HOMEWORK SET THEORY (02) 2022-10-03

2022/23

Hand in next week by 14:00 on 2022-10-11, either by hand in class (on 2022-10-10 of course), or by uploading to the course page on `elo.mastermath.nl`.

This homework is about well-orders. Two larger questions rather than many small ones.
We assume the Power Set Axiom.

1. This problem deals with the statement "for every two sets x and y there is an injection from x into y or an injection from y into x"; let us call it T.

a. Prove: the Axiom of Choice implies T.

b. Prove: for every set x there is an ordinal α such that there is no injective map from α into x. (The smallest such ordinal is written $\aleph(x)$ and called Hartogs' aleph.) *Hint*: Consider the set of pairs $\langle A, R\rangle$ where $A \subseteq x$, $R \subseteq x \times x$ and Rwell-orders A. Collect their order types in a set.

c. Prove that T implies the Axiom of Choice.

2. This problem leads you through Zermelo's second proof of the Well-Ordering Theorem. As in the first proof we start with a set M and a choice function $\gamma : \mathcal{P}(M) \setminus \{\emptyset\} \to M$.

A subfamily $\mathcal{A}$ of $\mathcal{P}(M)$ is a γ-chain if

(1) $M \in \mathcal{A}$,
(2) if $A \in \mathcal{A} \setminus \{\emptyset\}$ then $A' = A \setminus \{\gamma(A)\}$ belongs to $\mathcal{A}$ too, and
(3) if $\mathcal{A}' \subseteq \mathcal{A}$ then $\bigcap \mathcal{A}' \in \mathcal{A}$.

a. Verify that $\mathcal{P}(M)$ is a γ-chain.

b. Show if $\mathfrak{A}$ is a family of γ-chains then $\bigcap \mathfrak{A}$ is a γ-chain.

Now let $\mathcal{W}$ be the intersection of the collection of *all* γ-chains. Call $A \in \mathcal{W}$ *comparable* if for all $U \in \mathcal{W}$ we have $A \subseteq U$ or $U \subseteq A$.

c. Prove that every member of $\mathcal{W}$ is comparable. *Hint*: Show that the comparable elements of $\mathcal{W}$ form a γ-chain. When proving property (2) show that $\{U \in \mathcal{W} : U \subseteq A' \text{ or } A' \subseteq U\}$ is a γ-chain.

d. Let $N \subseteq M$ be nonempty. Show there is a unique member W of $\mathcal{W}$ such that $N \subseteq W$ and $\gamma(W) \in N$. *Hint*: Consider $\bigcap\{W \in \mathcal{W} : N \subseteq W\}$.

In particular we have for every $x \in M$ a unique $W_x \in \mathcal{W}$ such that $x \in W_x$ and $x = \gamma(W_x)$. Define $x \prec y$ iff $y \in W'_x$.

e. Prove that $\prec$ is a well-order of M.

f. Bonus: both proofs of Zermelo construct a well-order from γ in an unambiguous way. What is the relation (if any) between the resulting two well-orders?

Date: woensdag 05-10-2022 at 18:03:55 (cest).