\alpha : A \cap \alpha = A_\alpha \text{ and } B \cap \alpha = B_\alpha\}$ is stationary.

2. To prove that $\diamondsuit^*$ implies $\diamondsuit$ we need an equivalent of $\diamondsuit$ that looks more like $\diamondsuit^*$. The principle $\diamondsuit^-$ says: there is a sequence $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$ such that $\mathcal{A}_\alpha$ is a countable subfamily of $\mathcal{P}(\alpha)$ for all α, and for every subset A of ω_1 the set $\{\alpha : A \cap \alpha \in \mathcal{A}_\alpha\}$ is stationary.

a. Verify that $\diamondsuit$ and $\diamondsuit^*$ both imply $\diamondsuit^-$.

We now prove that $\diamondsuit^-$ implies $\diamondsuit$. Let $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit^-$-sequence.

b. Use a bijection to transform $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$ into $\langle \mathcal{B}_\alpha : \alpha < \omega_1 \rangle$ where $\mathcal{B}_\alpha$ is a countable subfamily of $\mathcal{P}(\alpha \times \omega)$ for all α and for every $B \subseteq \omega_1 \times \omega$ the set $\{\alpha : B \cap (\alpha \times \omega) \in \mathcal{B}_\alpha\}$ is stationary.

Enumerate every $\mathcal{B}_\alpha$ as $\langle B_\alpha(k) : k < \omega \rangle$.

c. Prove: for every $B \subseteq \omega_1 \times \omega$ there is a k such that $\{\alpha : B \cap (\alpha \times \omega) = B_\alpha(k)\}$ is stationary.

Define $B_\alpha(k, n) = \{\gamma \in \alpha : \langle \gamma, n \rangle \in B_\alpha(k)\}$.

d. Prove: there is an n such that $\langle B_\alpha(n, n) : \alpha < \omega_1 \rangle$ is a $\diamondsuit$-sequence. *Hint*: Assume not and choose a counterexample X_n for every n. Let $X = \bigcup_{n<\omega} (X_n \times \{n\})$ and show that part c would fail for X.

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3. We construct a Souslin line directly, using $\Diamond$, in two steps. First we build a linear order $\lhd$ on ω_1 with the following properties

(1) it is a dense order without first of last elements,
(2) all closed nowhere dense sets are countable, and
(3) it is not separable.

Then the Dedekind completion of $(\omega_1, \lhd)$ is a counterexample to Souslin's question. A set K is closed is for every $x \in \omega_1 \setminus K$ there is an interval (u, v) that contains x and is disjoint from K. A set closed K is nowhere dense is there is not non-trivial interval (u, v) that is contained in K.

a. Prove that property (2) implies that pairwise disjoint families of open intervals are countable.
Hint: By Zorn's Lemma assume that $\{(u_i, v_i) : i \in I\}$ is a *maximal* pairwise disjoint family of open intervals in $(\omega_1, \lhd)$. Show that $K = \omega_1 \setminus \bigcup_{i \in I}(u_i, v_i)$ is closed and nowhere dense, and that $u_i, v_i \in K$ for all i.

Let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\Diamond$-sequence. We construct linear orders $\lhd_\alpha$ on $\omega \cdot \alpha$ for all α such that $(\omega \cdot \alpha, \lhd_\alpha)$ is isomorphic to $\mathbb{Q}$, and such that $\lhd_\alpha$ extends $\lhd_\beta$ whenever $\beta < \alpha$.

To begin let $\lhd_1$ order ω via some bijection with $\mathbb{Q}$. If α is a limit let $\lhd_\alpha = \bigcup_{\beta < \alpha} \lhd_\beta$.

b. Show that $(\omega \cdot \alpha, \lhd_\alpha)$ is isomorphic with $\mathbb{Q}$.

For the successor step we need to look at $\mathbb{Q}$ and $\mathbb{R}$.

c. Prove: if $K \subseteq \mathbb{Q}$ is closed and nowhere dense *in* $\mathbb{Q}$ then its closure in $\mathbb{R}$ is closed and nowhere dense *in* $\mathbb{R}$.
d. Prove (or remember) the Baire Category Theorem: if $\{K_n : n < \omega\}$ is a family of closed and nowhere dense subsets of $\mathbb{R}$ then $\mathbb{R} \setminus \bigcup_n K_n$ is dense in $\mathbb{R}$.
e. Deduce: if $\{K_n : n < \omega\}$ is a family of closed and nowhere dense subsets of $\mathbb{R}$ then there are irrational numbers in $\mathbb{R} \setminus \bigcup_n K_n$. *Hint*: If $q \in \mathbb{Q}$ then $\{q\}$ is nowhere dense.

Assume $\lhd_\alpha$ is given such that $(\omega \cdot \alpha, \lhd_\alpha)$ is isomorphic to $\mathbb{Q}$. A *Dedekind cut* in $(\omega \cdot \alpha, \lhd_\alpha)$ is a pair of subsets such that such that $A \cup B = \omega \cdot \alpha$, if $a \in A$ and $b \in B$ then $a \lhd_\alpha b$, A has no maximum, and B has no minimum.

f. Show that there is a Dedekind cut (A, B) in $(\omega \cdot \alpha, \lhd_\alpha)$ such that for every $\beta \leqslant \alpha$: if A_β is closed and nowhere dense in $(\omega \cdot \alpha, \lhd_\alpha)$ then there are $a \in A$ and $b \in B$ such that $(a, b) \cap A_\beta = \emptyset$.
Hint: Via an isomorphism with $\mathbb{Q}$ find an irrational number that is not in the closure (in $\mathbb{R}$) of any A_β, with $\beta \leqslant \alpha$, that is closed and nowhere dense in $(\omega \cdot \alpha, \lhd_\alpha)$.

Now order the set $I_\alpha = \{\omega \cdot \alpha + n : n$
$in\omega\}$ isomorphic with $\mathbb{Q}$ and insert it between A and B: $a \lhd_{\alpha+1} x \lhd_{\alpha+1} b$ whenever $a \in A$, $x \in I_\alpha$, and $b \in B$.

g. Show that $\lhd_{\alpha+1}$ orders $\omega \cdot (\alpha + 1)$ isomorphic with $\mathbb{Q}$.

In the end let $\lhd = \bigcup_{\alpha < \omega_1} \lhd_\alpha$.

h. Show that $(\omega_1, \lhd)$ has properties (1), (2), and (3). *Hint*: For (2): is A is closed and nowhere dense then show that $\{\alpha : A \cap \omega \cdot \alpha \text{ is closed and nowhere dense in } (\omega \cdot \alpha, \lhd_\alpha)\}$ is cub. For (3): the interval I_α (and anything inserted into it later) shows that $\omega \cdot \alpha$ will not be a dense subset in the final order.