HOMEWORK 01

1 LET $A = \{ x \in Z_0 : x = \emptyset$ OR $x = \{y\}$ FOR SOME $y \in Z_0 \}$

THEN $\emptyset \in A$ AND IF $x \in A$ THEN $x \in Z_0$ AND SO $\{x\} \in A$.

SO A SATISFIES THE CONDITIONS THAT Z_0 SATISFIES MINIMALLY.

WE FIND $Z_0 \subseteq A$. AS $A \subseteq Z_0$ BY DEFINITION WE FIND $A = Z_0$

2 LET $B = \{ x \in Z_0 : A_x$ IS DEFINED $\}$

THEN - $\emptyset \in B$ BY DEFINITION

- IF $x \in B$ THEN $\{x\} \in B$ WITH $A_{\{x\}} = A_x \cup \{x\}$

AS IN PROBLEM 1 THIS SHOWS $B = Z_0$.

3 LET $C = \{ x \in Z_0 : A_x$ SATISFIES THE GIVEN CONDITION $\}$

THEN - $\emptyset \in C$ BECAUSE IF $y \in \emptyset$ THEN --- IS ALWAYS TRUE

- IF $x \in C$ AND $x \neq \emptyset$ THEN $x = \{u\}$ FOR SOME u

SO $A_x = A_u \cup \{u\}$

IF $y \in A_x$ THEN $y \in A_u$ OR $y = u$

IF $y \in A_u$ THEN $y = \emptyset$ OR $y = \{z\}$ FOR SOME $z \in A_u \subseteq A_x$

IF $y = u$ THEN $y = \emptyset$ OR $y = \{z\}$ FOR SOME $z \in Z_0$

WE MUST SHOW $z \in A_u$.

SO: AN EXTRA INDUCTION: IF $x = \{y\}$ THEN $y \in A_x$

$C' = \{ x \in Z_0 : x = \emptyset$ OR $x = \{y\}$ FOR SOME $y \in A_x \}$

- $\emptyset \in C'$

- IF $x \in C'$ THEN $x \in A_{\{x\}}$ AND $\{x\} = \{x\}$ SO $\{x\} \in C'$.

4 LET $D = \{ x \in Z_0 :$ IF $y \in A_x$ THEN $A_y \subseteq A_x \}$

- $\emptyset \in D$ BECAUSE $A_\emptyset = \emptyset$

- IF $x \in D$ AND $y \in A_{\{x\}}$ THEN

- $y \in A_x$ AND $A_y \subseteq A_x \subseteq A_{\{x\}}$, OR

- $y = x$ AND $A_x \subseteq A_{\{x\}}$

SO $D = Z_0$

5 LET $E = \{x \in Z_0 :$ FOR ALL $y \in Z_0$: $A_y \subseteq A_x$ OR $A_x \in A_y\}$

- $\emptyset \in E$ BECAUSE $\emptyset \subseteq A_y$ FOR ALL y.
- ASSUME $x \in Z_0$ AND LET $y \in Z_0$

  IF $A_y \subseteq A_x$ THEN ALSO $A_y \subseteq A_{\{x\}}$

  IF $A_x \in A_y$ THEN

  - $x \in A_y$ BUT THEN $A_{\{x\}} = A_x \cup \{x\} \in A_y$, OR
  - $x \notin A_y$ SO $x = \{u\}$ FOR SOME $u \in A_x$

    HENCE $u \in A_y$

    CLAIM $\{u\} \in A_y$ OR $\{u\} = y$

  $E' = \{x \in Z_0 :$ FOR ALL $u \in A_x$ EITHER $\{u\} \in A_x$ OR $\{u\} = x\}$

  - $\emptyset \in E'$ ~~AS~~ ALWAYS
  - IF $x \in E'$ AND $u \in A_{\{x\}}$

    THEN $u \in A_x$ AND SO $\{u\} \in A_x \cup \{x\}$

    OR $u = x$ AND SO $\{u\} = \{x\}$

  SO $E' = Z_0$.

  BUT $\{u\} \in A_y$ IMPLIES $x \in A_y$ CONTRADICTION

  AND $\{u\} = y$ IMPLIES $y = x$ AND $A_y = A_x$.

6 LET $F = \{x \in Z_0 :$ IF $A \subseteq A_x$ IS NONEMPTY THEN THERE IS $y \in A$ SUCH THAT $A_y \subseteq A_z$ FOR $z \in A\}$

- $\emptyset \in F$ AS ALWAYS
- ASSUME $x \in F$ LET $A \subseteq A_{\{x\}}$ BE NONEMPTY

  CASE 1 $A = \{x\}$ THEN WE ARE DONE

  CASE 2 $A \neq \{x\}$ SO $A \cap A_x \neq \emptyset$

  LET $y \in A \cap A_x$ BE AS IN THE ASSUMPTION

  SO $z \in A \cap A_x$ IMPLIES $A_y \subseteq A_z$

  BUT $A_y \in A_x$ BY 4

  SO $A_y \in A_z$ FOR $z \in A$.

7 IF $A \in Z_0$ IS NONEMPTY THEN TAKE $x \in A$

WE FIND $A \cap A_{\{x\}} \neq \emptyset$

SO THERE IS A $y \in A \cap A_{\{x\}}$ SUCH THAT

$A_y \subseteq A_z$ FOR $z \in A \cap A_{\{x\}}$

LET $z \in A \setminus A_{\{x\}}$ CLAIM $A_{\{x\}} \subseteq A_z$

PROOF CONSIDER E' IN 5

USE THAT TO SHOW $A_{\{x\}} \subseteq A_z$

- $\emptyset \in A_z$
- IF $y \in A_{\{x\}}$ AND $y \in A_z$ THEN $\{y\} \in A_z$ OR $\{y\} = z$

  BUT $\{y\} \in A_{\{x\}}$ OR $y = \{x\}$

  AND SO $\{y\} \neq z$: $\{y\} \in A_z$

8 $x \times y = \{z \in \mathcal{P}\mathcal{P}(x \cup y) : (\exists u \in x)(\exists v \in y)(z = \langle u, v \rangle)\}$